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DISKS ON A TREE: ANALYSIS OF A COMBINATORIAL GAME*

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Abstract. Anderson et al. [*Amer. Math. Monthly*, 96 (1989), pp. 481–493] studied a combinatorial game on an infinite path that is started with n disks at a vertex and ends with the disks spread between $k = \lfloor n/2 \rfloor$ vertices to the left and to the right of the initial vertex. They showed that the number of steps the game takes to converge to the final configuration is $ck^2 + o(k^2)$ for some constant c. We generalize this game to the case of an infinite rooted tree, where each vertex has degree $d + 1$ and where the earlier game corresponds to the case $d = 1$. We determine the final configuration when the game is started with n disks at the root and show that in this final configuration all disks are at depth at most $k = \Theta(\log_d n)$ for $d \geq 2$. We also show that the number of steps that the game takes to converge to the final configuration in this case is at most $O(k(1 + \log_d k))$, so that the convergence is faster than what it was for the case $d = 1$. We generalize the game to the case where the vertices at depth i in the tree have $d_i \geq 2$ children, where the d_i are not necessarily the same, and show that the convergence time in this case is at most $O(k^{1.5} + k \log_{d_{\min}} d_{\max})$, where $d_{\min}$ and $d_{\max}$ are the smallest and largest d_i, respectively.

Key words. disks, tree, chip-firing games

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1. Introduction.

In this article we study a very simple combinatorial game that can be played with several piles of disks arranged in a tree. At each unit of time, each pile, sitting at a vertex of degree d, is divided into d equal piles, which are moved to the d neighbors of the vertex, leaving a remainder of at most $d - 1$ disks at the original vertex.

We are interested in the case of an infinite rooted tree, where the root is at depth zero and has $d_0 + 1$ children, and in general each vertex at depth i has d_i children. The initial configuration has n disks.

This game was studied by Anderson et al. [1] in the case where all $d_i = 1$, so that the tree is an infinite path. They determined the final configuration and showed that this configuration is reached in $c(n/2)^2 + o(n^2)$ steps for some $1/3 \leq c \leq \pi^2/6 - 1$. Björner, Lovász, and Shor [3] studied the related slowed-down game on an arbitary graph with n vertices and m edges, where a single move consists of selecting a vertex of degree d with at least d disks and moving these disks to the d neighbors. They showed that the final configuration and the number of moves depend only on the initial configuration and that the game is infinite if the number of disks is greater than $2m - n$, is finite if the number of disks is smaller than m, and can be finite or infinite depending on the initial configuration if the number of disks is between m and $2m - n$. Tardos [27] showed that there exist graphs with an initial configuration for which the number of steps of this slowed-down game is $\Omega(n^4)$ and that the number of steps is always bounded by $2nmd = O(n^4)$, where d is the diameter of the graph.

Various versions of such games on graphs have been studied as chip-firing games and Abelian sandpile models, including the work of Goles et al. [20, 21, 22, 23, 24, 25, 26], Dhar et al. [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19], and others [2, 4].

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Back at the case of an infinite tree, if we denote by x_{ri} the number of disks at any one vertex at depth i after r steps, then we obtain the recurrence $x_{r0} = x_{(r-1)0} \bmod (d_0+1) + (d_0+1)\lfloor x_{(r-1)1}/(d_1+1)\rfloor$, and for $i \geq 1$, $x_{ri} = \lfloor x_{(r-1)(i-1)}/(d_{i-1}+1)\rfloor + x_{(r-1)i} \bmod (d_i+1) + d_i\lfloor x_{(r-1)(i+1)}/(d_{i+1}+1)\rfloor$. The base case is $x_{00} = n$ and $x_{0i} = 0$ for $i \geq 1$.

We first determine the final configuration for this game. For this final configuration, we denote by n_i the number of disks sitting at a subtree rooted at a vertex at depth i and by e_i the number of disks sitting at a vertex at depth i. Then $n_0 = n$, $e_0 = n \bmod (d_0+1)$, $n_1 = \lfloor n/(d_0+1)\rfloor$, $e_i = 1 + (n_i - 1) \bmod d_i$ for $1 \leq i \leq k$, and $n_{i+1} = \lfloor (n_i - 1)/d_i \rfloor$ for $1 \leq i \leq k$. Here k is the first i such that $n_{i+1} = 0$. In fact no disk ever reaches depth $k+1$.

We next study the number of steps it takes for the game to reach its final configuration. We focus on the case where all $d_i = d \geq 2$. We first consider the special case where $n = (d+1)(d^k-1)/(d-1)$, so that all $e_i = 1$ for $1 \leq i \leq k$. For this case, we show that the number of steps is bounded by $O(k) = O(\log_d n)$. The proof is based on a comparison with the fractional game where no remainder is left at a vertex.

For general n, repeated applications of the preceding result give a bound of $O(k(1+\log_d k)) = O(\log_d n \log_d \log_d n)$ on the number of steps.

