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Exponentially many 5-list-colorings of planar graphs

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Abstract

We prove that every planar graph with n vertices has at least $2^{n/9}$ distinct list-colorings provided every vertex has at least five available colors.

Keywords: List colorings

1. Introduction

If a graph is 4-list-colorable, then it is easy to see that it has exponentially many 5-list-colorings. Voigt [7] showed that a planar graph need not be 4-list-colorable. In [4] I proved that every planar graph is 5-list-colorable, and in the present paper I prove that it has exponentially many 5-list-colorings. Clearly, there are no more than 5^n distinct list-colorings if every vertex has precisely 5 available colors, so an exponential function is the best we can hope for. However, our exponential function is probably not the best possible. The following questions arise naturally from the result of the present paper.

Problem 1. *Does there exist a positive constant c_0 such that every planar graph with n vertices has at least $c_0 \cdot 2^n$ distinct 5-list-colorings?*

In case of an affirmative answer we may go even further in two different directions.

Problem 2. *Does every planar graph with n vertices have at least $60 \cdot 2^{n-3}$ distinct 5-list-colorings?*

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Birkhoff and Lewis [1] answered this in the affirmative for ordinary 5-colorings. It is best possible because of the planar triangulations obtained from a triangle by successively adding vertices of degree 3.

Problem 3. *Let S be a surface of Euler genus g. Does there exist a positive constant c_g such that every graph on S which is* 5*-list-colorable and which has n vertices has at least $c_g \cdot 2^n$ distinct L-colorings for each list assignment L with five colors in each list*?

For ordinary 5-colorings, this was proved in [5]. We also repeat a problem raised in [6].

Problem 4. *Does every planar triangle-free graph with n vertices have exponentially many distinct* 3*-colorings*?

In [6] this was verified for planar graphs of girth 5, even for the list-color version.

The notation and terminology are the same as in [2–4]. For the reader's convenience we repeat the most important definitions.

Let G be a graph. For every vertex v of G, let $L(v)$ be a list of colors which we call *available colors*. An *L-coloring* of G is a coloring of the vertex set such that every vertex v receives a color from $L(v)$, and neighbors always have distinct colors. If it is clear what L is, we just call it a *list-coloring* of G. If every list $L(v)$ has at least k colors, we call it a *k-list-coloring* of G. We say that G is *k-list-colorable* if it has a k-list-coloring for every possible choice of the list function L (with at least k colors in every list). We shall consider planar graphs only. If C is a cycle in a plane graph G, then *int*(C) denotes the set of vertices and edges in the interior of C. *Int*(C) denotes the graph $C \cup$ *int*(C). *ext*(C) and *Ext*(C) are defined analogously. If *ext*(C) is empty, then C is the *outer cycle* of G. If, in addition, every face (region) inside C is bounded by a triangle, then G is a *near-triangulation*. The proof of the 5-list-color theorem in [4] is about near-triangulations. For technical reasons two vertices on the outer cycle are precolored, and all other vertices on the outer cycle have only three available colors. Then a short argument shows that a list-coloring exists. The argument is tight in the sense that there is no choice for the color of the vertex which is deleted in the inductive argument. And simple examples of outerplanar near-triangulations show that indeed there need not be more than one list-coloring. To overcome that obstacle, we shall extend the result in [4] to a result where three vertices on the outer cycle are precolored. In that case a list-coloring need not exist. But, we characterize the exceptions, and we use that to provide a new proof of the 5-list color theorem which allows enough flexibility to imply exponentially many list-colorings.

2. 5-List-colorings with precolored vertices

Let G be a plane near-triangulation with outer cycle C: $v_1 v_2 \cdots v_k v_1$. We say that G is 3-*extendable with respect to the path $v_k v_1 v_2$* if the following statement holds:

For each vertex v in G, let $L(v)$ be a list of colors. Assume that the vertices v_k, v_1, v_2 are precolored, that is, if v is one of v_k, v_1, v_2, then $L(v)$ consists of one color only. If v is one of $v_3, v_4, \dots, v_{k-1}$, then $L(v)$ consists of at least three colors. Otherwise, $L(v)$ has at least five colors. Then G has an L-coloring.

2-*Extendability* is defined analogously. The result in [4] can be phrased as follows.

Theorem 1. *Any near-triangulation is* 2*-extendable with respect to any path* (*on the outer cycle*) *with two vertices.*

This implies the following.

Theorem 2. *Let H be a near-triangulation with prescribed and precolored outer cycle C of length at most* 5*. For each vertex v in $int(C)$, let $L(v)$ be a list of at least five colors. Then H has an L-coloring unless C has length* 5*, and $int(C)$ has a vertex joined to all vertices of C, and $L(v)$ consists of the colors of C.*

Proof. The proof is by induction on the number of vertices of H. If no vertex of $int(C)$ is joined to more than two vertices of C, then we consider the subgraph G induced by the vertices in $int(C)$. We delete from each list the colors of the neighbors in C. By Theorem 1, G is list-colorable with these reduced lists. (If G is not 2-connected, then we color the blocks of G successively.) So we may assume that some vertex v in $int(C)$ has at least three neighbors in C. If it is not possible to color v, then H satisfies the conclusion of Theorem 2. On the other hand, if it is possible to color v, then we color it and complete the proof by induction. If the exceptional case in Theorem 2 occurs for the reduced graph (that is, there is a colored cycle whose coloring cannot be extended to its interior), then v has precisely three consecutive neighbors in C, and we therefore have two possibilities for coloring v. So, the exceptional case in Theorem 2 can be avoided. □

3. Generalized wheels

In the next section we extend Theorem 1 to 3-extendability.

