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Graphs that triangulate a given surface and quadrangulate another surface

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Abstract

We show that for any closed surface F with $\chi(F) \leqslant -4$ (or $\chi(F) \leqslant -2$), there exist graphs that triangulate the torus or the Klein bottle (or the projective plane) and that quadrangulate F. We also give a sufficient condition for a graph triangulating a closed surface to quadrangulate some other surface.

Keywords: Triangulation; Quadrangulation; Closed surface

1. Introduction

A graph is said to *triangulate* a surface if it has a triangular embedding on the surface (that is, each face of the embedding is bounded by a triangle). Similarly, we say that a graph *quadrangulates* a surface if it embeds on the surface in such a way that each face is bounded by a cycle of length 4. For example, the graph of the octahedron triangulates the sphere but also admits a quadrangular embedding on the torus as shown in Fig. 1.

Recently, Negami, Nakamoto, Ota and Širáň [1] discussed graphs that triangulate the sphere and quadrangulate other closed surfaces, and established the following theorem.

Theorem 1.1. (Nakamoto, Negami, Ota and Širáň [1]) *Given a closed surface F except the sphere, there exists a graph that triangulates the sphere and quadrangulates F.*

This theorem motivates us to propose the following question. We denote the Euler characteristic of a closed surface F by $\chi(F)$.

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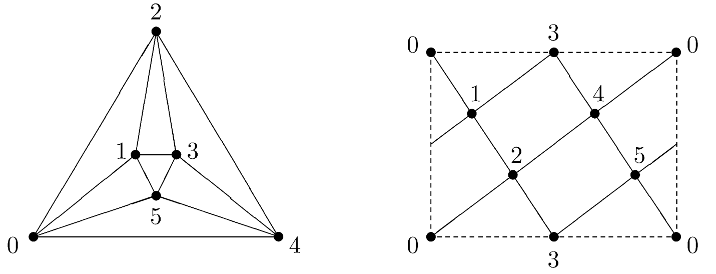

Fig. 1. The octahedron on the torus.

Question. Given two closed surfaces F_1 and F_2 with $\chi(F_1) > \chi(F_2)$, is there a graph that triangulates F_1 and quadrangulates F_2?

We need some additional conditions to give an affirmative answer to this question. If there exists such a graph G that triangulates F_1, the number of its vertices and edges is restricted by Euler's formula as follows:

$$\left|V(G)\right| = 3\chi(F_1) - 2\chi(F_2), \qquad \left|E(G)\right| = 6\chi(F_1) - 6\chi(F_2).$$

For example, if F_1 is the projective plane, then necessarily $\chi(F_2) \leqslant -2$; otherwise, G would have at most 5 vertices, but there is no such triangulation on the projective plane. Similarly, if F_1 is the torus or the Klein bottle, then we have $\chi(F_2) \leqslant -4$. In fact, in Section 4 we prove that these necessary conditions are also sufficient as follows:

Theorem 1.2. *There exists a graph that triangulates the projective plane and quadrangulates a closed surface F if and only if $\chi(F) \leqslant -2$.*

Theorem 1.3. *There exists a graph that triangulates the torus and quadrangulates a closed surface F if and only if $\chi(F) \leqslant -4$.*

Theorem 1.4. *There exists a graph that triangulates the Klein bottle and quadrangulates a closed surface F if and only if $\chi(F) \leqslant -4$.*

To construct concrete graphs that triangulate closed surfaces and quadrangulate other surfaces, we shall introduce the notions of "4-cycle double covers" and "slit-flip sum" in Sections 2 and 3, respectively. The first one corresponds to the set of cycles of length 4 bounding faces of a quadrangular embedding of a graph G, while slit-flip sum is a method of pasting two graphs to construct a new graph.