We next consider the case where all $d_i \geq 2$ are not necessarily equal. Again we first consider the special case where all $e_i = 1$ for $1 \leq i \leq k$. For this case, we show that the number of steps is bounded by $O(\log_{d_{\min}} n) = O(\sum_{i\leq k} \log_{d_{\min}} d_i) = O(k \log_{d_{\min}} d_{\max})$, where $d_{\min}$ and $d_{\max}$ denote the smallest and the largest d_i for $1 \leq i \leq k$.

We then obtain bounds for general n. If $2 \leq d_i \leq d_j$ for $1 \leq i \leq j$, then the number of steps is bounded by $O(k \log_{d_{\min}}(k d_{\max}))$.

If $d_i \geq d_j$ for $1 \leq i \leq j$, then the number of steps is at most $2k^2$.

In the general case of n arbitrary and all $d_i \geq 2$ for $i \geq 1$, the number of steps is bounded by $O(k^{1.5} + \log_{d_{\min}} n) = O(k^{1.5} + k \log_{d_{\min}} d_{\max})$.

We finally obtain a lower bound of $\Omega(k + \max_{1\leq i\leq k} \sum_{i\leq j\leq k} \log_{d_i} d_j)$ on the number of steps if all $d_i \geq 2$. Thus the upper bound for the case where all $e_i = 1$ for $0 \leq i \leq k$ is tight up to constant factors, provided $\log_{d_{\min}} d_1 = O(1)$, that is, $d_1 \leq d_{\min}^{O(1)}$.

The analysis of the game in the case where some of the d_i for $i \geq 1$ satisfy $d_i = 1$ and some satisfy $d_i \geq 2$ remains open.

The case of a tree is thus interesting because, unlike the case of a path, the number of steps depends only logarithmically on the number of disks, and the dependence seems to be essentially linear in the depth of the tree reached by the final configuration. This contrasts with the fact that in the case of a path, the dependence is quadratic in the length of the path, which equals in that case the number of initial disks. In fact, none of the previously studied cases in the literature shows dependence that is only logarithmic in the number of disks or linear in the diameter of the graph.

2. The final configuration. Recall that if we denote by x_{ri} the number of disks at any one vertex at depth i after r steps, then we obtain the recurrence $x_{r0} = x_{(r-1)0} \bmod (d_0+1) + (d_0+1)\lfloor x_{(r-1)1}/(d_1+1)\rfloor$, and for $i \geq 1$, $x_{ri} = \lfloor x_{(r-1)(i-1)}/(d_{i-1}+1)\rfloor + x_{(r-1)i} \bmod (d_i+1) + d_i\lfloor x_{(r-1)(i+1)}/(d_{i+1}+1)\rfloor$. The base case is $x_{00} = n$ and $x_{0i} = 0$ for $i \geq 1$.

LEMMA 1. *The combinatorial game terminates in some final configuration.*

Proof. Consider the potential function $\phi_r = \sum_i t_{ri} i^2$, where t_{ri} is the total number of disks at all vertices at depth i after r steps. This potential function increases at each step: if d_0+1 disks at depth 0 are moved to depth 1, it increases by d_0+1; if

d_i+1 disks at depth i are moved so that one disk goes to depth $i-1$ and d_i disks go to depth $i+1$, it increases by $(i-1)^2+d_i(i+1)^2-(d_i+1)i^2=2(d_i-1)i+d_i+1$.

Consequently no configuration ever repeats. Suppose that after some number r of steps, all depths up to depth k_r have been reached by disks. The number of disks at depth r cannot be more than $n/2^{k_r}\geq 1$. Therefore $k_r\leq \log_2 n$.

Since there is a finite number of configurations that never reach depth $2n$, and no configuration ever repeats because the potential function always increases, a final configuration must eventually be reached. □

We determine the final configuration for this game. For this final configuration, we denote by n_i the number of disks sitting at a subtree rooted at a vertex a depth i and by e_i the number of disks sitting at a vertex at depth i.

THEOREM 1. *The final configuration is given by* $n_0=n$, $e_0=n \bmod (d_0+1)$, $n_1=\lfloor n/(d_0+1)\rfloor$, $e_i=1+(n_i-1) \bmod d_i$ *for* $1\leq i\leq k$, *and* $n_{i+1}=\lfloor (n_i-1)/d_i\rfloor$ *for* $1\leq i\leq k$. *Here* k *is the first* i *such that* $n_{i+1}=0$. *In fact no disk ever reaches depth* $k+1$.

Proof. Since the d_0+1 subtrees rooted at depth 1 are identical, it follows that the total number of disks remaining in such subtrees is a multiple of d_0+1. Since $0\leq e_0\leq d_0$, it follows that $e_0=n \bmod (d_0+1)$, and therefore $n_1=\lfloor n/(d_0+1)\rfloor$.

Consider the n_i disks remaining at a subtree rooted at depth i. Since a vertex at depth i has d_i identical subtrees rooted at depth $i+1$, it follows that the total number of disks remaining in such subtrees is a multiple of d_i. Since $0\leq e_i\leq d_i$, it follows that $e_i=n_i \bmod d_i$ if n_i is not divisible by d_i, and otherwise either $e_i=0$ or $e_i=d_i$. We shall show that $e_i=0$ is not possible, so in this case $e_i=d_i$, and so in general $e_i=1+(n_i-1) \bmod d_i$, implying $n_{i+1}=\lfloor (n_i-1)/d_i\rfloor$ for $i\geq 2$. Since $n_{i+1}=0$, no disk ever reaches depth $k+1$.