First we describe some near-triangulations which are not 3-extendable. If the interior of the above near-triangulation G consists of the edges $v_1v_3, v_1v_4, \ldots, v_1v_{k-1}$, then we call G a *broken wheel*. We also call it a *generalized wheel*. We call v_1 its *major vertex* and $v_kv_1v_2$ its *principal path*. We also say that v_kv_1, v_1v_2 are the *principal edges* and that v_k, v_2 are the *principal neighbors* of v_1. If $k \geqslant 4$, then this generalized wheel is clearly not 3-extendable with respect to its principal path. If the interior of the above near-triangulation G consists of a vertex u and all edges from u to the outer cycle, then G is a *wheel*. We also call that a *generalized wheel*, and again, we call v_1 its *major vertex* and $v_kv_1v_2$ its *principal path*. It is easy to see that this generalized wheel is not 3-extendable with respect to its principal path when k is odd, $k \geqslant 5$. Finally, if G_1, G_2 are generalized wheels and we identify a principal edge in one of them with a principal edge in the other (in such a way that their major vertices are identified), then the resulting graph is also called a generalized wheel. Its two principal edges are those which are principal edges in one of the graphs but not part of the identification above. Again, it is easy to see that this generalized wheel is not 3-extendable with respect to its principal path unless it contains a vertex of even degree or degree 3 inside the outer cycle. As we shall not use the fact that a generalized wheel needs not be 3-extendable with respect to its principal path, we leave the proof for the reader. Instead, we shall now prove the converse, namely that a near-triangulation is 3-extendable provided it does not contain a generalized wheel as a spanning subgraph. For that we need some technical lemmas.

In Lemmas 1–3 below we refer to the near-triangulation G whose outer cycle C is described above.

Lemma 1. *Assume that G is a generalized wheel but not a broken wheel. Assume further that each vertex in $int(C)$ has at least five available colors and that each of the vertices $v_3, v_4, \dots, v_{k-1}$ has at least three available colors. Then there is at most one coloring of v_k, v_1, v_2 which cannot be extended to a list-coloring of G.*

Proof. We prove the lemma by induction on k. Consider first the case where G is a wheel. Let v be the vertex not in C. Suppose v_k, v_1, v_2 are colored $c(v_k), c(v_1), c(v_2)$, respectively, and that this coloring cannot be extended to G. Then $L(v_3) \setminus \{c(v_2)\}$ consists of precisely two colors, say α, β, since otherwise we can color v and extend that coloring to G by applying Theorem 1 to $G - v_1 - v_2$. Similarly, $L(v_{k-1}) \setminus \{c(v_k)\}$ consists of precisely two colors, say γ, δ. If $L(v) \setminus \{c(v_k), c(v_1), c(v_2)\}$ has a color ϵ distinct from α, β, then we can give v that color, we give v_3 the list $\{\alpha, \beta, \epsilon\}$, and then extend the resulting coloring to G by applying Theorem 1 to $G - v_1 - v_2$, a contradiction. So we may assume that $L(v) \setminus \{c(v_k), c(v_1), c(v_2)\} = \{\alpha, \beta\}$. In particular, $c(v_k), c(v_1), c(v_2)$ are distinct. Similarly, $L(v) \setminus \{c(v_k), c(v_1), c(v_2)\} = \{\gamma, \delta\}$. Thus $L(v_3), L(v_{k-1})$ have precisely two colors in common, $c(v_2)$ is the unique color of $L(v_3) \setminus L(v_{k-1})$, and $c(v_k)$ is the unique color of $L(v_{k-1}) \setminus L(v_3)$, and $c(v_1)$ is the unique color of $L(v) \setminus (L(v_3) \cup L(v_{k-1}))$. This shows that the coloring of v_k, v_1, v_2 is unique.

Consider next the case where v_1 is joined to v_3. If there are two distinct colorings of v_k, v_1, v_2 that cannot be extended to G, then each of these can be extended to v_3, and we thereby get at least two distinct colorings of v_k, v_1, v_3 that cannot be extended to G, a contradiction to the induction hypothesis.

We use a similar argument if v_1 is joined to v_{k-1}. So we assume that v_1 is joined to none of v_3, v_{k-1}.

Assume finally that v_1 is joined to a vertex v_i, $4 \leqslant i \leqslant k-2$. The edge $v_1 v_i$ divides G into two generalized wheels G_1, G_2 none of which is a broken wheel. Suppose there is a coloring $c(v_k), c(v_1), c(v_2)$ of v_k, v_1, v_2, respectively, that cannot be extended to G. Then $L(v_i) \setminus \{c(v_1)\}$ has precisely two colors α, β. For if there were three such colors, then we can apply Theorem 1 to G_1 where the three available colors of v_i consists of $c(v_1)$ and two of the three colors in $L(v_i) \setminus \{c(v_1)\}$. This implies that G_1 can be colored such that there are two possible choices for the color of v_i. Similarly for G_2. But then the color of v_i can be chosen such that the coloring of v_k, v_1, v_2, v_i can be extended to G, a contradiction.

