A SHORT NOTE ON NUMERATION SYSTEMS WITH NEGATIVE DIGITS ALLOWED

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Abstract. Several numeral systems are known, that are based on Fibonacci numbers. The best known is the Zeckendorf representation. Another one is the lazy Fibonacci representation of natural numbers. Both of them use 0-1 digits. A more recent one is due to Alpert. He allows negative digits in his representation. We introduce two more systems that use negative digits.

1. Introduction

$(F_i)_{i=0}^{\infty}$ is the Fibonacci sequence ([12] A000045) defined as $F_n = F_{n-1} + F_{n-2}$ ($n \geq 2$) with $F_0 = 0$ and $F_1 = 1$. We refer to $F_2 < F_3 < F_4 < \dots$ as the Fibonacci numbers. Using the recursion formula one can easily extend the sequence to negative indices:

$$\dots, 13, -8, 5, -3, 2, -1, 1, 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, \dots.$$

Zeckendorf [15] introduced a numeral system based on Fibonacci numbers. It became very popular, igniting many lines of research. They are considering for example algorithmic problems of the arithmetic of numbers given by the Zeckendorf representation (for example [1], [14], [7]) and probabilistic questions on the distribution of the number of digit 1's among the numbers of given Zeckendorf length (for example [11], [9], [5]).

Other numerical systems were introduced, and they became popular too. In Section 2 we give a short overview of previous notions. For a more detailed account see [13]. The first systems used non-negative digits, but later negative digits were allowed too. We go further in this direction.

The main topic of this paper is to introduce two new numeral systems. Our two main results are the following two theorems.

1991 *Mathematics Subject Classification.* 11B39.

Key words and phrases. Fibonacci numbers, numeral systems.

Partially supported by SZTE, Smart Institute, Hungary grant 20391-3/2018/FEKUSTRAT by the Ministry of Human Capacities.

Theorem 1.1 (Alternating representation)**.** *Every natural number n can be written uniquely as a sum*

$$n = F_\ell - F_{i_1} + F_{i_2} - F_{i_3} + \ldots + (-1)^{t-1} F_{i_{t-1}},$$

with $i_0 = \ell \gg i_1 \gg i_2 \gg \ldots \gg i_{t-3} \gg i_{t-2} \ggg i_{t-1} \geq 2$, where $i \gg j$ denotes $i > j+1$ (i.e. $i \geq j+2$) and $i \ggg j$ denotes $i > j+2$ (i.e. $i \geq j+3$).

t denotes the number of terms. Note that $n = F_\ell$ is an alternating representation, but $n = F_\ell - F_{\ell-2}$ is not (in the case $t = 2$ the requirement on the last two indices: $i_{t-2} \ggg i_{t-1}$ must be fulfilled).

Theorem 1.2 (Even representation)**.** *Every natural number n can be written uniquely as a sum*

$$n = F_{2\ell} + \epsilon_{i_1} F_{2i_1} + \epsilon_{i_2} F_{2i_2} + \epsilon_{i_3} F_{2i_3} + \ldots + \epsilon_{t-1} F_{2i_{t-1}},$$

where $i_0 = \ell > i_1 > i_2 > \ldots > i_{t-1} \geq 1$,$\epsilon_i \in \{-1, 1\}$ and in the sequence $\epsilon_{i_0} = 1, \epsilon_{i_1}, \epsilon_{i_2}, \ldots \epsilon_{i_{t-1}}$ there are no two consecutive -1's.

The two theorems are proven in Section 3 and Section 4.

Throughout the paper the set $\{0, 1, 2, 3, \ldots\}$, i.e. the set of natural numbers is denoted as $\mathbb{N}$. The set of positive integers is denoted by $\mathbb{N}_+$. The intervals are always intervals of $\mathbb{Z}$, so $]2, 6] = (2, 6] = \{3, 4, 5, 6\}$, $[2, 2] = \{2\}$. $A \dot\cup B$ denotes $A \cup B$ and contains the additional information that A and B are disjoint. $A \dot\cup B \dot\cup C \dot\cup$ denotes $A \cup B \cup C \cup \ldots$ and contains the additional information that $A, B, C, \ldots$ are pairwise disjoint.