These methods have been already introduced in [1], but we shall refine them for our purpose. With their help and with the help of Theorems 1.2–1.4, we will prove in Section 5 the main result of this paper:

Theorem 1.5. *Given two closed surfaces F_1 and F_2 with $2\chi(F_1) - \chi(F_2) \geqslant 4$, then there exists a graph that triangulates F_1 and quadrangulates F_2.*

2. 4-Cycle double cover

We begin with introducing a notion from [1] to present examples of graphs that triangulate a surface and quadrangulate another surface. Let G be a graph that triangulates a closed surface and quadrangulates another closed surface. Let $\mathcal{C} = \{C_1, C_2, \ldots\}$ be the family of cycles of length 4 in G each of which corresponds to the boundary cycle of a face in a fixed quadrangular embedding of G. Then $\mathcal{C}$ is what is called a *cycle double cover* of G. That is, each edge of G is contained in exactly two distinct cycles of $\mathcal{C}$. Since each cycle in $\mathcal{C}$ has length 4, we call it a 4-*cycle double cover*.

For example, let G be the graph that triangulates the projective plane shown in Fig. 2 (left) where each pair of vertices with the same labels should be identified. Consider the family $\mathcal{C}$ of directed cycles

1243, 1374, 1425, 1572, 2345, 2653, 2736, 3546, 4756.

Then $\mathcal{C}$ can be regarded as a 4-cycle double cover of G, since each edge is covered by exactly two cycles in $\mathcal{C}$. Further, one can check that the following conditions are satisfied:

(i) Two cycles containing an edge induce different directions on the edge.
(ii) Cycles passing through each vertex induce a rotation on edges incident with the vertex.

Clearly, these conditions imply that $\mathcal{C}$ corresponds to a quadrangular embedding of G on the orientable closed surface of genus 2.

We call a cycle $C_i \in \mathcal{C}$ a *rhombus* if it bounds a quadrilateral region obtained as a union of two triangular faces of the triangular embedding of G, and a *rhombus cover* of G if all cycles in $\mathcal{C}$ are rhombi. It is clear that a rhombus cover of the triangular embedding of G corresponds to a quadrangular embedding of G on a closed surface if and only if the rhombi incident to each vertex $v \in V(G)$ induce a cyclic order over the neighbors of v, which corresponds to the *rotation* around v in the quadrangulation on the surface.

Let G be a graph that triangulates a closed surface, except K_4 on the sphere. For a 4-cycle double cover $\mathcal{C} = \{C_1, C_2, \ldots\}$, we define a graph $R_{\mathcal{C}}$ as follows and call it the *rhombus cover graph* of $\mathcal{C}$. Let G^* denote the dual of the triangular embedding of G on the surface and set $V(R_{\mathcal{C}}) = V(G^*)$. Join a pair of vertices f_1 and $f_2 \in V(R_{\mathcal{C}})$ with an edge whenever two faces corresponding to f_1 and f_2 form a quadrilateral region bounded by a rhombus $C_i \in \mathcal{C}$. Since G^* is a 3-regular graph, each vertex of $R_{\mathcal{C}}$ has degree 0, 1, 2 or 3.

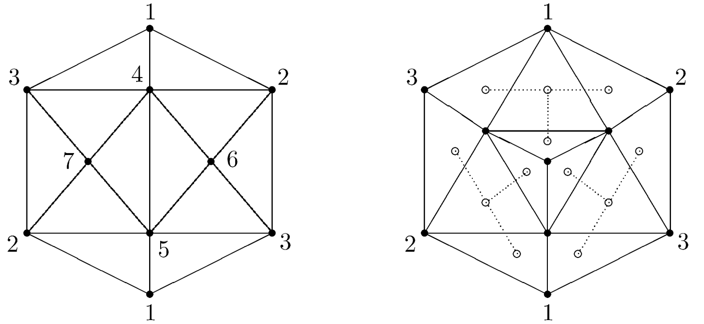

Fig. 2. Triangulations with seven vertices on the projective plane.

It is easy to see that there are two types of rhombus covers of the triangular embedding of G; either $R_{\mathcal{C}}$ consists of only vertices of degree 0 and 2, or of degree 1 and 3. We will say that the first cover is of *even type* and the second one of *odd type*. Similarly, a rhombus cover $\mathcal{C}$ is said to be of *even type* (or *odd type*) if $R_{\mathcal{C}}$ is even (or odd). Furthermore, it has been shown in [1] that:

Lemma 2.1. *Any rhombus cover of the triangular embedding of G of odd type corresponds to a quadrangular embedding of G on a nonorientable closed surface.*

Let K be a connected graph. A spanning subgraph F of G is called a *K-factor* if each component of F is isomorphic to K. Clearly, the rhombus cover graph $R_{\mathcal{C}}$ of odd type is a $K_{1,3}$-factor of the dual G^* of a triangular embedding of G on a closed surface. The following corollary is an immediate consequence of the above lemma.