By Theorem 1, the coloring of v_k, v_1, v_2 can be extended to G_i for $i = 1, 2$. We may assume that v_i has the color α in the coloring of G_1 and the color β in the coloring of G_2. Thus $c(v_k), c(v_1), \alpha$ is the unique coloring of v_k, v_1, v_i that cannot be extended to G_2, and $\beta, c(v_1), c(v_2)$ is the unique coloring of v_i, v_1, v_2 that cannot be extended to G_1. This completes the proof of Lemma 1. $\square$

Lemma 2. *Assume that G is a generalized wheel. Assume further that each vertex in $int(C)$ has at least five available colors and that each of the vertices $v_3, v_4, \dots, v_{k-1}$ has at least three available colors. Assume that v_k, v_1, v_2 are precolored. Let e be any edge not on the outer cycle. Then $G - e$ has a list coloring.*

Lemma 2 can be proved by induction on the number of edges from v_1 to the outer cycle. The proof is easily reduced to the case where G is a wheel or the union of two wheels where e is their common edge. We leave the details for the reader.

Lemma 3. *Assume that the interior of C has precisely two vertices u, v, and there exists a natural number i, where $3 \leqslant i \leqslant k-1$, such that u is joined to $v, v_1, v_2, \ldots, v_i$, and v is joined to $u, v_i, \ldots, v_k, v_1$. Then G is 3-extendable with respect to the path $v_k v_1 v_2$.*

Proof. We give u a color α such that $L(v_3) \setminus \{c(v_2), \alpha\}$ has at least two colors. If $i = k-1$, then we color $v_{k-1}, v_{k-2}, \ldots, v_3, v$ in that order. So assume that $i \leqslant k-2$ and, similarly, $i \geqslant 4$.

If it is now possible to color v such that v_i still has two available colors, then it is easy to complete the coloring by coloring $v, v_{k-1}, v_{k-2}, \ldots, v_3$ in that order. So we may assume that such a coloring of v is not possible. That is, after u has received color α, both v and v_i have the same two available colors, say β, γ. So before coloring u, both v and v_i have the same three available colors, namely α, β, γ. By a similar argument, u, v_i have the same three available colors. Again, we give u the color α, and we give v the color β. If β is not in $L(v_{k-1})$, then we color $v_i, v_{i+1}, \ldots, v_{i-1}, \ldots$. So we may assume that β, γ are the only available colors at v_{k-1} distinct from the color of v_k.

Again, we give u the color α, we give v the color β, and we color $v_{k-1}, v_{k-2}, \ldots, v_{i+1}$. If this coloring can be extended to v_i, it is easy to complete the coloring. So we may assume that v_{i+1} has color γ. We now try another coloring. We give v the color α, we give u the color γ, and we color $v_3, v_4, \ldots, v_{i-1}$. We may assume that this cannot be extended to v_i, that is, v_{i-1} has color β. Now we keep the colors of $v_3, v_4, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{k-1}$. And we give v_i, v, u the colors α, β, γ, respectively.

This completes the proof of Lemma 3. □

4. 3-Extendability

We now characterize the near-triangulations that are not 3-extendable.

Theorem 3. *Let G be a plane near-triangulation with outer cycle C: $v_1 v_2 \cdots v_k v_1$. For each vertex v in G, let $L(v)$ be a list of colors. Assume that the vertices v_k, v_1, v_2 are precolored, that is, if v is one of v_k, v_1, v_2, then $L(v)$ consists of one color only. If v is one of $v_3, v_4, \ldots, v_{k-1}$, then $L(v)$ consists of at least three colors. Otherwise, $L(v)$ has at least five colors. Then G has an L-coloring unless G contains a subgraph G' which is a generalized wheel whose principal path is $v_k v_1 v_2$, and all vertices on the outer cycle of G' are on C and have precisely three available colors.*

Proof. The proof is by induction on the number of vertices of G. For $k \leqslant 5$, Theorem 3 follows from Theorem 2. So assume that $k > 5$. Suppose (reductio ad absurdum) that the theorem is false, and let G be a smallest counterexample.

Claim 1. *C has no chord.*

Proof. Suppose (reductio ad absurdum) that $v_i v_j$ is a chord of C, where $1 \leqslant i < j \leqslant k$. Then $v_i v_j$ divides G into near-triangulations G_1, G_2, respectively. If G_2, say, does not contain v_1, then any L-coloring of v_i, v_j can be extended to G_2, by Theorem 1. Therefore G_1 has no L-coloring. Now we apply the induction hypothesis to G_1 and obtain a contradiction. So assume that $i = 1$.

