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THE ERDŐS–PÓSA PROPERTY FOR LONG CIRCUITS

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We show that, for every l, the family $\mathcal{F}_l$ of circuits of length at least l satisfies the Erdős–Pósa property, with $f(k)=13l(k-1)(k-2)+(2l+3)(k-1)$, thereby sharpening a result of C. Thomassen. We obtain as a corollary that graphs without k disjoint circuits of length l or more have tree-width $O(lk^2)$.

1. Introduction

Let G be a graph and $\mathcal{F}$ a family of graphs. A *transversal* of $\mathcal{F}$ is a set X of vertices of G such that $G-X$ contains no member of $\mathcal{F}$. The family $\mathcal{F}$ is said to have the *Erdős–Pósa property* if there exists a function $f:\mathbb{N}\to\mathbb{N}$ such that every graph G contains either k vertex-disjoint members of $\mathcal{F}$ or a transversal of $\mathcal{F}$ of size at most $f(k)$. This concept originated in [6], where Erdős and Pósa established the existence of such a function f when $\mathcal{F}$ is the family of circuits. For the rest of the paper, we abbreviate *vertex-disjoint* to *disjoint*.

In this paper, we show that, for every l, the family $\mathcal{F}_l$ of circuits of length at least l satisfies the Erdős–Pósa property, with $f(k)=13l(k–1)(k–2)+(2l+3)(k-1)$. This sharpens a result of Thomassen [11], who obtained a doubly exponential bound on $f(k)$. Applying a result of Birmelé [1], we obtain as a corollary that graphs without k disjoint circuits of length l or more have tree-width $O(lk^2)$.

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We now discuss our results in more detail. Let k and l be integers, with $k \geq 1$ and $l \geq 3$. By a *long circuit*, we mean a circuit of length at least l. Let G be a graph which does not contain k disjoint long circuits. We shall bound the size of a minimum transversal of the long circuits of G. The case $k=2$ is of particular importance. The proof of this base case is much more complicated than the inductive argument which allows us to extend it to arbitrary k. Moreover, it is reasonable to hope that an exact result can be obtained when $k=2$. Indeed, we propose:

Conjecture 1. Let G be a graph containing no two disjoint long circuits. Then there exists a transversal of the long circuits of G of size at most l.

The complete graph on $2l-1$ vertices shows that the bound l cannot be reduced. Lovász [8] proved that it is sharp when $l=3$. Moreover, he characterized the graphs containing no two disjoint circuits. Birmelé [2] proved that the bound is also sharp for $l=4$ and $l=5$. We obtain the bound $2l+3$, valid for all $l \geq 3$.

Theorem 1. *Let G be a graph containing no two disjoint long circuits. Then there exists a transversal of the long circuits of size at most $2l+3$.*

To obtain a bound in the general case, we delete an appropriate set of vertices X of bounded size and then apply Theorem 1 to $G-X$. We prove:

Theorem 2. *Let G be a graph containing no k disjoint long circuits. Then there exists a transversal of the long circuits of size at most $13l(k-1)(k-2)+(2l+3)(k-1)$.*

In Section 2, we show how to derive Theorem 2 from Theorem 1. The proof of Theorem 1 is given in Section 3. Consequences of our results for tree-width are given in Section 4. We close the paper with a discussion of related questions.

To close this section, we note a number of other families which enjoy the Erdős–Pósa property. Let H be a graph and $\mathcal{F}_H$ the family of graphs containing an H-minor. It is easily seen that if H is not planar, $\mathcal{F}_H$ does not have the Erdős–Pósa property. Robertson and Seymour [9] proved, on the other hand, that if H is planar, $\mathcal{F}_H$ does have the Erdős–Pósa property. Thomassen [11] showed that the family $\mathcal{F}_{l,m}$ of circuits of length l modulo m satisfies the Erdős–Pósa property if $l=0$. On the other hand, if $l \neq 0$, this is not necessarily true, as was shown by Dejter and Neumann-Lara [4]; in particular, the family of odd circuits do not satisfy the Erdős–Pósa property. Reed [10] proved, however, that there exists a function $f:\mathbb{N}\to\mathbb{N}$ such that, for any positive integer k, a graph either contains k odd circuits using each vertex at most twice or has an odd circuit transversal of size at most $f(k)$.

2. Transversals of graphs without k pairwise-disjoint long circuits

We give here a proof of Theorem 2, based on Theorem 1:

Let G be a graph which does not contain k disjoint long circuits. Then there exists a transversal of the long circuits of G of size at most $13l(k-1)(k-2)+(2l+3)(k-1)$.