It remains to show that $e_i=0$ is not possible for $1\leq i\leq k$. Suppose $e_i=0$. The last time disks left depth i, each vertex at depth $i-1$ received at least d_{i-1} disks from its children, so $e_{i-1}=d_{i-1}$. Similarly, there were 0 disks at depth $i-1$ before these d_{i-1} disks arrived from depth i; otherwise we would later get a nonzero number of disks at depth i, so the last time disks left depth $i-1$ happened before, and each vertex at depth $i-2$ received at least d_{i-2} disks from its children, so $e_{i-2}=d_{i-2}$. Proceeding inductively, we obtain $e_1=d_1$, and there were 0 disks at depth 1 before these d_1 disks arrived from depth 2, so the last time disks left depth 1, the root at depth 0 received at least d_0+1 disks from its children. This would give $e_0\geq d_0+1$, contrary to the fact that $e_0\leq d_0$. This completes the proof. □

3. The case of all $d_i=d\geq 2$. We shall study the number of steps it takes for the game to reach its final configuration. In this section, all d_i have the same value $d_i=d\geq 2$. If we denote by x_{ri} the number of disks at a vertex at depth i after r steps, then we obtain the recurrence $x_{r0}=x_{(r-1)0} \bmod (d+1)+(d+1)\lfloor x_{(r-1)1}/(d+1)\rfloor$, and for $i\geq 1$, $x_{ri}=\lfloor x_{(r-1)(i-1)}/(d+1)\rfloor+x_{(r-1)i} \bmod (d+1)+d\lfloor x_{(r-1)(i+1)}/(d+1)\rfloor$. The base case is $x_{00}=n$ and $x_{0i}=0$ for $i\geq 1$.

There is a closely related fractional game where no remainder is left at a vertex. For this fractional game, we study the recurrence $y_{ri}=y_{(r-1)(i-1)}/(d+1)+dy_{(r-1)(i+1)}/(d+1)$. The base case is $y_{00}=n$ and $y_{0i}=0$ for $i\neq 0$. Here we are allowing i to be negative.

LEMMA 2. *The solution of the recurrence is* $y_{r(2i-r)}=n(1/d)^i(d/(d+1))^r\binom{r}{i}$ *and* $y_{ri}=0$ *for* $i+r$ *odd.*

Proof. Clearly $y_{ri}=0$ unless r and i are either both even or both odd. Let $z_{r(2i-r)}=d^iy_{r(2i-r)}$. Then $z_{(r+1)(2i-(r+1))}/d=z_{r(2(i-1)-r)}/(d+1)+z_{r(2i-r)}/(d+1)$.

Let $w_{r(2i-r)} = ((d+1)/d)^r z_{r(2i-r)}$. Then $w_{(r+1)(2i-(r+1))} = w_{r(2(i-1)-r)} + w_{r(2i-r)}$.

Then $w_{r(2i-r)} = n\binom{r}{i}$. Therefore $z_{r(2i-r)} = n(d/(d+1))^r\binom{r}{i}$, and so $y_{r(2i-r)} = n(1/d)^i(d/(d+1))^r\binom{r}{i}$. □

We shall use the concept of slowed-down versions of the combinatorial game. Here not all disks that could be moved at a given point in time are moved, so that moving these disks is delayed until later. This means that the slowed-down game takes longer to reach the final configuration than the original game. See also [3, 27]. Thus in a slowed-down game, we still move the same number of disks to each neighbor, but we may choose a smaller number of such disks to move, so that a number larger than the smallest possible remainder is left at each chosen vertex. This results in partially postponing the full move that would happen at a step, so that the rest of the move will happen later. The result is that the number of steps is increased when we go to the slowed-down game, yet the same final configuration is eventually reached. We also consider at times a fractional game, where fractions of disks may be moved to all neighbors, in the same quantity to each neighbor, as opposed to moving only full disks, which results again in postponing the move of the remaining fraction, while eventually reaching the same final configuration.

LEMMA 3. *In the combinatorial game with all $d_i = d \geq 2$ and $n = (d+1)(d^k-1)/(d-1)$, so that $e_0 = 0$ and all $e_i = 1$ for $1 \leq i \leq k$, depth k is reached in $O(k)$ steps (independently of d).*

Proof. We slow down the combinatorial game by requiring that if there are at least d disks at a vertex at depth i after $r-1$ steps, then exactly d disks are left at depth i for the rth step; if there are at most d disks at a vertex at depth i, then none of these disks is moved. We show that this slowed-down fractional game reaches depth k within $O(k)$ steps. This implies that the original combinatorial game, which is not slowed down, will reach depth k as well.

The numbers of disks t_{ri} for the slowed down game are upper bounded by $t_{ri} \leq d + y_{ri}$, where the y_{ri} are the quantities from the recurrence for the preceding lemma, since disks in excess of d are moved according to fractional game defining the y_{ri}, and so the claim follows by induction. That is, the game played above d disks always has $t_{ri} - d \leq y_{ri}$, since those excess disks satisfy the recurrence for the y_{ri}, except that some disks may be lost if they reach a pile with fewer than d disks.