2. Representations of natural numbers based on Fibonacci numbers

Observation 2.1. (o) *Every natural number can be written as a sum of Fibonacci numbers.*

(i) *Every natural number can be written as a sum of distinct Fibonacci numbers.*

The first claim is obvious since 1 is a Fibonacci number (0 is considered as an empty sum). To see (i) consider an arbitrary positive integer n and take all the possible terms $1 < 2 < 3 < 5 < \ldots < F_\ell - 1 < F_\ell$ ($\leq n < F_{\ell+1}$). We sketch two possible strategies/algorithms to come up with a representation:

(Eager/Greedy) Take F_ℓ as the first term and try to complete the representation, continuing with $n - F_\ell$.

(Lazy) If the sum of the numbers $F_{\ell-1} > \ldots > 5 > 3 > 2 > 1$ is at least n we throw away F_ℓ and try to complete the representation. If the sum is smaller than n, we are forced to take F_ℓ as the first term and proceed with $n - F_\ell$.

When we have a representation of n as in (i) we can code n using place-value notation. The places/positions correspond to the Fibonacci numbers: $\ldots, 8, 5, 3, 2, 1$ and the Fibonacci digits are $0, 1$. For example

$$\begin{aligned}2021 =& F_{17} + F_{14} + F_9 + F_7 = F_{16} + F_{15} + F_{14} + F_9 + F_7\\ =& F_{16} + F_{15} + F_{13} + F_{11} + F_{10} + F_8 + F_7 + F_6 + F_5 + F_4 + F_3 + F_2\end{aligned}$$

can be coded as:

$$2021 = 1001000010100000_F = 111000010100000_F = 110101101111111_F.$$

Two representations play a central role in further research.

Theorem 2.2 ((i): Brown [3], (ii): Zeckendorf [15]).
(i) *Every natural number has a unique representation as a sum Fibonacci numbers such that till the highest indexed term no two consecutive Fibonacci numbers are missing.*
(ii) *Every natural number has a unique representation as a sum of nonconsecutive Fibonacci numbers.*

The existence in (i) can be proven by applying the (Lazy) rule repeatedly. The proof of uniqueness is standard (for example see [3]). The representation is called lazy Fibonacci representation. The existence in (ii) can be proven by applying the (Eager) rule repeatedly. The proof of uniqueness is standard (for example see [15]). The representation is called Zeckendorf representation.

Sometimes these representations are called dense, resp. sparse representations.

To emphasize the two special forms we use Z, ℓF subscripts in the case of Zeckendorf, resp. lazy Fibonacci representation:

$$2021 = 1001000010100000_Z = 110101101111111_{\ell F}.$$

In 1992 Bunder [4] introduced a Zeckendorf type representation using Fibonacci numbers with negative indices. He proved that every integer number has a unique representation as a sum of nonconsecutive Fibonacci numbers with negative indices. It is an easy and well-known fact that $F_{-i} = (-1)^{i+1} F_i$. So Bunder's theorem is equivalent to the following statement.

Theorem 2.3 (Bunder [4]). *Every integer number n can be written uniquely as a sum*

$$n = i_0 F_{i_0} + \epsilon_{i_1} F_{i_1} + \epsilon_{i_2} F_{i_2} + \epsilon_{i_3} F_{i_3} + \ldots + \epsilon_{t-1} F_{i_{t-1}},$$

where $i_0 \gg i_1 \gg i_2 \gg \ldots \gg i_{t-1} \geq 1$ *and* $\epsilon_{i_j} = 1$*, when* i_j *is odd,* $\epsilon_{i_j} = -1$*, when* i_j *is even.*

In 2009 Alpert [2] proved a new way to represent natural numbers (in fact integers), that will be important for us. He also uses Fibonacci numbers as places, but the digits are $-1, 0, 1$ (as in the case of our interpretation of Bunder's theorem).

