TWO GEOMETRICAL APPLICATIONS OF THE SEMI-RANDOM METHOD

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Abstract

The semi-random method was introduced in the early eighties. In its first form of the method lower bounds were given for the size of the largest independent set in hypergraphs with certain uncrowdedness properties. The first geometrical application was a major achievement in the history of Heilbronn's triangle problem. It proved that the original conjecture of Heilbronn was false. The semi-random method was extended and applied to other problems. In this paper we give two further geometrical applications of it.

First, we give a slight improvement on Payne and Wood's upper bounds on a Ramsey-type parameter, introduced by Gowers. We prove that any planar point set of size $\Omega\left(\frac{n^2\log n}{\log\log n}\right)$ contains n points on a line or n independent points.

Second, we give a slight improvement on Schmidt's bound on Heilbronn's quadrangle problem. We prove that there exists a point set of size n in the unit square that doesn't contain four points with convex hull of area $\mathcal{O}(n^{-3/2}(\log n)^{1/2})$.

1. Introduction

The semi-random method was introduced for graphs in [1]. Later it was extended to 3-uniform hypergraphs in [8]. The method was further extended in [2] and [4].

A hypergraph $\mathcal{H}$ on the vertex set V is a subset of $\mathcal{P}(V)$, the power set of V. I.e. $\mathcal{H}$ is a collection of certain subsets of V, called edges. If the edges have a common size, say k, then we say that $\mathcal{H}$ is k-uniform. In a hypergraph $\mathcal{H}$ a vertex set $I \subset V$ is called an independent set iff it doesn't contain any edge as a subset. The

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maximum size of the independent sets of $\mathcal{H}$ is denoted by $\alpha(\mathcal{H})$. There are several results concerning independent sets in 3-uniform uncrowded hypergraphs. From hypergraph theory we recall that the degree of a vertex x ($deg(x)$) is the number of edges, containing x. Also a k-cycle ($k \geq 2$) in $\mathcal{H}$ is a sequence of k different vertices: $x_1, \ldots, x_{k-1}, x_k = x_0$ and a sequence of k different edges: $E_1, \ldots, E_k$ such that $x_{i-1}, x_i \in E_i$ for $i = 1, 2 \ldots, k$. The cycle, above is called a simple cycle iff $E_i \cap \left(\cup_{j:j \neq i} E_j\right) = \{x_{i-1}, x_i\}$ for $i = 1, 2 \ldots, k$. We quote the earliest result on hypergraphs using the semi-random method.

Theorem 1 ([8], Lemma 1)**.** *Let $\mathcal{H}$ be a 3-uniform hypergraph on v vertices. Let $\overline{d}$ denote the average degree of $\mathcal{H}$. Assume that $\overline{d} \leq t^2$ and $1 \ll t \ll v^{1/10}$.*

If $\mathcal{H}$ doesn't contain simple cycles of length at most 4, then

$$\alpha(\mathcal{H}) = \Omega\left(\frac{v}{t}\sqrt{\log t}\right).$$

In our applications we might have many simple cycles of length 3 and 4. We need the following strengthening of the basic bound:

Theorem 2 ([4], Theorem 2)**.** *Let $\mathcal{H}$ be a k-uniform hypergraph on v vertices. Let Δ be the maximum degree of $\mathcal{H}$. Assume that $\Delta \leq t^{k-1}$ and $1 \ll t$. If $\mathcal{H}$ doesn't contain a 2-cycle (two edges with at least two common vertices), then*

$$\alpha(\mathcal{H}) = \Omega\left(\frac{v}{t}(\log t)^{\frac{1}{k-1}}\right).$$

We give two new geometrical applications of the above bound.

In the first application we consider a question asked by Gowers [5]. Given a planar point set $\mathcal{P}$, what is the minimal size of $\mathcal{P}$ that guarantees that one can find n points on a line or n independent points (no three on a line) in it? He noted that the grid shows that $\Omega(n^2)$ many points are necessary, and in the case of $2n^3$ many points without n points on a line a simple greedy algorithm finds n independent points. Payne and Wood [9] improved the upper bound to $\mathcal{O}(n^2 \log n)$. They also considered an arbitrary point set with much fewer points than n^3 and without n points on a line. But instead of the greedy algorithm they used Spencer's lemma, which is based on a simple probabilistic sparsification. They also used the Szemerédi–Trotter theorem in order to bound the number of edges of their hypergraph.

We improve the previous upper bound methods. We also start with a random sparsification. After some additional preparation (we get rid of 2-cycles) we are able to use a semi-random method (see [4]) to find a large independent set.

Theorem 3. *Let $\mathcal{P}$ be an arbitrary planar point set of size $\Omega\left(\frac{n^2 \log n}{\log \log n}\right)$. Then we can find n points in $\mathcal{P}$, that are incident to a line or independent.*

Our second application is closely related to Heilbronn's triangle problem [10], [15], [11], [12], [13], [14], [7]. Take a "nice" unit area domain D (usually a square, disc or a regular triangle). Place n points into D and find the smallest area among the triangles determined by the chosen points. Let $H_\triangle(n)$ denote the maximum of this parameter over all possible choices of n points.