We bound the $y_{r(2i-r)}$ for $i \geq r/2$ by

$$y_{r(2i-r)} \leq n(1/d)^i(2d/(d+1))^r \leq n(4d/(d+1)^2)^{r/2}.$$

If we let $r = ck$ for a large constant c, then for $i \geq r/2$ we have $y_{r(2i-r)} \leq n(1/d)^{c'k}$ for another large constant c' depending on c.

If all vertices at depth $0 \leq i \leq k-1$ have d disks, then this accounts for exactly $n-1$ disks. The excess $y_{r(2i-r)} \leq n(1/d)^{c'k}$ in the bound $t_{ri} \leq d + y_{ri}$ for $0 \leq i \leq k-1$ accounts for strictly less than 1 disk if c' is large enough. Therefore some fraction of one disk must have reached depth k by step $r = ck$ in the slowed-down fractional game, so at least one disk will have reached depth k by step $r = ck$ in the combinatorial game. □

Define a *special configuration* to be a configuration where the sequence $x_{r0}x_{r1}\cdots x_{rk}$ is given by $01^*((d+1)d^*01^*)^*1$ or by $((d+1)d^*01^*)^+1$. Here x^* denotes any nonnegative number of copies of x, and x^+ denotes any positive number of copies of x.

LEMMA 4. *In the combinatorial game with all $d_i = d \geq 2$ and $n = (d+1)(d^k - 1)/(d-1)$, so that $e_0 = 0$ and all $e_i = 1$ for $1 \leq i \leq k$, after depth k is reached, we have a special configuration.*

Proof. After depth k is reached, there will be 1 disk at each vertex at depth k. A vertex at depth $k-1$ has at most 1 disk by a count on the total number of disks. If there is 1 disk at depth $k-1$, then we proceed inductively on k. If there are 0 disks at depth $k-1$, then the last time disks were moved from depth $k-1$ we obtained at least d disks at a vertex at depth $k-2$ and at most $d+1$ disks at such a vertex by a count on the number of disks. If there are exactly d disks, then again the last time disks were moved from depth $k-2$ we obtained at least d disks at a vertex at depth $k-3$, and so on. This accounts for the sequence ending in $((d+1)d^*01^*)1$. The number of disks accounted by such a sequence is the same as for a sequence of the same length of the form 1^*, so we may again proceed inductively to obtain again a sequence ending in $((d+1)d^*01^*)^2 1$, and so on for a sequence ending in $((d+1)d^*01^*)^* 1$. The resulting number of disks for the root at depth 0 will be either $d+1$ or 0, giving one of the two kinds of special configuration. □

LEMMA 5. *A special configuration with $k \geq 2$ takes at most $2k-3$ steps to reach the configuration 01^* with $e_0 = 0$ and $e_i = 1$ for $1 \leq i \leq k$.*

Proof. We show that each step decreases by at least 1 the number of $x_i = d$, which is at most $k-2$, in some slowed-down game. To see this, if some d is preceded by a 1, then we must in particular have a subsequence $1(d+1)(0(d+1))^r d$ for some r, which gives rise in one step to the subsequence $(d+1)(0(d+1))^{r+1}$, decreasing the number of d's by 1. If the first d is not preceded by a 1, then the initial sequence is either $(0(d+1))^r d$, giving in one step $(d+1)(0(d+1))^r$, or $(d+1)(0(d+1))^r d$, giving in one step $(0(d+1))^{r+1}$, again decreasing the number of d's by 1.

Once there remain no d's, each step increases the number of 1's at the end by 1, since the sequence must be of one of the two forms $01^*((d+1)01^*)^* 1$, or by $((d+1)01^*)^+ 1$. It thus takes at most $k-1$ steps to reach 01^* for a total of $(k-2)+(k-1) = 2k-3$ steps. □

Combining Lemmas 3, 4, and 5, we have that the combinatorial game takes $O(k)$ steps to reach depth k by Lemma 3, at which point we have a special configuration by Lemma 4, and the remaining steps that take this special configuration to a final configuration are bounded in a slowed-down game analysis of these remaining steps by $2k-3$, for a total of $O(k)$ steps. We thus obtain the following.

THEOREM 2. *In the combinatorial game with all $d_i = d \geq 2$ and $n = (d+1)(d^k - 1)/(d-1)$, so that $e_0 = 0$ and all $e_i = 1$ for $1 \leq i \leq k$, it takes $O(k)$ steps to reach the final configuration, independent of d.*

For the rest of the section, it will be convenient to change the value of d_0. This will be justified by the following.

LEMMA 6. *The combinatorial game with n disks and some value of d_0 is equivalent to the game with $\lfloor n/(d_0+1) \rfloor$ disks on a tree modified to have a degree 1 root; that is, both games take the same number of steps. Thus there is a correspondence between different possible values of d_0 via the value $d_0 = 0$.*

Proof. In both games, the first step moves $\lfloor n/(d_0+1) \rfloor$ disks from the root at depth 0 to each vertex at depth 1. In subsequent pairs of steps $2i$ and $2i+1$, if the root receives r disks from each vertex at depth 1 in step $2i$, then it sends r disks back to each vertex at depth 1 in step $2i+1$. □

Assume still that all $d_i = d \geq 2$ for $1 \leq i \leq k$ but set $d_0 = d-2$.