By Theorem 1, G_2 has an L-coloring. That coloring cannot be extended to G_1. The induction hypothesis implies that G_1 satisfies the conclusion of Theorem 3. A similar argument shows that G_2 satisfies the conclusion of Theorem 3. It only remains to be proved that $L(v_i)$ has only three available colors. But, if $L(v_i) \setminus \{c(v_1)\}$ has a subset consisting of three colors, then, by Theorem 1, each of G_1, G_2 can be list-colored, and the color of v_i can be chosen in two distinct ways (among these three colors) for each of G_1, G_2. Hence G can be L-colored, a contradiction which proves Claim 1. $\square$

Claim 2. *G has no separating triangle and no separating 4-cycle.*

Proof. Suppose (reductio ad absurdum) that G has a separating cycle C' of length 3 or 4. We consider the case where C' has length 4. (The case where C' has length 3 is similar and easier.) Replace *int*(C') by a single edge e, say, and denote the resulting graph by G'. If G' can be list-colored, then so can G, by Theorem 2. So we may assume that G' cannot be list-colored. Then G' contains a generalized wheel by the induction hypothesis. This generalized wheel contains e because we assume that G does not contain such a generalized wheel. If we delete the edge e from G', then the resulting graph can be list-colored by Lemma 2. By Theorem 2, G can be list-colored, a contradiction which proves Claim 2. $\square$

Claim 3. *If u is a vertex in int(C) which is joined to both of v_i, v_j where $2 \leqslant i \leqslant j-2 \leqslant k-2$, then u is joined to each of $v_i, v_{i+1}, \ldots, v_j$.*

Proof. Let C' be the cycle $uv_iv_{i+1}\cdots v_ju$. Suppose (reductio ad absurdum) that *Int*(C') is not a broken wheel. We apply the induction hypothesis, first to *Ext*(C') and then to *Int*(C'). If *Int*(C') is a generalized wheel, then, by Lemma 1, there is at most one coloring of its principal path that cannot be extended to *Int*(C'). Before we apply the induction hypothesis to *Ext*(C'), we delete from $L(u)$ the color u in the above mentioned coloring of the principal path of *Int*(C'). The resulting list-coloring of G gives a contradiction which proves Claim 3. $\square$

Claim 4. *G has no vertex u in int(C) which is joined to both of v_2, v_k.*

Proof. Suppose (reductio ad absurdum) that some vertex u in *int*(C) is joined to both of v_2, v_k. By Claim 3, u is joined to all vertices of C except possibly v_1. However, Claim 2 implies that u is joined to v_1, too. Hence G is a wheel. If some vertex of C has more than three available colors, then it is easy to list-color G. This contradiction proves Claim 4. $\square$

Claim 5. *v_3 has degree at least 4.*

Proof. Suppose (reductio ad absurdum) that v_3 has degree at most 3. By Claim 1, v_3 has degree precisely 3, and G has a vertex u in *int*(C) joined to v_2, v_3, v_4. Let i be the largest number such that u is joined to v_i. The path v_2uv_i divides G into two near-triangulations G_1, G_2 where G_1 contains v_1. By Claims 2, 3, G_2 is a broken wheel. By Claim 4, $i < k$.

Now we use the argument of the proof of Theorem 1 in [4]. We delete from $L(u)$ two colors of $L(v_3) \setminus L(v_2)$ and denote the new list assignment of $G - v_3$ by L'. We may assume that G_1 has no L'-coloring. For otherwise, that coloring could be extended to $G - v_3$ and hence also to G. Therefore the induction hypothesis implies that G_1 contains a generalized wheel. By Claims 1, 2, G_1 is a generalized wheel.

As C is chordless, by Claim 1, there is a vertex w in *int*(C) which is joined to $v_k, v_{k-1}, \ldots, v_i, u, v_1$. By Claim 4, w is not joined to v_2. Then u is joined to v_1. But then Lemma 3 implies that G has an L-coloring, a contradiction which proves Claim 5. $\square$

By a similar argument we get

Claim 6. v_{k-1} *has degree at least* 4.

We now claim that

Claim 7. v_3 *and* v_{k-1} *have a common neighbor in* $int(C)$.

Proof. Suppose (reductio ad absurdum) that Claim 7 is false. Let $v_2, u_1, \dots, u_q, v_4$ be the neighbors of v_3 in clockwise order. Then $q \geqslant 2$, by Claim 5. There is a similar neighborhood around v_{k-1}. Let $u_i v_j$ be the unique edge such that i is minimum and j is maximum. By Claim 3, $i = q$, and $j \leqslant k-2$. As in the proof of Claim 5 we let L' be obtained from L by deleting two colors of $L(v_3) \setminus L(v_2)$ from each neighbor of v_3 in $int(C)$. Now $G - v_3$ does not satisfy the conclusion of Theorem 3 because v_{k-1} has two neighbors in $int(C)$ joined by an edge, and a generalized wheel does not have that property. (Note that the generalized wheel above must contain v_{k-1} because $j \neq k$.) Therefore $G - v_3$ has an L'-coloring, and that can be extended to G, a contradiction which proves Claim 7. □

We are now ready for the final contradiction. As in the proof of Claim 7, we let $v_2, u_1, \dots, u_q, v_4$ be the neighbors of v_3 in clockwise order. Now Claims 2, 3, 7 imply that u_q is joined to $v_3, v_4, \dots, v_{k-1}$. By Claim 3, v_k is not joined to any of $u_1, \dots, u_{q-1}$, and by Claims 2, 6, v_k is not joined to u_q.