We shall need to apply the following well-known result of Erdős and Szekeres [7].

Theorem 3. *Let S be a sequence of $(m-1)(n-1)+1$ distinct integers. Then S has either an increasing subsequence of m terms or a decreasing subsequence of n terms.*

Proof of Theorem 2. We proceed by induction on k, and may therefore assume that G contains $k-1$ disjoint long circuits $C_1,\dots,C_{k-1}$.

For $i<j$, consider the graph $G_{ij}:=G-\bigcup_{r\neq i,j}C_r$. If there are $26l$ disjoint paths linking C_i and C_j in G_{ij}, there are 26 such paths whose ends in C_i are separated by segments of length at least l. By Theorem 3, the ends of some six of these paths appear in the same order on C_i and C_j. The union of these six paths with C_i and C_j contains three disjoint long circuits (see Figure 1).

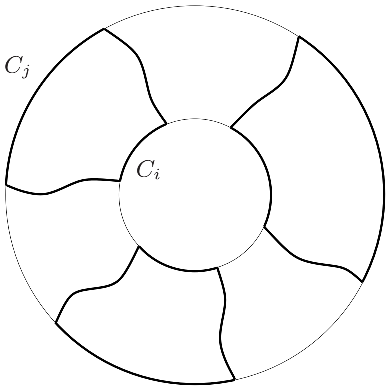

Figure 1. Three disjoint long circuits.

These, together with the $k-3$ circuits C_r, $r\neq i,j$, define a set of k disjoint long circuits, contradicting the hypothesis on G. Consequently, there exists a set X_{ij} of at most $26l$ vertices separating C_i and C_j in G_{ij}. We set $X:=\cup\{X_{ij}:1\leq i<j\leq k-1\}$.

We claim that each component of $G-X$ intersects at most one of the circuits C_i, $1 \le i \le k-1$. Suppose, to the contrary, that some component H of $G-X$ intersects two of these circuits, C_i and C_j. Then there is a path P in H connecting C_i and C_j; without loss of generality, we may assume that no internal vertex of P lies on any of the circuits $C_1,\dots,C_{k-1}$. But then C_i and C_j belong to the same component of $G_{ij}-X$, hence to the same component of $G_{ij}-X_{ij}$, a contradiction.

It follows that no component H of $G-X$ contains two disjoint long circuits, for if it did, these two circuits, together with $k-2$ of the circuits C_i, $1 \le i \le k-1$, not meeting H, would constitute a set of k disjoint long circuits in G. By Theorem 1, each component of $G-X$ has a transversal of size at most $2l+3$. Since G has only $k-1$ disjoint long circuits, $G-X$ has a transversal of size at most $(2l+3)(k-1)$. Therefore G has a transversal of size at most $13l(k-1)(k-2)+(2l+3)(k-1)$. ∎

3. Transversals of graphs without two disjoint long circuits

This section is devoted to a proof of our main theorem, Theorem 1:

Let G be a graph containing no two disjoint long circuits. Then there exists a set X of at most $2l+3$ vertices that hits all long circuits.

The notation $P[x,y]$ will be used to indicate a path P with initial vertex x and terminal vertex y. Likewise, for a given path P or circuit C, we denote by $P[x,y]$ or $C[x,y]$ the xy-segment of the path P or the circuit C (with respect to its prescribed sense of orientation).

Let X and Y be two subsets of vertices of G. An (X,Y)-*path* is a path which starts at a vertex of X, ends at a vertex of Y, and has no internal vertex in either X or Y.

Proof. Let C be a shortest long circuit in G, with a prescribed sense of orientation. Because C intersects every long circuit of G, its vertex set is a transversal for the long circuits. We may thus assume that C has length at least $2l+4$ and is induced (that is, has no chord).

The following concept is the key to the proof of Theorem 1. A path P which connects two vertices, u and v, of C, and which is internally-disjoint from C, will be called a *long path* if both uv-segments of C are of length at least $l/2$. By the minimality of C and the fact that C is of length at least $2l$, a long path P necessarily has length at least $l/2$, and thus forms a long circuit with each of the uv-segments of C it defines.