LEMMA 7. *A slowed-down game reaches a configuration with* $x_{ri} \leq (d-1)(k+1)$ *and* $x_{ri} \geq x_{rj}$ *for* $i \leq j$ *in* $r = O(k)$ *steps.*

Proof. Repeatedly subtract the largest $n' \leq n$ from n that can be replaced by a sequence of the form 1^l with $l \leq k$ by the result of Theorem 2, in $O(k)$ steps. Each value of l will be chosen at most $d-1$ times, since the sequence d^l would give instead the sequence 1^{l+1}. Notice that this takes a total of $O(k)$ steps, since a slowed-down game can simultaneously carry out the different steps that lead to each 1^l.

The result, after $O(k)$ steps of this slowed-down game, is thus at most $k+1$ sequences s_l^l with $0 \leq s_l \leq d-1$, for $0 \leq l \leq k$, and these sequences together prove the lemma. □

LEMMA 8. *In a slowed-down game, a configuration with* $x_{ri} \leq (d-1)(k+1)$ *and* $x_{ri} \geq x_{rj}$ *for* $i \leq j$ *leads to a configuration with* $x_{ri} = O(d(1+\log_d k))$ *in* $O(k(1+\log_d k))$ *steps.*

Proof. Subtract from each x_{ri} at most d elements so that each x_{ri} is a multiple of d. Now decompose the configuration of resulting x'_{ri} into sequences of the form d^l, and replace each such sequence by a sequence of the form 1^{l+1} in at most $2k$ steps by an application of Lemma 5. This reduces the largest x_{ri} by a factor of d.

Performing this transformation $O(1+\log_d k)$ times, we will be left just with the $O(1+\log_d k)$ remainders of at most d elements for $x_{r'i}$, so that $x_{r'i} = O(d(1+\log_d k))$ after $O(k(1+\log_d k))$ steps. □

LEMMA 9. *In a slowed-down game, a configuration with* $x_{ri} = O(d(1+\log_d k))$ *leads to the final configuration in* $O(k(1+\log_d k))$ *steps.*

Proof. There exists a special configuration $v_i \leq x_{ri}$ such that $v_i = d$ or $v_i = d+1$ whenever $x_{ri} \geq d+1$, and if $v_i = 0$, then $x_{ri} \leq d-1$. To see this, replace any sequence of entries x_{ri} that are at least $d+1$ by a sequence $(d+1)d^l$ of v_{ri}, adding some extra v_i set to d at the end for x_{ri} that are equal to d as well. Insert in between blocks of the form 01^l, noting that each v_i set to 0 will then correspond to x_{ri} that are at most $d-1$, since otherwise a d would have been used for v_i. This gives a special configuration.

Such a special configuration of v_i leads in $2k$ steps to a sequence of the form 01^l by Lemma 5, thus reducing the largest x_{ri} by at least $d-1$.

Performing this transformation $O(1+\log_d k))$ times will ensure that all resulting x_{ri} have value at most d, and we thus have a final configuration in $O(k(1+\log_d k))$ steps. □

Combining Lemmas 6, 7, 8, and 9, we obtain the following.

THEOREM 3. *In the combinatorial game with all* $d_i = d \geq 2$ *and arbitrary* n*, it takes* $O(k(1+\log_d k))$ *steps to reach the final configuration.*

4. The case of arbitrary $d_i \geq 2$. In this section, the d_i may have different values, but all $d_i \geq 2$ for $1 \leq i \leq k$. Recall that if we denote by x_{ri} the number of disks at a vertex at depth i after r steps, then we obtain the recurrence $x_{r0} = x_{(r-1)0} \bmod (d_0+1) + (d_0+1)\lfloor x_{(r-1)1}/(d_1+1)\rfloor$, and for $i \geq 1$, $x_{ri} = \lfloor x_{(r-1)(i-1)}/(d_{i-1}+1)\rfloor + x_{(r-1)i} \bmod (d_i+1) + d_i\lfloor x_{(r-1)(i+1)}/(d_{i+1}+1)\rfloor$. The base case is $x_{00} = n$ and $x_{0i} = 0$ for $i \geq 1$.

Let $d = d_{\min}$ denote the mininum d_i for $1 \leq i \leq k$. By Lemma 6, we may assume $d_0 = d_{\min}$. We again define a closely related fractional game with no remainders, with recurrence $s_{ri} = s_{(r-1)(i-1)}/(d_{i-1}+1) + d_i s_{(r-1)(i+1)}/(d_{i+1}+1)$ for $i \geq 1$, $s_{r0} = (d_0+1)s_{(r-1)1}/(d_1+1)$. The base case is $s_{00} = n$, $s_{0i} = 0$ for $i \geq 1$.