As in the proof of Claim 7, we define L' and conclude that $G - v_3$ has no L'-coloring. Therefore there is no L'-coloring of the cycle $v_1 v_2 u_1 \cdots u_q v_{k-1} v_k v_1$ and its interior. By the induction hypothesis, that graph contains a generalized wheel, and by Claim 1 it follows that there is a vertex w in $int(C)$ joined to u_q, v_{k-1}, v_k, v_1. By Claim 4, w is not joined to v_2.

We repeat the above argument considering v_{k-1} instead of v_3. Hence $q = 2$, and u_1 is joined to v_1.

By Claim 3, the cycle $u_2 v_3 v_4 \cdots v_{k-1} u_2$ together with its interior is a broken wheel. By Claim 2, G contains the edge $u_1 w$. (Note that G cannot contain the edge $u_2 v_1$ because of Claim 2.)

We may assume that $L(v_3) \setminus \{c(v_2)\}$ has precisely two colors and that $L(v_4)$ has precisely three colors since otherwise we just delete some colors from $L(v_3), L(v_4)$. If $L(v_4)$ intersects $L(v_3) \setminus \{c(v_2)\}$, we give u_2 a color not in any of these sets. Otherwise, we give u_2 any available color not in $L(v_3) \setminus \{c(v_2)\}$. Then we color $v_{k-1}, w, u_1, v_3, v_{k-2}, v_{k-3}, \dots, v_4$.

This contradiction completes the proof of Theorem 3. □

5. Further list-color properties of generalized wheels

Lemma 4. *Let G be a generalized wheel. Assume that each vertex in $int(C)$ has at least five available colors and that each of the vertices $v_3, v_4, \dots, v_k$ has at least three available colors. Assume that v_2 is precolored with color $c(v_2)$.*

Then it is possible to color v_k such that any coloring of v_1 can be extended to G.

Proof. We prove Lemma 4 by induction on the number of vertices of G.

If G is not a broken wheel, then Lemma 4 follows easily from Lemma 1. So assume that G is a broken wheel. In particular, v_1 is joined to v_3. Let α, β be two colors in $L(v_3) \setminus \{c(v_2)\}$.

Let γ, δ, ϵ be three colors in $L(v_k)$. Suppose (reductio ad absurdum) that, for each of these three colors it is possible to color v_1 such that the coloring cannot be extended to G. The color at v_1 must be one of α, β since otherwise, the coloring can be extended by Theorem 1 applied to $G - v_2$. So for two of the colors γ, δ, ϵ, say γ, δ, it is the same color, say α, which is used at v_1. But now we get a contradiction to Theorem 1 applied to G, where v_1 has color α, and v_k has the available colors γ, δ, α.

This completes the proof of Lemma 4. □

We now define a *generalized wheel string* as follows.

Let $G_1, G_2, \ldots, G_m$ be generalized wheels. (In particular, some of them may be triangles.) Identify a principal neighbor of the major vertex in G_1 with a principal neighbor of the major vertex in G_2. Identify the other principal neighbor of the major vertex in G_2 with a principal neighbor of the major vertex in G_3, etc. In other words, each principal neighbor of the major vertex in G_i has been identified with precisely one neighbor of the major vertex in G_{i-1} or G_{i+1}, $i = 2, 3, \ldots, m-1$. One principal neighbor of the major vertex in G_1 (respectively G_m) has not been identified with any other vertex. We call these the two *clean vertices*. If each of the graphs $G_1, G_2, \ldots, G_m$ is a broken wheel, then G is a *broken wheel string*.

Lemma 5. *Let G be a generalized wheel string. Assume further that each vertex not on the outer face boundary has at least five available colors and that each non-clean vertex on the outer face boundary has at least three available colors. Assume that the two clean vertices have two available colors each. Then it is possible to color the two clean vertices and all the cutvertices of G such that any coloring of the major vertices can be extended to G.*

Proof. We prove Lemma 5 by induction on the number of vertices of G. Suppose (reductio ad absurdum) that G is a smallest counterexample.

Consider first the case where $m \geqslant 2$. Let x (respectively y) be the clean vertex in G_1 (respectively G_m). Let z be the common vertex of G_1 and G_2. Assume that $L(z) = \{\alpha, \beta, \gamma\}$. We now apply the induction hypothesis to G_1. We may assume that x, z can be colored such that the conclusion of Lemma 5 holds. Assume the color of z is α. Then we again apply the induction hypothesis to G_1 but now we only allow colors β, γ at z. So the coloring of x, z can be chosen in two ways in which z has two distinct colors. Applying the induction hypothesis to $G_2 \cup G_3 \cup \cdots \cup G_m$ there are two distinct colorings of z, y (with z getting distinct colors) such that the conclusion of Lemma 5 holds. Now we let z receive a color that appears in both a coloring of x, z and a coloring of y, z. So we may assume that $m = 1$.