Theorem 2.4 (Alpert [2])**.** *Every natural number* n *can be written uniquely as a sum*

$$n = F_\ell + \epsilon_{i_1} F_{i_1} + \epsilon_{i_2} F_{i_2} + \epsilon_{i_3} F_{i_3} + \ldots + \epsilon_{t-1} F_{i_{t-1}},$$

where $i_0 = \ell > i_1 > i_2 > \ldots > i_{t-1} \geq 2$ *and* $\epsilon_{i_0} = 1, \epsilon_{i_1}, \epsilon_{i_2}, \ldots \epsilon_{t-1} \in \{-1, 1\}$*. Furthermore if* $\epsilon_{i_j} = \epsilon_{i_{j+1}}$*, then* $i_j - i_{j+1} \geq 4$*; if* $\epsilon_{i_j} = -\epsilon_{i_{j+1}}$*, then* $i_j - i_{j+1} \geq 3$*.*

t, in the Theorem, is the number of terms in the representation. Note that $t = 0$ and $t = 1$ are possible. Hence 0 and the Fibonacci numbers can be represented as stated in the theorem. Alpert claims that any natural number can be represented and the Alpert representation is unique. We will denote the -1 digit as $\bar{1}$.

For example

$$2021 = 2584 - 610 + 55 - 8 = F_{18} - F_{15} + F_{10} - F_6, \quad 2021 = 100\bar{1}0000100\bar{1}0000_A.$$

We mention that Alpert allowed the first digit to be -1, so he included representations for negative numbers too.

3. Alternating representation, proof of Theorem 1.1

Let

$$n = F_\ell - F_{i_1} + F_{i_2} - F_{i_3} + \ldots + (-1)^{t-1} F_{i_{t-1}},$$

obeying the rules of Theorem 1.1, i.e. $\ell \gg i_1 \gg \ldots \gg i_{t-2} \ggg i_{t-1} \geq 2$.

Observation 3.1. $n \leq F_\ell$*.*

Indeed. The first term is F_ℓ, the remainder terms can be paired. Each pair (a sum of two signed Fibonacci numbers) and the possible last term gives a negative contribution to the first term.

Observation 3.2. $n \geq F_\ell - F_{\ell-2} + 1 = F_{\ell-1} + 1$*.*

Indeed. The cases $\ell = 2, 3, 4$ are easy. If we have one term in the representation the estimate is correct. If we have two terms ($n = F_\ell - F_{i_1}$), then $\ell \ggg i_1$ and the bound is obvious. If we have more than three terms we can pair the terms after $F_\ell - F_{i_1} (\geq F_\ell - F_{\ell-2})$.

Each term gives a positive contribution to the value. The observation is proven.

Note that

$$\mathbb{N}_+ = [1,1]\dot\cup[2,2]\dot\cup[3,3]\dot\cup[4,5]\dot\cup[6,8]\dot\cup[9,13]\dot\cup[14,21]\dot\cup\ldots\dot\cup[F_{\ell-1}+1,F_\ell]\dot\cup\ldots$$

The two observations give us that if n can be written as required, then the first term of the representation can be determined.

Existence: We use induction on n. For $n = 0, 1, 2, 3$ the claim is obvious. Assume that $n \in [F_{\ell-1} + 1, F_\ell]$ ($\ell \geq 5$). Consider $F_\ell - n$. We know that $0 \leq F_\ell - n \leq F_\ell - (F_{\ell-1} + 1) = F_{\ell-2} - 1$. We distinguish two cases.

Case 1: $0 \leq F_\ell - n \leq F_{\ell-3}$. Then by the induction hypothesis, we know that the representation of $F_\ell - n$ exists and its first term F_k, where k is at most $\ell - 3$. Rearranging the equality we obtain a representation of n.

Case 2: $F_{\ell-3} + 1 \leq F_\ell - n \leq F_{\ell-2} - 1$. Again by the induction hypothesis we know that the representation of $F_\ell - n$ exists and its first term $F_{\ell-2}$. Furthermore, we also know that the number of terms is at least two. The end of the proof of existence is as above.