Lemma 10. *The solution of the recurrence has*

$$s_{r(2i-r)} \leq n(d_{2i-r}+1)(1/d)^i (d/(d+1))^r \binom{r}{i}$$

for $i \geq r/2$*; otherwise* $u_{ri} = 0$.

Proof. Define $u_{ri} = s_{ri}/(d_i+1)$. We obtain the recurrence $u_{ri} = u_{(r-1)(i-1)}/(d_i+1) + d_i u_{(r-1)(i+1)}/(d_i+1)$ for $i \geq 1$, $u_{r0} = u_{(r-1)1}$. Setting $d = d_{\min}$, it suffices to show $u_{r(2i-r)} \leq y_{r(2i-r)}$, with y_{ri} given as in Lemma 2.

We show this by induction on r. If $i > (r+1)/2$, then $u_{(r+1)(2i-(r+1))} = u_{r(2(i-1)-r)}/(d_{2i-(r+1)}+1)+d_{2i-(r+1)}u_{r(2i-r)}/(d_{2i-(r+1)}+1) \leq y_{r(2(i-1)-r)}/(d_{2i-(r+1)}+1) + d_{2i-(r+1)}y_{r(2i-r)}/(d_{2i-(r+1)}+1) \leq y_{r(2(i-1)-r)}/(d+1) + dy_{r(2i-r)}/(d+1) = y_{(r+1)(2i-(r+1))}$, since $y_{r(2(i-1)-r)} \geq y_{r(2i-r)}$ by Lemma 2. If $i = (r+1)/2$, then $u_{(r+1)(2i-(r+1))} = u_{r(2i-r)} \leq y_{r(2i-r)} \leq y_{(r+1)(2i-(r+1))}$. ∎

Lemma 11. *In the combinatorial game with all* $d_i \geq 2$*, and with* $e_0 = 0$ *and all* $e_i = 1$ *for* $1 \leq i \leq k$*, depth* k *is reached in* $O(\log_{d_{\min}} n)$ *steps.*

Proof. The proof is similar to that of Lemma 3. We slow down the combinatorial game by requiring that if there are at least d_i disks at a vertex at depth i after $r-1$ steps, then exactly d_i disks are left at depth i for the rth step; if there are at most d_i disks at a vertex at depth i, then none of these disks is moved. We show that this slowed-down fractional game reaches depth k within $O(k)$ steps. This implies that the original combinatorial game, which is not slowed down, will reach depth k as well.

The number of disks t_{ri} for the slowed-down game are upper bounded by $t_{ri} \leq d_i+s_{ri}$, where the s_{ri} are the quantities from the recurrence from the preceding lemma, since disks in excess of d are moved according to the fractional game defining the s_{ri}.

We bound the $s_{r(2i-r)}$ for $i \geq r/2$ by $s_{s(2i-r)} \leq n(d_{2i-r}+1)(4d/(d+1)^2)^{r/2}$ for $d = d_{\min}$. If we let $r = c \log_d n$ for a large constant c, then for $i \geq r/2$, we have $s_{r(2i-r)} \leq (1/n)^{c'}$ for another large constant c'.

If all vertices at depth $0 \leq i \leq k-1$ have d_i disks, then this accounts for exactly $n-1$ disks. The excess $s_{r(2i-r)} \leq (1/n)^{c'}$ in the bound $t_{ri} \leq d_i + s_{ri}$ for $0 \leq i \leq k-1$ accounts for strictly less than 1 disk if c' is large enough. Therefore some fraction of one disk must have reached depth k by step $r = c \log_d n$ in the slowed-down fractional game, so at least one disk will have reached depth k by step $r = c \log_d n$ in the combinatorial game. ∎

Define a *special configuration* to be a configuration where the sequence $x_{r0}x_{r1}\cdots x_{rk}$ is given by $01^*((d_i+1)d_i^*01^*)^*1$ or by $((d_i+1)d_i^*01^*)^+1$, where the corresponding d_i is chosen for position x_{ri}. The arguments of Lemmas 4 and 5 yield the following two lemmas.

Lemma 12. *In the combinatorial game with all* $d_i \geq 2$ *with* $e_0 = 0$ *and all* $e_i = 1$ *for* $1 \leq i \leq k$*, after depth* k *is reached, we have a special configuration.*

Lemma 13. *A special configuration with* $k \geq 2$ *takes at most* $2k-3$ *steps to reach the configuration* 01^* *with* $e_0 = 0$ *and* $e_i = 1$ *for* $1 \leq i \leq k$.

Combining Lemmas 11, 12, and 13 yields the following.

Theorem 4. *In the combinatorial game with all* $d_i \geq 2$*, and with* $e_0 = 0$ *and all* $e_i = 1$ *for* $1 \leq i \leq k$*, it takes* $O(\log_{d_{\min}} n) = O(\sum_{i \leq k} \log_{d_{\min}} d_i) = O(k \log_{d_{\min}} d_{\max})$ *steps to reach the final configuration, where* $d_{\min}$ *and* $d_{\max}$ *denote the smallest and the largest* d_i *for* $1 \leq i \leq k$.

We now consider cases with arbitrary n. By Lemma 6, we may set $d_0 = d_{\min} - 2$.