We use the same notation as in the definition of a generalized wheel. Let $x = v_2$, $y = v_k$ be the clean vertices in $G_1 = G$. Let $L(v_2) = \{\alpha, \beta\}$. We may assume that G is a broken wheel since otherwise, we apply Lemma 1. Then we may assume that $L(v_3) = \{\alpha, \beta, \gamma\}$ since otherwise we give v_2 a color not in $L(v_3)$ and we give v_k any available color and complete the proof by applying Theorem 1 to $G - v_2$ (no matter what the color of v_1 is). Now, there are four possible ways of coloring v_2, v_k. We may assume that none of them works. In other words, for each of those four colorings, it is possible to color v_1 such that the resulting coloring cannot be extended to G. The color of v_1 must be a color in $L(v_3) = \{\alpha, \beta, \gamma\}$ since otherwise, the coloring can be extended by applying Theorem 1 to $G - v_2$. So, for two of the colorings of v_2, v_k, the color at v_1 is the same, and that color must be γ. For if it was β, say, then v_2 would have the color α in the above two colorings of v_2, v_k but that would contradict Theorem 1 applied to G with v_2, v_1 colored α, β, respectively. So we may assume that $L(v_k) = \{\alpha', \beta'\}$ and that the colorings α, β', γ

and β, α', γ of v_2, v_k, v_1, respectively cannot be extended to G. We now try the coloring α, α' of v_2, v_k, respectively. The color at v_1 which prevents the extension must be β. (For, if it was γ, then we would get a contradiction to Theorem 1 with v_1, v_2 being precolored.) If v_2, v_k, v_1 are colored such that there is no list-color-extension to G, then the color of v_1 must be present in all lists of the uncolored vertices. So all these lists must contain β, γ. Since we can interchange between β, α, all the lists with three colors must be the same, namely $\{\alpha, \beta, \gamma\}$. They must also equal $\{\alpha', \beta', \gamma\}$. So v_2 and v_k have the same available colors, namely α, β. But then one of the above-mentioned colorings α, β', γ or α, α', β of v_2, v_k, v_1 will work, a contradiction which proves Lemma 5. □

6. Exponentially many 5-list-colorings of planar graphs

In this section we prove the main result.

Theorem 4. *Let G be a plane near-triangulation with outer cycle C: $v_1 v_2 \cdots v_k v_1$. For each vertex v in G, let $L(v)$ be a list of colors. Assume that the vertices v_k, v_1, v_2 or the vertices v_1, v_2 are precolored, that is, if v is one of v_k, v_1, v_2 (respectively v_1, v_2), then $L(v)$ consists of one color only. If v is one of $v_3, v_4, \dots, v_{k-1}$ (respectively $v_3, v_4, \dots, v_k$), then $L(v)$ consists of at least three colors. Otherwise, $L(v)$ has at least five colors. Let n denote the number of non-precolored vertices, and let r denote the number of vertices with precisely three available colors. Assume that G has an L-coloring. Then the number of distinct L-colorings of G is at least $2^{n/9-r/3}$, unless G has three precolored vertices and also contains a vertex with precisely four available colors which is joined to the three precolored vertices and has only one available color distinct from the colors of the three precolored vertices.*

Proof. The proof is by induction on n. It is easy to verify the statement if $n \leqslant 1$ so we proceed to the induction step. Let f denote the number of vertices with precisely four available colors.

We assume that G is a counterexample such that n is minimum and, subject to this, r is maximum, and, subject to these conditions, f is minimum. We shall establish a number of properties of G which will lead to a contradiction. Clearly, $n > 3r$.

Claim 8. *G has no separating triangle.*

Proof. Suppose (reductio ad absurdum) that $xyzx$ is a separating triangle which divides G into near-triangulations G_1, G_2, respectively, where G_1 contains C. Then any L-coloring of x, y, z can be extended to G_2, by Theorem 2. Let n_1 be the number of non-precolored vertices in G_1, and let n_2 be the number of vertices in $G_2 - x - y - z$. By the minimality of n, G_1 has at least $2^{n_1/9-r/3}$ distinct list-colorings. Each such coloring has at least $2^{n_2/9}$ extensions to G_2. As $n = n_1 + n_2$, this proves Claim 8. □

Claim 9. *C has no chord.*

Proof. Suppose (reductio ad absurdum) that $v_i v_j$ is a chord of C, where $1 \leqslant i < j \leqslant k$. Then $v_i v_j$ divides G into near-triangulations G_1, G_2, respectively.

Consider first the case where G_2, say, does not contain a precolored vertex distinct from v_i, v_j. Then any L-coloring of v_i, v_j can be extended to G_2, by Theorem 1. We now obtain a contradiction by repeating the proof of Claim 8.

Assume next that $i = 1$ and that v_k is precolored. If each of G_1, G_2 is a generalized wheel such that each non-precolored vertex on the outer cycle has precisely three available colors, then $r \geqslant n/3$, and there is nothing to prove. So assume that G_2, say, is not such a generalized wheel. Moreover, it does not contain such a generalized wheel because G has no separating triangle, by Claim 8, and every chord of G_2, if any, is incident with v_1, by the first part of the proof of Claim 9. Now we repeat the proof of Claim 8. This proves Claim 9 unless G has an edge from v_1 to v_{k-1}. So assume that this edge is present.