Uniqueness: We use induction on n. For small values the claim is obvious. 0 can be represented only as of the empty sum. For the induction step assume that $n \in [F_{\ell-1}+1, F_\ell]$. We prove by contradiction. Assume that we have two different representations. Since the first term of both representations are F_ℓ, we have

$$n = F_\ell - F_{i_1} + F_{i_2} - F_{i_3} + \ldots + (-1)^{t-1}F_{i_{t-1}} = F_\ell - F_{j_1} + F_{j_2} - F_{j_3} + \ldots + (-1)^{s-1}F_{j_{s-1}},$$

$$F_\ell - n = F_{i_1} - F_{i_2} + F_{i_3} - \ldots + (-1)^{t-2}F_{i_{t-1}} = F_{j_1} - F_{j_2} + F_{j_3} - \ldots + (-1)^{s-2}F_{j_{s-1}}.$$

We obtained two different representations of $F_\ell - n$ that is a natural number smaller than n: $F_\ell - n \leq F_\ell - (F_{\ell-1} + 1) \leq F_{\ell-2} - 1 < n$. That is a contradiction with the induction hypothesis.

This finishes the proof of Theorem 1.1.

The Theorem 1.1 (existence and uniqueness) leads to a new numeration system, where the positions correspond to Fibonacci numbers and the digits are $\bar{1} \equiv -1$, 0 and 1. We use the subscript *Alt* to emphasize that we use this representation of numbers. We finish the section with examples

$$1 = 1_{Alt}, 2 = 10_{Alt}, 3 = 100_{Alt}, 4 = 100\bar{1}_{Alt}, 5 = 1000_{Alt}, 6 = 100\bar{1}0_{Alt},$$

$$7 = 1000\bar{1}_{Alt}, 8 = 10000_{Alt}, 9 = 10\bar{1}001_{Alt}, 10 = 100\bar{1}00_{Alt}, 11 = 1000\bar{1}0_{Alt}.$$

4. Even representation, proof of Theorem 1.2

Let

$$n = F_{2\ell} + \epsilon_{i_1} F_{2i_1} + \epsilon_{i_2} F_{2i_2} + \epsilon_{i_3} F_{2i_3} + \ldots + \epsilon_{t-1} F_{2i_{t-1}}, \tag{1}$$

obeying the rules of Theorem 1.2, i.e. $\epsilon_{i_j} \in \{-1, 1\}$ and there are no two consecutive -1's in the sequence of ϵ's.

Observation 4.1. $n \leq F_{2\ell} + F_{2\ell-2} + F_{2\ell-4} + \ldots + F_2 = F_{2\ell+1} - 1$.

Indeed. We can upper bound n with the sum of all possible positive terms. The last equality is well-known (see [10], Corollary 5.1, page 83).

Observation 4.2. $n \geq F_{2\ell} - F_{2\ell-2} = F_{2\ell-1}$.

Indeed. Pair each negative term with the previous (necessarily positive) term. We imagine these pairs as one bracket, one term. With this view, the first term in the representation of n is at least $F_{2\ell} - F_{2\ell-2}$, the further terms are positive. The observation is proven.

Note that

$$\mathbb{N}_+ = [1,1] \dot\cup [2,4] \dot\cup [5,12] \dot\cup [13,33] \dot\cup [34,88] \dot\cup [89,232] \dot\cup \ldots \dot\cup [F_{2\ell-1}, F_{2\ell+1}-1] \dot\cup \ldots$$

The two observations give us that if n can be written as required, then ℓ (see (1)) can be determined.

Existence: We use induction on n. For $n = 0, 1, 2, 3$ the claim is obvious. Assume that $n \in [F_{2\ell-1}, F_{2\ell+1} - 1]$ ($\ell \geq 2$). We distinguish two cases for proving the induction step.

Case 1: $n \in [F_{2\ell}, F_{2\ell+1} - 1]$. Write n as $F_{2\ell} + n'$. Note that $0 \leq n' \leq F_{2\ell-1} - 1 (< n)$. Hence by induction n' has a representation that starts with F_{2i}, where $i < \ell$. The representation of n' easily give us the representation of n.