Corollary 2.1. *If the dual of a triangular embedding of G on a closed surface has a $K_{1,3}$-factor, then G admits a quadrangular embedding on a nonorientable closed surface.*

For example, the right-hand side of Fig. 2 presents a graph that triangulates the projective plane with 7 vertices. Since the dual of the triangular embedding of it has a $K_{1,3}$-factor, the graph quadrangulates the nonorientable closed surface of genus 4.

A graph G embedded on a closed surface F is said to be *r-representative* if any non-contractible simple closed curve on F meets G in at least r points. (See [3] for the details on "representativity.")

Since any cycle of length 4 in a 5-connected and 5-representative triangulation must induce a rhombus, we have the following theorem. This theorem presents basically the same fact as Theorem 18 in [1]. Negami and Suzuki [2] have already given more detailed information on such triangulations on the sphere.

Theorem 2.1. *Let G be a 5-connected graph that admits a 5-representative triangulation on a closed surface. Then, G can quadrangulate another closed surface if and only if the dual of its triangular embedding has a $K_{1,3}$-factor.*

3. Slit-flip sums

Let G_i be a graph that can be embedded on closed surfaces Σ_i as a triangulation T_i and on F_i as a quadrangulation Q_i, and suppose that a path $u_i v_i w_i$ in G_i forms a corner of a face in T_i and also does in Q_i for $i = 1, 2$.

Cut open the two closed surfaces Σ_1 and Σ_2, each including T_1 and T_2, along the edges u_1v_1 and u_2v_2, respectively, and paste them along the resulting boundaries so that u_1 is identified with v_2 and v_1 with u_2. Then the union of two faces $u_1v_1w_1$ and $u_2v_2w_2$ forms a rectangular region with a diagonal $u_1v_1 = v_2u_2$. Replace this diagonal with w_1w_2 to eliminate the multiple edges between u_1 and v_1. Let $T_1 \ddagger T_2$ denote the resulting triangulation on the closed surface Σ obtained from the two slitted surfaces. In a similar way we can construct a new quadrangulation $Q_1 \ddagger Q_2$ on a closed surface F obtained from F_1 and F_2 slitted along u_1v_1 and u_2v_2.

It is clear that underlying graphs of $T_1 \ddagger T_2$ and $Q_1 \ddagger Q_2$ are isomorphic with the same graph, denoted by $G_1 \ddagger G_2$. We say that $G_1 \ddagger G_2$ is obtained from G_1 and G_2 by *slit-flip sum* and call the path $u_1v_1w_1$ (or $u_2v_2w_2$) in the above assumption for G_1 (or G_2) a *useful corner* at v_1 (or v_2).

Lemma 3.1. *Let G be a graph that triangulate a closed surface and quadrangulates another closed surface. If G has a vertex v of degree 4 or 3, then it has a useful corner at v.*

Proof. Let v be a vertex of degree 4, and let v_1, v_2, v_3 and v_4 be its neighbors lying around v in this cyclic order in the triangular embedding of G. Then, there are only three types of 4-cycle double covers around v corresponding to a quadrangular embedding of G. They contain:

(i) $v_1vv_2u_0$, $v_2vv_3u_1$, $v_3vv_4u_2$ and $v_4vv_1u_3$;
(ii) $v_1vv_3u_0$, $v_2vv_4u_1$, $v_1vv_2u_2$ and $v_3vv_4u_3$;
(iii) $v_1vv_3u_0$, $v_2vv_4u_1$, $v_2vv_3u_2$ and $v_4vv_1u_3$.

