Transversals for ordinal intervals

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A *transversal* of a family of sets is a one to one choice function.

If I is a closed interval $[\alpha, \beta]$ of odinals, we write $\iota(I) = \alpha$ and $\tau(I) = \beta$. We also write $\mu(I) = \beta - \alpha$, i.e., $\mu(I)$ is the ordinal satisfying $\tau(I) = \iota(I) + \mu(I)$. The *length* $\lambda(I)$ of I is $\mu(I) + 1$. If $\gamma < \iota(I)$ then $I - \gamma$ is defined as $[\iota(I) - \gamma, \tau(I) - \gamma]$.

The aim of this paper is to prove a conjecture of A. P. Huhn (Szeged, Hungary).

Theorem. If $\mathcal{I}$ is a set of closed intervals of ordinals such that $\lambda(I) \neq \lambda(I)$ whenever $I, J \in \mathcal{I}$ and $I \neq J$, then $\mathcal{I}$ has a transversal.

(This result was proved in [1, p.12] under the assumption that $\lambda(I) < \omega^k$ for each $I \in \mathcal{I}$ for some natural number k.)

Proof. Note first that the result is easy for a set of finite intervals: choose elements from intervals by order of the length of the intervals; since the k-th interval contains at least k elements, it has an element different from those chosen so far.

Let $I \in \mathcal{I}$. Write

$$\mu(I) = \omega^{\alpha_1} \cdot k_1 + \omega^{\alpha_2} \cdot k_2 + \ldots + \omega^{\alpha_{t-1}} \cdot k_{t-1} + k_t,$$

where $\alpha_1 > \alpha_2 > \ldots > \alpha_{t-1} > 0$ and $k_i < \omega$. We define the set

$$\Xi(I) = \{\, \omega^{\alpha_1} \cdot l_1 + \omega^{\alpha_2} \cdot l_2 + \ldots + \omega^{\alpha_{t-1}} \cdot l_{t-1} + l_t :\ l_i \leq k_i \ for\ 0 < i \leq t \,\}$$

and $\theta(I) = \{\, \iota(I) + \xi : \in \Xi(I) \,\}$. Since $\theta(I) \subset I$ for each $I \in \mathcal{I}$ it suffices to prove:

(A) *The set* $\theta(\mathcal{I}) = \{\, \theta(I) :\ I \in \mathcal{I} \,\}$ *has a transversal.*

Since the sets $\theta(I)$ are finite, it suffices, by compactness, to show that $\theta(\mathcal{F})$ has a transversal for each finite subfamily $\mathcal{F}$ of $\mathcal{I}$. For any such $\mathcal{F}$ let $\zeta(\mathcal{F}) = \min\{\, \alpha :\ \tau(I) < \omega^\alpha \ for\ each\ I \in \mathcal{F} \,\}$. Since $\mathcal{F}$ is finite, $\zeta(\mathcal{F})$ is a successor ordinal. We prove our assertion by induction on $\zeta(\mathcal{F})$. Let $\zeta(\mathcal{F}) = \alpha + 1$ and assume that for any finite family $\mathcal{H}$ of closed intervals of different length such that $\zeta(\mathcal{H}) \leq \alpha$, the family $\theta(\mathcal{H})$ has a transversal.

For each $I \in \mathcal{F}$ define:

$$m^\alpha(I) = \max\{\, k :\ \iota(I) \geq \omega^\alpha \cdot k \,\} \quad \text{and} \quad n^\alpha(I) = \min\{\, k :\ \tau(I) < \omega^\alpha \cdot (k+1) \,\}.$$

Also write $\hat{I} = [m^\alpha(I), n^\alpha(I)]$. For every $\beta < \omega^\alpha$ let

$$\mathcal{F}_\beta = \{\, I \in \mathcal{F} :\ \mu(I) = \omega^\alpha \cdot k + \beta \ for\ some\ \ k < \omega \,\}$$

and $\mathcal{K}_\beta = \{\, \hat{I} :\ I \in \mathcal{F}_\beta \,\}$. Notice that if $I \in \mathcal{F}_\beta$ then

$$\mu(I) = \omega^\alpha(n^\alpha(I) - m^\alpha(I)) + \beta.$$

If $I, J \in \mathcal{F}_\beta$ and $I \neq J$ then $n^\alpha(I) - m^\alpha(I) \neq n^\alpha(J) - m^\alpha(J)$, since otherwise clearly $\lambda(I) = \lambda(J)$. Thus $\mathcal{K}_\beta$ is a set of finite intervals of different lengths, and hence has a transversal f_β. For each $I \in \mathcal{F}_\beta$ let $Z_\beta(I) = [\iota(I), \iota(I) + \beta]$ if $f_\beta(\hat{I}) = m^\alpha(I)$ and $Z_\beta(I) = [\omega^\alpha \cdot f_\beta(\hat{I}) + \beta]$ otherwise. Also define $W_\beta(I) = Z_\beta(I) - \omega^\alpha \cdot f_\beta(\hat{I})$. For each $k < \omega$ let

$$\mathcal{A}_k = \{\, W_\beta(I) :\ \beta < \omega^\alpha \ \ and\ \ I \in \mathcal{F}_\beta \ \ and\ \ f_\beta(\hat{I}) = k \,\}.$$

Since each f_β is a transversal, and since $\lambda(W_\beta(I)) = \beta + 1$ whenever $I \in \mathcal{F}_\beta$, it follows that all lengths of intervals in $\mathcal{A}_k$ are different. Also, $\zeta(\mathcal{A}_k) < \alpha$. Hence, by induction hypothesis, $\theta(\mathcal{A}_k)$ has a transversal g_k. We define a transversal h of $\theta(\mathcal{F})$ as follows: let $I \in \mathcal{F}$, and write $\mu(I) = \omega^\alpha \cdot l + \beta$, where $\beta < \omega^\alpha$. If $k = f_\beta(\hat{I})$, define:

$$h(\theta(I)) = \omega^\alpha \cdot k + g_k(\theta(W_\beta(I))).$$

It is easily cheked that $h(\theta(I)) \in \theta(I)$ and that h is a transversal, proving (A) and thus proving the theorem.

Reference

[1] P.Hajnal, One-one mappings between well ordered sets, prepint (Szeged, 1983) (in Hungarian)