Lemma 14. *A slowed-down game reaches a configuration with* $x_{ri} \leq (d_{\max}-1)(k+1)$ *and* $x_{ri} \geq x_{rj}$ *for* $i \geq j$ *in* $O(\log_{d_{\min}} n)$ *steps.*

Proof. The proof is as in Lemma 7. Repeatedly subtract the largest $n' \le n$ from n that can be replaced by a sequence of the form 1^l with $r \le k$ by the result of Theorem 4 in $O(\log_{d_{\min}} n)$ steps. Each value of l will be chosen at most $d_{\max} - 1$ times, since the sequence $(d_0+2)d_1 \cdots d_l$ would give instead the sequence 1^{l+1}.

The result, after $O(\log_{d_{\min}} n)$ steps of this slowed-down game, is thus at most $k+1$ sequences s_l^l with $0 \le s_l \le d_{\max}-1$, for $0 \le l \le k$, and these sequences together prove the lemma. ∎

LEMMA 15. *Suppose $2 \le d_i \le d_j$ for $1 \le i \le j$. A configuration with $x_{ri} \le (d_{\max}-1)(k+1)$ and $x_{ri} \ge x_{rj}$ for $i \le j$ leads to a configuration with $x_{r'i} = O((d_i+1)\log_{d_{\min}}(kd_{\max}))$ in $O(k\log_{d_{\min}}(kd_{\max}))$ steps.*

Proof. The proof is as in Lemma 8. Subtract from each x_{ri} at most d_i elements so that each x_{ri} is a multiple of d_i; for x_{r0}, subtract at most d_0+1 elements so that x_{r0} is a multiple of d_0+2. Note that if $x_{ri} < d_i$, then $x_{r(i+1)} < d_{i+1}$, since $x_{r(i+1)} \le x_{ri}$ and $d_i \le d_{i+1}$. Now decompose the configuration of resulting x'_{ri} into sequences of the form $(d_0+2)d_1 \cdots d_l$ and replace each such sequence by a sequence of the form 1^{l+1} in at most $2k$ steps by an application of Lemma 13. This reduces the largest x_{ri} by a factor of $d_{\min}$.

Performing this transformation $O(\log_{d_{\min}}(kd_{\max}))$ times, we will be left just with the $O(\log_{d_{\min}}(kd_{\max}))$ remainders of at most d_i elements for $x_{r'i}$ or d_0+1 elements for $x_{r'0}$, so that $x_{r'i} = O((d_i+1)\log_{d_{\min}}(kd_{\max}))$ after $O(k\log_{d_{\min}}(kd_{\max}))$ steps. ∎

LEMMA 16. *For any value V, a configuration with $x_{r'i} = O((d_i+1)V)$ leads to the final configuration in $O(kV)$ steps.*

Proof. The proof is as in Lemma 9. There exists a special configuration $v_i \le x_{ri}$ such that $v_i = d_i$ or $v_i = d_i+1$ whenever $x_{ri} \ge d_i+1$, and if $v_i = 0$, then $x_{ri} \le d_i - 1$. To see this, note that any $x_{ri} = 0$ will be preceded by a sequence $(d_i+1)d_i^l$, and blocks of the form $(d_i+1)d_i^l$ can be separated by blocks of the form 01^l, thus giving a special configuration.

Such a special configuration of v_i leads in $2k$ steps to a sequence of the form 01^l by Lemma 13, thus reducing each $x_{ri} \ge d_i+1$ by at least $d_i - 1$.

Performing this transformation $O(V)$ times will ensure that all resulting x_{ri} have value at most d_i, and we thus have a final configuration in $O(kV)$ steps. ∎

Combining Lemmas 6, 14, 15, and 16, we obtain the following.

THEOREM 5. *If $2 \le d_i \le d_j$ for $1 \le i \le j$, then the number of steps is bounded by $O(k\log_{d_{\min}}(kd_{\max}))$.*

THEOREM 6. *If $d_i \ge d_j$ for $1 \le i \le j$, then the number of steps is at most $2k^2$.*

Proof. We may assume $d_0 = d_{\max}$ by Lemma 6. We wish to reach the configuration $x_i = e_i$. Suppose more generally we wish to reach a configuration $x_0 = e_0+(d_0+1)u$, $x_i = e_i+(d_i-1)u$ for $1 \le i \le k$. This configuration can be reached from $x'_0 = e_0+(d_0+1)(u+1+(d_k-1)u)$, $x'_i = e_i+(d_i-1)(u+1+(d_k-1)u)$ for $1 \le i \le k-1$, $x'_k = e_k - 1$ in $2k-1$ steps by Lemma 13 that transforms $(d_0+1)d_1 \cdots d_{k-1}$ into 01^k. The configuration x'_i can in turn be reached from $x'_0 = e_0+(d_0+1)(u+e_k+(d_k-1)u)$, $x'_i = e_i+(d_i-1)(u+e_k+(d_k-1)u)$ for $1 \le i \le k-1$, $x'_k = 0$ in $2k-1$ steps by Lemma 13 again.