In each type, G has at least two useful corners at v. If v is a vertex of degree 3, then all of three corners at v form useful corners. □

Lemma 3.2. *Any graph $G_1 \ddagger G_2$ obtained by a slit-flip sum has two useful corners.*

Proof. Let $u_iv_iw_i$ be a useful corner in a graph G_i and suppose that a slit-flip sum flips $u_1v_1 = v_2u_2$ to w_1w_2 to obtain $G_1 \ddagger G_2$. Then both $u_1w_2w_1$ and $v_1w_1w_2$ form two corners in $T_1 \ddagger T_2$ and also in $Q_1 \ddagger Q_2$. Thus, they are useful corners in $G_1 \ddagger G_2$. □

Now we shall consider performing slit-flip sums in "two places." If both G_1 and G_2 has two disjoint useful corners, say $u_iv_iw_i$ and $u_i'v_i'w_i'$, then we can perform the slit-flip sums at these corners at the same time. Denote the resulting graph, triangulation and quadrangulation by $G_1 \ddagger\ddagger G_2$, $T_1 \ddagger\ddagger T_2$ and $Q_1 \ddagger\ddagger Q_2$, respectively. Then $T_1 \ddagger\ddagger T_2$ triangulates a closed surface homeomorphic to $\Sigma_1 \# \Sigma_2$ with one handle attached while $Q_1 \ddagger\ddagger Q_2$ quadrangulates a closed surface homeomorphic to $F_1 \# F_2$ with one handle attached. It should be noticed that such surfaces may not be orientable even if all Σ_i and F_i are orientable. Let $\Sigma_1 \ddagger\ddagger \Sigma_2$ and $F_1 \ddagger\ddagger F_2$ denote these surfaces. To control their orientability, we define the "type" of a pair of useful corners as follows.

Let G be a graph that triangulates Σ and quadrangulates F and suppose that both surfaces Σ and F are orientable. Two disjoint useful corners uvw and $u'v'w'$ are said to be *coherent* on Σ (or F) if they induce the same orientation over Σ_1 (or F), and *incoherent* otherwise. Such a pair of useful corners is said to be of *type* $(\pm, \pm)$, where "$+$" (or "$-$") mean that they are coherent (or incoherent) on Σ and F in that order.

Lemma 3.3. *The graph of the octahedron has four pairs of disjoint useful corners of all types.*

Proof. Let T and Q denote the triangulation and the quadrangulation in Fig. 1, each of which is an embedding of the octahedron. Then the pairs $\{051, 342\}$, $\{051, 423\}$, $\{051, 324\}$ and $\{051, 243\}$ are of type $(+,+)$, $(+,-)$, $(-,+)$ and $(-,-)$ in order. □

Let G_i, T_i, Q_i, Σ_i and F_i be as above. It is clear that $\Sigma_1 \ddagger\ddagger \Sigma_2$ is orientable if and only if both Σ_1 and Σ_2 are orientable and the pair of useful corners for G_1 has the same coherency as that for G_2. In particular, when the triangular embedding of G_2 is the octahedron, we say that $G_1 \ddagger\ddagger G_2$ is obtained from G_1 by adding a *slit-flip handle*. By Lemma 3.3, a slit-flip handle decreases $\chi(\Sigma_1)$ by 2 and $\chi(F_1)$ by 4. If both Σ_1 and F_1 are orientable, then we can control the orientability of $\Sigma_1 \ddagger\ddagger \Sigma_2$ and of $F_1 \ddagger\ddagger F_2$ by attaching the slit-flip handle at a suitable pair of useful corners listed in the proof of Lemma 3.3.

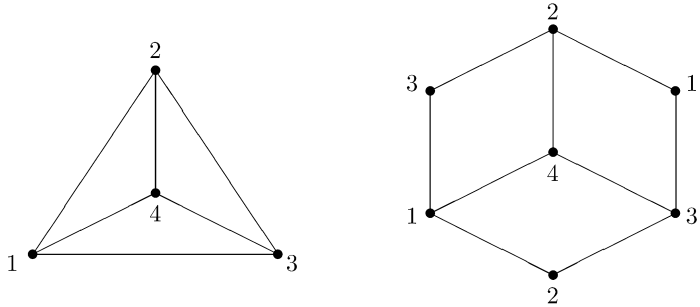

Fig. 3. The tetrahedron on the projective plane.

Figure 3 presents two embeddings of K_4. The left-hand side exhibits a tetrahedron, which is the minimal triangulation on the sphere, and identifying antipodal points on the hexagon in the right-hand side yields a quadrangulation on the projective plane. It is easy to check that all corners in the tetrahedron are useful corners. A slit-flip sum with this K_4 adds one crosscap to F_1, not changing the homeomorphism type of Σ_1. On the other hand, a slit-flip sum with the graph of the octahedron adds one orientable handle to F_1. We call these slit-flip sums simply *adding a crosscap* and *a handle*.