Claim. *If there are no long paths, there exists a set X of at most l vertices that hits all long circuits.*

Suppose that there are no long paths. As noted above, every long circuit D intersects C. Define C_D to be a shortest segment of C containing $V(C)\cap V(D)$, and choose a long circuit D for which C_D is minimal (see Figure 2). Let x_D and y_D be the first and last vertices of C_D, respectively (with respect to the sense of orientation of C). Let X_1 be the set of $\lfloor l/2 \rfloor$ vertices of C immediately preceding x_D, and let X_2 consist of y_D and the set of $\lceil l/2 \rceil - 1$ vertices of C immediately following y_D. Set $X := X_1 \cup X_2$. We shall show that X hits all long circuits. If not, there is a long circuit D' in $G-X$. By the choice of D, D' intersects the segment C' of $C-X$ which is disjoint from C_D. But D' also intersects D. It follows that there is a long path, made up of a segment of D' and a segment of D, connecting a vertex of C' and a vertex of C_D. This contradiction establishes the claim.

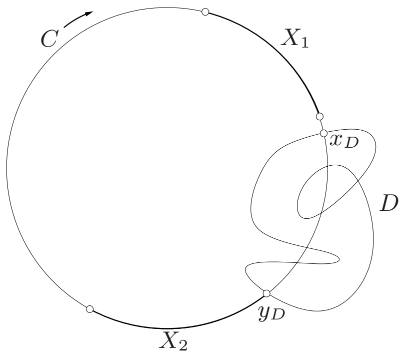

Figure 2. A long circuit D with $C[x_D, y_D]$ minimal.

We may henceforth assume that there is a long path. Let H be the component of $G-C$ containing the internal vertices of a long path. Suppose, first, that H is 2-connected. Choose a long path $P[x,y]$ whose internal vertices are in H, with $C[x,y]$ minimal. Let X_1 consist of the vertex x and the set of $\lceil l/2 \rceil - 1$ vertices of C immediately preceding x, and let X_2 consist of the vertex y and the set of $\lceil l/2 \rceil - 1$ vertices of C immediately preceding y. Set $X := X_1 \cup X_2$. Denote by A the segment of $C - X$ contained in $C[x,y]$, and by B the other segment (see Figure 3); note that A might be empty.

Then, in $G-X$:

- *There is no $(A, P - \{x,y\})$-path.* If there were such a path $P'[x',y']$, the path $P'[x',y']P[y',y]$ would be a long path contradicting the choice of P.

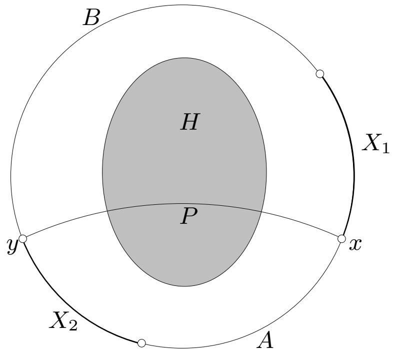

Figure 3. A long path $P[x,y]$.

- *Every* (A,B)*-path is disjoint from* H. If not, there would be an $(A,P-\{x,y\})$-path.
- *Every* (A,B)*-path is long.*
- *There are at most two disjoint* (A,B)*-paths.* Two 'parallel' (A,B)-paths would yield, when combined with appropriate segments of C, two disjoint long circuits; likewise, three mutually 'crossing' (A,B)-paths, together with the path P, would yield two disjoint long circuits (see Figure 4).

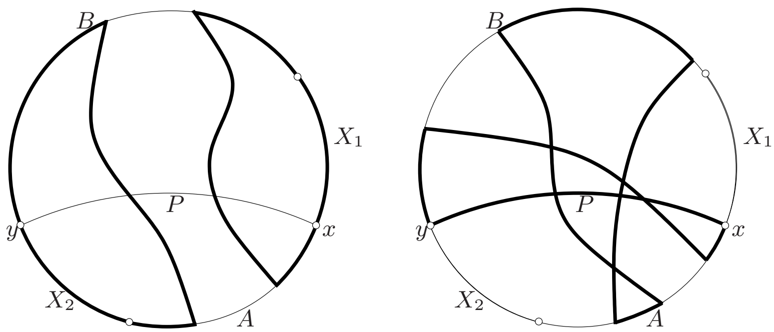

Figure 4. Parallel and crossing AB-paths.

Menger's theorem thus implies that there is a set X_3 of at most two vertices meeting all (A,B)-paths in $G-X$. We now set $X := X_1 \cup X_2 \cup X_3$. Then, in $G-X$:

- *There is no* (A,B)*-path.*
- *Every long circuit intersects* A *or* B*, but not both.* A long circuit meeting A and B would contain an (A,B)-path.