Then we color v_{k-1}, and we apply the induction hypothesis to $G - v_k$. If v_{k-1} has at least four available colors, then n decreases, but there are at least two choices for the color of v_{k-1}. So we only need to consider the exceptional case in Theorem 4, namely that G has a vertex with precisely four available colors joined to v_{k-1}, v_1, v_2. Then $k = 5$, and $n = 2$. If v_3 has precisely three available colors, then $r = 1$, and there is nothing to prove. On the other hand, if v_3 has at least four available colors, then G has at least two list-colorings. This proves Claim 9. □

Claim 10. *Each non-precolored vertex on C has precisely three available colors.*

Proof. Suppose (reductio ad absurdum) that Claim 10 is false. Select a set L' of four available colors in $L(v_i)$ for some vertex v_i of C. Let L'' be one of the four 3-element subsets of L'. Now replace $L(v_i)$ by L''. By the maximality of r, the new G has at least $2^{n/9-(r+1)/3}$ distinct list-colorings. As L'' can be chosen in four ways, this results in $4 \cdot 2^{n/9-(r+1)/3}$ list-colorings and each of these is counted three times. Thus we get at least $4 \cdot 2^{n/9-(r+1)/3}/3$ distinct L-colorings, a contradiction which proves Claim 10 unless G is a wheel with precisely two vertices with more than three available colors. But then either $n = r + 2$ and $r > 0$ in which case there is nothing to prove, or else $n = 2$, $r = 0$ in which case G has at least two distinct L-coloring. □

Claim 11. *v_k is precolored.*

Proof. Suppose (reductio ad absurdum) that Claim 11 is false. The coloring of v_1, v_2 can be extended to G. We give v_k the color in that coloring. This decreases each of n, r by 1 and hence we obtain a contradiction to the minimality of n. Note that, by Claim 10, the new G cannot have a vertex with precisely four available colors joined to the three colored vertices. □

Claim 12. *If v is a vertex in int(C) joined to v_i, v_j, where $2 \leqslant i < j \leqslant k$, then v is also joined to each of $v_{i+1}, v_{i+2}, \ldots, v_{j-1}$.*

Proof. Let G_1 be the cycle $v_1 v_2 \cdots v_i v v_j \cdots v_k v_1$ and its interior, and let G_2 be the cycle $v v_i v_{i+1} \cdots v_j v$ and its interior. Suppose (reductio ad absurdum) that Claim 12 is false. Then G_2 is not a broken wheel. Then we apply induction first to G_1 and then to G_2. This proves Claim 12 unless G_2 is a generalized wheel. Then we may assume that G_2 is a wheel (by choosing a larger i and a smaller j if necessary). Then there is at most one coloring of v_i, v, v_j which cannot be extended to G_2 by Lemma 1. Let α be the color of v if such a coloring exists. In that case we delete α from $L(v)$ before we apply induction to G_1. If n' (respectively r') is the number of non-precolored vertices (respectively non-precolored vertices with precisely three available colors) of G_1, then it is easy to see that $n'/9 - r'/3 \geqslant n/9 - r/3$. Only one problem remains when we apply induction to G_1, namely that v has precisely four available colors and is joined to v_k, v_1, v_2. But then we color v (and we have two choices for that), and we apply the induction hypothesis to $G - v_1$ unless $G - v_1$ is a generalized wheel. But, if $G - v_1$ is a generalized wheel, then $r > n/3$, and there is nothing to prove.

This contradiction proves Claim 12. □

We may assume that

Claim 13. $k > 4$.

For, if $k = 3$, then we delete the edge v_2v_3. And if $k = 4$, then we color v_3 and delete it and use induction.

We now split the proof up into the following two cases.

Case 1. G does not contain a path $v_2u_1u_2\cdots u_qv_k$ with the properties that

(i) each of $u_1, u_2, \ldots, u_q$ is a vertex in *int*(C) joined to at least two vertices of $v_3, v_4, \ldots, v_{k-1}$,

and

(ii) the cycle $v_1v_2u_1u_2\cdots u_qv_kv_1$ and its interior form a generalized wheel.

Case 2. G contains a path $v_2u_1u_2\cdots u_qv_k$ with the above-mentioned properties *(i)* and *(ii)*.

We first do Case 1. We shall prove that the number of list-colorings is not just at least $2^{n/9-r/3}$ as required in Theorem 4, but even at least $2^{(n+1)/9-r/3}$. This will be important in Case 2 which we shall reduce to Case 1 by deleting an appropriate vertex.

Let R be the set of vertices in *int*(C) which are joined to at least two vertices of the path $C - v_k - v_1 - v_2$. By Claim 12, the union of the path $C - v_k - v_1 - v_2$ and R and the edges from R to C form a broken wheel string which we call W.

Subcase 1.1. No two consecutive blocks in W are triangles.

We use Lemma 5 to color all the principal neighbors of the major vertices in W in such a way that, regardless of how the major vertices in W are colored, the coloring can be extended to W. This means that we can apply induction to $G' = G - v_3 - v_4 - \cdots - v_{k-1}$. Any list-coloring of G' can be extended to G. By the induction hypothesis, the number of list-colorings of G' is at least $2^{n'/9-r'/3}$ where $n' = n - k + 3 = n - r$ and $r' = |R|$. The assumption of Subcase 1.1 implies that $r' \leqslant (2r - 1)/3$. Hence the number of list-colorings of G' is at least $2^{(n+1)/9-r/3}$.