Case 2: $n \in [F_{2\ell-1}, F_{2\ell} - 1]$. Write n as $F_{2\ell} - F_{2\ell-2} + n'$. Note that $0 \leq n' \leq F_{2\ell-2} - 1 (< n)$. Hence by induction n' has a representation that starts with F_{2i}, where $i < \ell$:

$$n' = F_{2i} + \epsilon_{i_1} F_{2i_1} + \epsilon_{i_2} F_{2i_2} + \epsilon_{i_3} F_{2i_3} + \ldots + \epsilon_{t-1} F_{2i_{t-1}}.$$

Hence

$$n = F_{2\ell} - F_{2\ell-2} + F_{2i} + \epsilon_{i_1} F_{2i_1} + \epsilon_{i_2} F_{2i_2} + \epsilon_{i_3} F_{2i_3} + \ldots + \epsilon_{t-1} F_{2i_{t-1}}.$$

$2i = 2\ell - 2$ or $2i < 2\ell - 2$. We are done in both cases. The two cases cover all possibilities, hence the proof of existence is complete.

Uniqueness: We use induction on n. For small values the claim is obvious. For the induction step assume that $n \in [F_{2\ell-1}, F_{2\ell+1} - 1]$. We prove by contradiction. Again we distinguish two cases:

1. case: $n \in [F_{2\ell}, F_{2\ell+1}-1]$. Write n as $F_{2\ell}+n'$. Both representations of n start with $F_{2\ell}$ and is followed with a representation of n'. This leads to two different representations of $n'(< n)$. This contradicts the hypothesis of the induction step.

2. case $n \in [F_{2\ell-1}, F_{2\ell}-1]$.

$$\begin{aligned} n =& F_{2\ell} + \epsilon_{i_1} F_{2i_1} + \epsilon_{i_2} F_{2i_2} + \epsilon_{i_3} F_{2i_3} + \ldots + \epsilon_{t-1} F_{2i_{t-1}} \\ =& F_{2\ell} + \epsilon'_{i'_1} F_{2i'_1} + \epsilon'_{i'_2} F_{2i'_2} + \epsilon'_{i'_3} F_{2i'_3} + \ldots + \epsilon'_{t-1} F_{2i'_{t-1}}. \end{aligned}$$

We can conclude that

$$\begin{aligned} n - F_{2\ell} + F_{2\ell-2} =& F_{2\ell-2} + \epsilon_{i_1} F_{2i_1} + \epsilon_{i_2} F_{2i_2} + \epsilon_{i_3} F_{2i_3} + \ldots + \epsilon_{t-1} F_{2i_{t-1}} \\ =& F_{2\ell-2} + \epsilon'_{i'_1} F_{2i'_1} + \epsilon'_{i'_2} F_{2i'_2} + \epsilon'_{i'_3} F_{2i'_3} + \ldots + \epsilon'_{t-1} F_{2i'_{t-1}}. \end{aligned}$$

We see two different representations of $n - F_{2\ell} + F_{2\ell-2}(< n)$, a contradiction.

The two cases cover all possibilities, hence the proof of uniqueness is complete.

This finishes the proof of Theorem 1.2.

The Theorem 1.2 (existence and uniqueness) leads to a new numeration system, where the positions correspond to Fibonacci numbers with even indices $(\ldots, 121, 55, 21, 8, 3, 1)$ and the digits are $\bar{1} \equiv -1$, 0 and 1. We use the subscript *Even* when we use this representation of numbers. We finish the section with examples

$$1 = 1_{Even}, 2 = 1\bar{1}_{Even}, 3 = 10_{Even}, 4 = 11_{Even}, 5 = 1\bar{1}0_{Even}, 6 = 1\bar{1}1\bar{1}0_{Even},$$

$$7 = 10\bar{1}_{Even}, 8 = 100_{Even}, 9 = 101_{Even}, 10 = 11\bar{1}_{Even}, 11 = 110_{Even},$$

$$12 = 111_{Even}, 13 = 1\bar{1}00_{Even}, 14 = 1\bar{1}01_{Even}, 15 = 1\bar{1}1\bar{1}_{Even}, 16 = 1\bar{1}10_{Even}.$$

5. Conclusion

We presented two numeral systems based on Fibonacci numbers. There are not too many of these. They are natural systems but one must recognize their existence. As they are recognized they start to live a life of their own. The simplicity and beauty usually ignite further research and lead to applications, connections to other fields.

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