- *No long circuit intersects A.* Such a circuit would be disjoint from the long circuit $P[x,y]C[y,x]$.
- *Every long circuit intersects B.*

Let X_4 be the set of $\lceil l/2 \rceil - 1$ vertices of C immediately following y. Set $X := X_1 \cup X_2 \cup X_3 \cup X_4$. Let B' be the segment of $C - X$contained in B. For every long circuit D of $G - X$, define $C_D[x_D, y_D]$ to be a shortest segment of B' containing $V(C) \cap V(D)$, and choose a long circuit D (if any) such that the vertex y_D occurs as soon as possible after the interval X_4. Let X_5 be the set of $\lfloor l/2 \rfloor$ vertices of C immediately following y_D, and including y_D. We now set $X := X_1 \cup X_2 \cup X_3 \cup X_4 \cup X_5$. If there is a long circuit in $G - X$ then, as in the proof of the Claim above, there is a long (B,B)-path $Q[u,v]$ linking the two segments of $B' - X_5$. Note that Q is at least as long as the segment $C[u,v]$, and thus certainly of length at least three. Moreover, Q intersects P, because otherwise $C[u,v]Q[v,u]$ and $C[x,y]P[y,x]$ would be disjoint long circuits. Hence the internal vertices of Q lie in H. Denote the neighbours on P of x and y by x' and y', respectively, and the neighbours on Q of u and v by u' and v', respectively (see Figure 5). Note that the

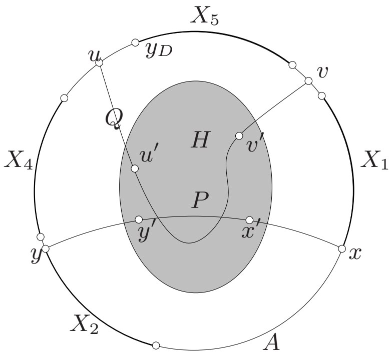

Figure 5. The long path $Q[u,v]$.

four vertices x', y', u' and v' are distinct, as otherwise the long path Q would be too short. Because H is 2-connected, there are two disjoint paths in H linking the sets $\{x', u'\}$ and $\{y', v'\}$. These paths, together with the edges xx', yy', uu', vv' and appropriate segments of C, yield two disjoint long circuits, a contradiction (see Figure 6). Thus X intersects all long circuits of G, and $|X| \le 2l + 2$.

We now consider the case where H is not 2-connected. Here, it is necessary to define the long path $P[x,y]$ differently. We assume, as before, that H contains the internal vertices of a long path. We root H at an arbitrary

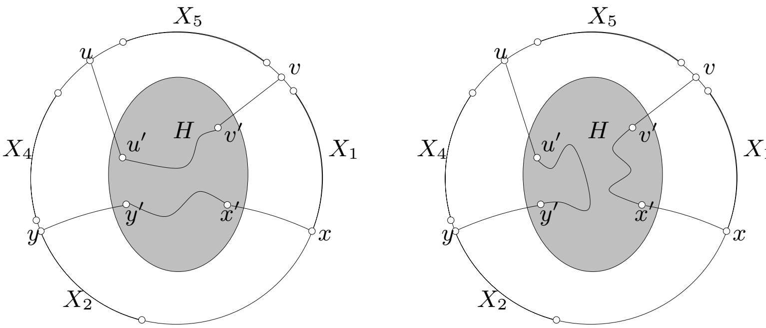

Figure 6. Two disjoint long circuits.

block R, and thereby associate, to each block Z of H, a subgraph H_Z of H, namely the union of Z and the blocks of H 'below' Z, relative to R (see Figure 7).

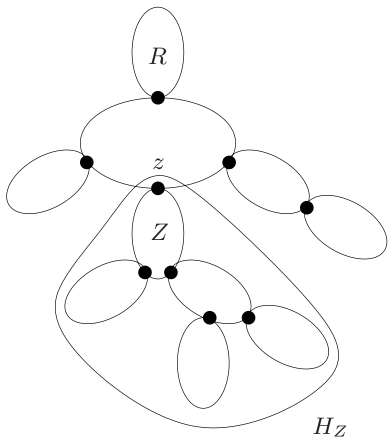

Figure 7. The rooted block tree of H.