4. Individual cases

The necessity parts of Theorems 1.2–1.4 have been shown in the introduction. In this section we present constructions of graphs that triangulate the projective plane, the torus and the Klein bottle and quadrangulate other closed surfaces.

Proof of Theorem 1.2. First, we shall construct a graph that triangulates the projective plane and quadrangulates the orientable closed surface of genus $g \geqslant 2$. We have already obtained such a graph for $g = 2$, say P_0, the triangular embedding of which is in the left part of Fig. 2. There are many useful corners in P_0 and hence we can carry out a slit-flip sum with the graph that triangulates the sphere as the octahedron and quadrangulates the torus. The resulting graph that triangulates the projective plane quadrangulates the orientable closed surface of genus 3. By Lemma 3.2, we can repeat slit-flip sums to obtain the desired genus of the quadrangulated surface as we want.

To construct a graph that triangulates the projective plane and quadrangulates the nonorientable closed surface of genus $k \geqslant 4$, it suffices to use the right one of Fig. 2 as P_0 and the tetrahedron, instead of the octahedron. Slit-flip sums with the graph of the tetrahedron increase the number of crosscaps on the quadrangulated surface by any amount we want. □

Since the same argument works for the torus and the Klein bottle with little change, we shall prove Theorems 1.3 and 1.4 simultaneously.

Proof of Theorems 1.3 and 1.4. First take n copies of the graph of the octahedron, say $G_1, \ldots, G_n$ with $n \geqslant 2$ and carry out a slit-flip sum using the useful corner 342 in G_i and 015 in G_{i+1} for $i = 1, \ldots, n-1$. This results in a graph $G_1 \ddagger \cdots \ddagger G_n$ that triangulates the sphere and quadrangulates the orientable closed surface of genus n. The pair of 015 in G_1 and 342 (or 324) in G_n is of type $(+,+)$ while the pair of 015 in G_1 and 423 (or 243) in G_n is of type $(+,-)$.

Performing the same deformation at these corners as in a slit-flip sum, we obtain two triangulations T_n and $\tilde{T}_n$ on the torus (or the Klein bottle), the graphs of them quadrangulate the orientable closed surface of genus $n+1$ and the nonorientable closed surface of genus $2(n+1)$. Both these surfaces have Euler characteristic $2-2(n+1) \leqslant 4$. We can add a crosscap to decrease the Euler characteristic by 1 since $G_1 \ddagger \cdots \ddagger G_n$ has useful corners by Lemma 3.2. $\square$

5. General cases

We divide Theorem 1.5 into four statements depending on the orientability of two closed surfaces that a graph triangulates and quadrangulates.

Theorem 5.1. *Given two orientable closed surfaces F_1 and F_2 with $2\chi(F_1) - \chi(F_2) \geqslant 4$, there exists a graph that triangulates F_1 and quadrangulates F_2.*

Proof. We use the triangulation T_2 on the torus constructed in the proof of Theorem 1.3. This graph quadrangulates the orientable closed surface of genus 3 and has two useful corners. By Lemma 3.2, we can repeat adding slit-flip handles as we like. Repeating this $g_1 - 1$ times, we obtain the triangulated closed surface F_1 of genus g_1 while the corresponding quadrangulated surface has genus $2(g_1-1)+3 = 2g_1+1$. Since the genus of the latter can be increased by adding handles, we can construct a quadrangulation on any closed surface F_2 of genus $g_2 \geqslant 2g_1 + 1$. This inequality follows from $2\chi(F_1) - \chi(F_2) \geqslant 4$. $\square$

Theorem 5.2. *Given an orientable closed surface F_1 and a nonorientable closed surface F_2 with $2\chi(F_1) - \chi(F_2) \geqslant 4$, then there exists a graph that triangulates F_1 and quadrangulates F_2.*

Proof. This time we use the triangulation $\tilde{T}_2$ of the torus constructed in the proof of Theorem 1.3. This graph quadrangulates the nonorientable closed surface of genus 6. Applying the slit-flip handle addition to this embedding $g_1 - 1$ times yields a triangulated orientable closed surface F_1 of genus g_1 and a quadrangulated nonorientable closed surface F_2 of genus $4(g_1-1)+6 = 4g_1+2$. By further addition of crosscaps we may increase the genus of F_1 to any value $k_2 > 4g_1+2$. This inequality is equivalent to $2\chi(F_1) - \chi(F_2) \geqslant 4$. $\square$