Subcase 1.2. Two consecutive blocks in W are triangles. Let w_1, w_2 be two vertices in R each joined to precisely two consecutive vertices of C. That is, there is a natural number i such that W contains the blocks $w_1v_{i-1}v_iw_1$ and $w_2v_iv_{i+1}w_2$. We now color successively v_3 and the cutvertices of W with increasing indices until we color v_{i-1}. Whenever we color a cutvertex, we do it such that the corresponding block of W can be colored regardless of how we color the major vertex. This is possible by Lemma 4. There are even two possibilities for coloring such a cutvertex of W whenever the preceding cutvertex is a neighbor of the cutvertex that is being colored. Then we color successively v_{k-1} and the cutvertices of W with decreasing indices until we color v_{i+1}. Again, there are even two possibilities for coloring such a cutvertex of W whenever the preceding cutvertex is a neighbor of the cutvertex that is being colored. Finally we color v_i and apply the induction hypothesis to $G - v_3 - v_4 - \cdots - v_{k-1}$. Let r' be the number of vertices of R, and let n' be the number of uncolored vertices of $G - v_3 - v_4 - \cdots - v_{k-1}$. Then $n' = n - k + 3 = n - r$.

The number of colorings of the vertices of W in the path $v_2 v_3 \cdots v_{k-1}$ is at least 2^t, where t is the number of blocks of W which are triangles.

For each of these there are at least $2^{n'/9 - r'/3}$ list-colorings of $G - v_3 - v_4 - \cdots - v_{k-1}$, by the induction hypothesis. Let s be the number of blocks of W which are not triangles. Then $r' = s + t$ and $r \geqslant 2s + t + 1$. So the total number of colorings of G is at least $2^{n'/9 - r'/3 + t}$ which is greater than $2^{(n+1)/9 - r/3}$. This completes the proof in Case 1.

We now do Case 2. Let m be the smallest number such that u_q is joined to v_m. By Claim 12, u_q is joined to $v_m, v_{m+1}, \ldots, v_k$ (and possibly also to v_1). Again, we split up into two cases.

Subcase 2.1. v_1 is joined to u_q. We select two colors in $L(v_{k-1})$ distinct from the color of v_k. We delete these colors from $L(u_q)$ and we delete the vertex v_{k-1} from G. Then we color u_q and delete also v_k. By the induction hypothesis, if the resulting graph G' has at least one list-coloring, then it has at least $2^{(n-2)/9 - (r-1)/3}$ list-colorings. Each such coloring can be extended to v_{k-1} and the proof is complete. So assume that G' has no list-coloring. By Theorem 3, G' contains a generalized wheel. Hence either $q = 1$ in which case G is a wheel by Claim 12, or else $q = 2$ in which case $G' - v_{m+1} - v_{m+2} - \cdots - v_{k-2}$ is a wheel. But then $n - r \leqslant 2$ and there is nothing to prove. □

Subcase 2.2. v_1 is not joined to u_q. Now G has a vertex w joined to v_1, v_k, u_q, u_{q-1} by the definition of a generalized wheel.

If $m < k - 2$, then we select two colors in $L(v_{k-1})$ distinct from the color of v_k. We delete these colors from $L(u_q)$ and we delete the vertices $v_{k-1}, v_{k-2}, \ldots, v_{m+1}$ from G. Then we use the induction hypothesis to obtain a contradiction. So assume that $m = k - 2$.

If $q = 1$, then both of w and u_1 are joined to all of $v_3, v_4, \ldots, v_{k-1}$ which is impossible. So assume that $q > 1$.

If u_{q-1} is joined to v_{k-2}, then we select two colors in $L(v_{k-1})$ distinct from the color of v_k. We delete these colors from $L(u_q)$ and we delete the vertex v_{k-1} from G. The resulting graph G' satisfies Claim 12 and it also satisfies the assumption in Case 1. Therefore we may repeat the proof in Case 1. The proof in Case 1 gives a number of list-colorings which is larger than what we need, and therefore the proof in Case 2 is complete. Therefore we may assume that u_{q-1} is not joined to v_{k-2}. In this case u_{q-1} does not satisfy Claim 12, and therefore a different argument is needed.

Let i be the smallest number such that u_{q-1} is joined to v_i, and let j be the largest number such that u_{q-1} is joined to v_j. Then $j < k - 2$. We select two colors in $L(v_{k-1})$ distinct from the color of v_k. We delete these colors from $L(u_q)$ and we delete the vertex v_{k-1}. The path $v_i u_{q-1} u_q$ divides the resulting graph into two graphs G_1, G_2, where G_1 contains v_1. If G_2 is a generalized wheel, then we obtain a contradiction by applying the induction hypothesis to G_1 (which has a smaller r than G has). We also use the fact that, by Lemma 1, there is at most one coloring of the path $v_i u_{q-1} u_q$ which cannot be extended to G_2. We delete the color of u_{q-1} in this coloring from $L(u_{q-1})$ before we apply the induction hypothesis to G_1. In this case the number list colorings of G_1 is greater than $2^{n/9 - r/3}$, and any such coloring can be extended to G_2.

On the other hand, if G_2 is not a generalized wheel, then we obtain a contradiction by applying the induction hypothesis first to G_1 and then to G_2. We lose a multiplicative factor $2^{1/9}$ because of the deleted vertex v_{k-1}. We make up for that before we apply induction to G_1 since we can delete one of the available colors of u_{q-1} in at least five different ways. In this way we gain a multiplicative factor $5/4$, and now the proof is complete, because $5/4 > 2^{1/9}$. $\square$

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