We now select a block Z of H such that H_Z contains the internal vertices of some long path and is minimal with respect to this property. The path $P[x,y]$ is then chosen to be a long path in H_Z for which $C[x,y]$ is minimal. If $Z \neq R$, we denote by z the cutvertex of H which separates H_Z from the rest of H.

We define X_1, X_2, X, A, B as before. As before, there is a set S of at most two vertices meeting all (A,B)-paths in $G - X - H_Z$. We set $X_3 := S$ if $Z = R$, and $X_3 := S \cup \{z\}$, otherwise. We now set $X := X_1 \cup X_2 \cup X_3$. Then, in $G - X - H_Z$:

- *There is no (A,B)-path.*
- *Every long circuit intersects A or B, but not both.*

However, by the choice of P, no edge links A to a vertex of H_Z. Thus, in $G-X$:

- *No long circuit intersects A.*
- *Every long circuit intersects B.*

We define X_4, B', D, y_D, X_5 as before, and set $X := X_1 \cup X_2 \cup X_3 \cup X_4 \cup X_5$. We may assume, as before, that there is a long (B,B)-path Q in $G-X$. This path necessarily intersects P. By the choice of P and the definition of X_3, the path Q must include an edge of Z. We define x', y' as the first and last vertices of P in Z (necessarily distinct), and u', v' as the first and last vertices of Q in Z (also necessarily distinct), and obtain two disjoint long circuits, as before. This final contradiction shows that X is a set of vertices meeting all long circuits, with $|X| \le 2l+3$. ∎

4. Tree-width

The notion of tree-width was introduced by Robertson and Seymour, who showed in [9] that, given a graph H, the graphs with no H-minor have bounded tree-width if and only if H is planar. In the case where H is planar, however, the bound given in [9] for the tree-width is exponential in m, where m is such that H can be embedded in the $m \times m$-grid. The authors of [5] gave a somewhat simpler proof of the Robertson–Seymour theorem, but their bound is exponential, too. The results in this paper imply a polynomial bound for the tree-width when H is a disjoint union of l-circuits.

Note that if $X \subset V(G)$, and $(T, \mathcal{W})$ is a tree-decomposition of $G-X$, a tree-decomposition of G can be obtained by simply adjoining the set X to each element of $\mathcal{W}$. It follows that

$$tw(G) \le tw(G-X) + |X|. \tag{1}$$

In order to deduce our bound, we also need the following result of Birmelé [1].

Theorem 4. *Let G be a graph containing no long circuit. Then $tw(G) \le l$–2.*

This bound is best possible, the complete graph on $l-1$ vertices having tree-width $l-2$.

Corollary 1. *Let G be a graph which does not contain k disjoint long circuits. Then $tw(G) \le 13l(k-1)(k-2)+3l+1$.*

Proof. If $k=2$, this follows from Theorem 4 on applying inequality (1) with X the transversal of $2l+3$ vertices guaranteed by Theorem 1. If $k>2$, it follows from the case $k=2$ on applying inequality (1) with X the set of $13l(k-1)(k-2)$ vertices defined in the proof of Theorem 2. ∎

5. Related questions

We conclude by briefly mentioning several questions suggested by the results presented in this paper.

1. Let us first reiterate Conjecture 1:
   Let G be a graph containing no two disjoint long circuits. Then there exists a transversal of the long circuits of G of size at most l. The first open case of this conjecture is the case $l=6$.
2. One may consider versions of Conjecture 1 for particular classes of graphs. For instance, one might ask for the smallest function f such that, in any planar graph G containing no two disjoint long circuits, there exists a transversal of the long circuits of G of size at most $f(l)$.
3. Consider the family $\mathcal{G} := \mathcal{G}(l,m)$ of all graphs containing no m disjoint long circuits. Denote by $f(l,m)$ the largest tree-width of a member of $\mathcal{G}$, and by $g(l,m)$ the largest size of a minimum transversal of a member of $\mathcal{G}$. What are the correct orders of magnitude of the functions f and g? We have shown here that $f(l,m) = O(lm^2)$. Erdős and Pósa [6] proved that $c_1 m \log m \leq g(3,m) \leq c_2 m \log m$ for appropriate constants c_1 and c_2.
4. The above questions have edge analogues, obtained on replacing 'disjoint' by 'edge-disjoint' and 'vertices' by 'edges'.
5. The *l-prism* is the cartesian product $C_l \times K_2$. In [3], a quadratic upper bound on the tree-width of the graphs containing no l-prism minor is established.

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