We have thus obtained a configuration $x'_0 = e_0+(d_0+1)v$, $x_i = e_i+(d_i-1)v$ for $1 \le i \le k-1$ in $2(2k-1)$ steps, and this configuration has $x'_k = 0$. Repeatedly applying the same argument, we may set $x''_{k-1} = 0$, $x'''_{k-2} = 0, \ldots$ in turn, until the initial configuration is reached. The number of steps is $2(2k-1) + 2(2(k-1)-1) + 2(2(k-3)-1) + \cdots = 2k^2$. ∎

THEOREM 7. *In the general case of* n *arbitrary and all* $d_i \geq 2$ *for* $i \geq 1$, *the number of steps is bounded by* $O(k^{1.5} + \log_{d_{\min}} n) = O(k^{1.5} + k \log_{d_{\min}} d_{\max})$.

Proof. Consider the slowed-down fractional game of Lemma 11 that reaches depth k within $O(\log_{d_{\min}} n) = O(k \log_{d_{\min}} d_{\max})$ steps. At the end of this fractional game, we have $t_{ri} \leq d_i + s_{ri}$ as before, and the excesses s_{ri} account for strictly less than one disk as before. The d_i for $i < j$ together with the extra disk coming from the s_{ri} account for less than one disk that is in the subtree rooted at a vertex at depth j in the final solution. The reason is that a sequence $(d_0 + 1)d_1 d_2 \cdots d_l$ can at most transform to 01^{l+1}, with one disk moving to a subtree rooted at depth $l + 1$.

In the extreme case, suppose each subtree is missing one disk from its parent. If a leaf at depth k is going to receive 1 disk from its parent, then the parent at depth $k - 1$ must also send 1 disk to its parent and thus receive a total of 2 disks from its parent; then the parent at depth $k - 2$ must also send 2 disks to its parent and thus receive a total of 3 disks from its parent. In general, a vertex at depth $k - i$ will send at most i disks to its parent and receive at most $i + 1$ disks from its parent.

Consequently, in the combinatorial game, after the initial

$$O(\log_{d_{\min}} n) = O(k \log_{d_{\min}} d_{\max})$$

steps follows a second phase during which each vertex sends at most k disks to each of its neighbors, for a total of k^2 disks when adding over all depths i. As long as a vertex at some depth i has at least $\sqrt{k}(d_i + 1)$ disks, such a vertex will send $\sqrt{k}$ disks simultaneously to each of its neighbors; this can happen for at most $k^2/\sqrt{k} = k^{1.5}$ steps.

Once the vertices at each depth i have at most $\sqrt{k}(d_i + 1)$ disks, a final configuration can be reached in $O(k^{1.5})$ steps by Lemma 16, completing the proof. ◻

THEOREM 8. *Assume that* $d_k = e_k$. *If* $d_i \geq 2$ *for* $i \geq 1$, *then there is a lower bound of* $\Omega(k + \max_{1 \leq i \leq k} \sum_{i \leq j \leq k} \log_{d_i} d_j)$ *on the number of steps. Thus the upper bound of Theorem* 4 *is tight up to constant factors, provided* $\log_{d_{\min}} d_1 = O(1)$, *that is,* $d_1 \leq d_{\min}^{O(1)}$ *(since in that case* $O(\sum_{i \leq k} \log_{d_{\min}} d_i) = O(\sum_{i \leq k} \log_{d_1} d_i)$*).*

Proof. There is an immediate lower bound of $\Omega(k)$ to reach depth k. Consider the n_i disks that may reach depth i. Only a fraction $d_i/(d_i + 1)$ of these disks will be moved from a vertex at depth i to depth $i + 1$ at one time, since a fraction $1/(d_i + 1)$ must be moved to depth $i - 1$. Thus after one step, we are still left with at least $(n_i - e_i)/(d_i + 1)$ disks that have not yet reached depth $i + 1$; after two steps we are still left with at least $(n_i - e_i)/(d_i + 1)^2$ disks that have not yet reached depth $i + 1$; and after r steps we are still left with at least $(n_i - e_i)/(d_i + 1)^r$ disks that have not yet reached depth $i + 1$. It will thus take at least $r = \log_{d_i+1}(n_i - e_i)$ steps to move the $n_i - e_i$ disks to depth $i + 1$. The result follows from the bound $n_i \geq d_i d_{i+1} \cdots d_k$. ◻

5. Conclusion. We have analyzed a combinatorial game played on an infinite rooted tree where all the vertices at depth i have the same number of children d_i. The analysis determines the final configuration in the general case and bounds the number of steps needed to reach this final configuration when $d_i \geq 2$ for $i \geq 1$. The case where all $d_i = 1$ for $i \geq 1$ was previously studied by Anderson et al. [1]. It remains open to analyze the combinatorial game when some of the d_i for $i \geq 1$ satisfy $d_i = 1$ and some satisfy $d_i \geq 2$.

The fact that the dependence of the number of steps depends essentially linearly on the depth of the tree and logarithmically in the number of disks, instead of being quadratic as in the case of a path, indicates that the particular structure of each graph under consideration greatly affects the number of steps that the game takes. It thus seems that trees are the graphs for which the convergence to the final configuration is fastest, as it is in many cases only linear in the diameter reached by the game.

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