Theorem 5.3. *Given two nonorientable closed surfaces $\dot{F}_1$ and $\dot{F}_2$ with $2\chi(\dot{F}_1) - \chi(\dot{F}_2) \geqslant 4$, then there exists a graph that triangulates $\dot{F}_1$ and quadrangulates $\dot{F}_2$.*

Proof. Let k_1 and k_2 be the genus of $\dot{F}_1$ and $\dot{F}_2$, respectively. First suppose that $k_1 = 2g_1 + 1$ with $g_1 \geqslant 0$. Adding a slit-flip handle to the right-hand side in Fig. 2 g_1 times, we obtain a triangulated nonorientable closed surface of genus $2g_1 + 1$ and a quadrangulated one of genus $4g_1 + 4 = 2k_1 + 2$. Adding crosscaps, we can increase the genus of the latter to be any value $k_2 \geqslant 2k_1 + 2$. This inequality is equivalent to

$$2\chi(\dot{F}_1) - \chi(\dot{F}_2) \geqslant 4.$$

Now suppose that $k_1 = 2g_1 > 0$. By Theorem 1.4, we have a triangulation $\tilde{T}_2$ on the Klein bottle the graph of which quadrangulates the nonorientable closed surface of genus 6. Adding slit-flip handles $g_1 - 1$ times, we obtain a triangulation on the nonorientable closed surface $\dot{F}_1$

of genus k_1 and a quadrangulation on $\dot{F}_2$ of genus $2(g_1-1)+6=k_1+4$. Adding crosscaps, we can increase the genus of $\dot{F}_2$ to be any value $k_2 \geqslant k_1+4$. This inequality is equivalent to

$$\chi(\dot{F}_1)-\chi(\dot{F}_2)\geqslant 4.$$

Since this inequality is implied by the one given in the previous case, the theorem follows. □

Theorem 5.4. *Given a nonorientable closed surface $\dot{F}_1$ and an orientable closed surface $\dot{F}_2$ with $2\chi(\dot{F}_1)-\chi(\dot{F}_2)\geqslant 4$, then there exists a graph that triangulates $\dot{F}_1$ and quadrangulates $\dot{F}_2$.*

Proof. Let k_1 and g_2 be the genus of $\dot{F}_1$ and $\dot{F}_2$, respectively. If $k_1=2g_1+1\geqslant 1$, then we apply adding slit-flip handle g_1 times to the triangulation on the projective plane given in Fig. 2 and obtain a triangulation on $\dot{F}_1$ of genus k_1 and a quadrangulation on $\dot{F}_2$ of genus $2g_1+2=k_1+1$. Adding handles, we can increase the genus of $\dot{F}_2$ to be any value $g_2\geqslant k_1+1$. Thus, we have $2\chi(\dot{F}_1)-\chi(\dot{F}_2)\geqslant 4$.

If $k_1=2g_1\geqslant 2$, then we use the triangulation T_2 on the Klein bottle, constructed in the proof of Theorem 1.4 so that its graph quadrangulates the orientable closed surface of genus 3. Adding slit-flip handles g_1-1 times and then adding handles, we obtain a triangulation on $\dot{F}_1$ of genus k_1 and a quadrangulation on $\dot{F}_2$ of genus $g_2\geqslant 2(g_1-1)+3=k_1+1$. Thus, we have the same inequality as in the previous case. □

References

[1] A. Nakamoto, S. Negami, K. Ota, Širáň, Planar triangulations which quadrangulate other surfaces, European J. Combin. 25 (2004) 817–833.

[2] S. Negami, Y. Suzuki, 5-Connected planar triangulations quadrangulating other surfaces, Yokohama Math. J. 47 (2000) 187–193 (special issue).

[3] N. Robertson, R. Vitray, Representativity of surface embeddings, in: Paths, Flows, and VLSI-Layout, Bonn, 1988, in: Algorithms Combin., vol. 9, Springer, Berlin, 1990, pp. 293–328.