Instead of triangles we can take k-tuples of our point set and consider the area of the convex hull of the k chosen points. We denote the corresponding parameter by $H_k(n)$ (so $H_3(n) = H_\triangle(n)$). The best lower bound on $H_\triangle(n)$ [8], and some trivial observations are summarized in the next line:

$$\Omega\left(\frac{\sqrt{\log n}}{n^2}\right) = H_\triangle(n) \le H_4(n) \le H_5(n) \le \ldots = \mathcal{O}\left(\frac{1}{n}\right).$$

We mention two major open problems: Is it true that $H_\triangle(n) = O(1/n^{2-o(1)})$ and $H_4(n) = o(1/n)$?

Our interest is in the lower bound on $H_4(n)$. Schmidt [15] proved that $H_4(n) = \Omega(n^{-3/2})$. The proof is a construction of a point set by a simple greedy algorithm. In [3] the authors provide a new proof, and extensions of this result. They also proposed an open question, which they have not yet been able to resolve: that is whether Schmidt's bound can be improved by a logarithmic factor. With the help of the semi-random method we are able to improve Schmidt's bound and settle the problem of [3].

Theorem 4. *There exists a point set of size n in the unit square that doesn't contain four points with convex hull of area* $\mathcal{O}(n^{-3/2}(\log n)^{1/2})$.

In some cases we closely follow the preceding papers. Since it is hard to refer technical details we repeat the necessary arguments. This way our paper is self-contained. Throughout the paper we will use log to denote the logarithm of base 2, we omit all floor and ceiling signs, whenever these are not essential, and assume that n is large enough.

2. The proof of Theorem 3

Let $\mathcal{P}$ be a planar point set of size $N = \frac{n^2 \log n}{\log\log n}$, not containing n points on a line. We must show that it contains a large independent set. The collinear triples of $\mathcal{P}$ form a 3-uniform hypergraph $\mathcal{H}_3$. Independent subsets of the point set are the independent sets of the hypergraph. The collinear quadruples of $\mathcal{P}$ form a 4-uniform $\mathcal{H}_4$ hypergraph. Edges correspond to $K_4^{(3)}$'s (four vertices with all four triples as edges) in $\mathcal{H}_3$.

First we consider the size of $\mathcal{H}_3$, and $\mathcal{H}_4$. A line with i incident points determines $\binom{i}{3}$ many edges of $\mathcal{H}_3$ and $\binom{i}{4}$ many edges of $\mathcal{H}_4$. Let t_i denote the number of lines that contain exactly i points of $\mathcal{P}$. Our assumption gives that $0 = t_n = t_{n+1} = \ldots$. Similarly let $t_{\ge i}$ denote the number of lines that contain at least i points of $\mathcal{P}$

$(t_{\geq i} = t_i + t_{i+1} + \ldots + t_{n-1})$. Then

$$|\mathcal{H}_3| = \sum_{i=2}^{n-1} \binom{i}{3} t_i \leq \sum_{i=2}^{n-1} \left(i^2 \sum_{j=i}^{n-1} t_j \right) = \sum_{i=2}^{n-1} i^2 t_{\geq i}.$$

The Szemerédi–Trotter theorem [16] says that $t_{\geq i} = \mathcal{O}(|\mathcal{P}|^2/i^3 + |\mathcal{P}|/i) = \mathcal{O}(N^2/i^3)$ (in our case $N^2/i^3 \gg N/i$). For a suitable constant see [6] (Theorem 18.6 and Theorem 18.7): $t_{\geq i} \leq 1000N^2/i^3$. Thus,

$$|\mathcal{H}_3| \leq \sum_{i=2}^{n-1} i^2 t_{\geq i} \leq \sum_{i=2}^{n-1} i^2 1000 \frac{N^2}{i^3} = 1000N^2 \sum_{i=2}^{n-1} \frac{1}{i} \leq 2000N^2 \log n = 2000 \frac{n^4 (\log n)^3}{(\log \log n)^2}.$$

Similarly

$$|\mathcal{H}_4| \leq \sum_{i=2}^{n-1} i^3 t_{\geq i} \leq \sum_{i=2}^{n-1} i^3 1000 \frac{N^2}{i^3} = 1000N^2 \sum_{i=2}^{n-1} 1 \leq 1000N^2 n = 1000 \frac{n^5 (\log n)^2}{(\log \log n)^2}.$$

Consider a random subset of $\mathcal{P}$, that we obtain keeping each point with probability p, and throwing away with probability $1 - p$ (and doing this independently).

Let

$$p = \frac{1}{100} \left(\frac{1}{n^{1/3} N^{1/3}} \right) = \frac{(\log \log n)^{1/3}}{100 n (\log n)^{1/3}}.$$

After the random sparsification let $\boldsymbol{\mathcal{P}}$, $\boldsymbol{\mathcal{H}}_3$, and $\boldsymbol{\mathcal{H}}_4$ be the set of the surviving points, collinear triples, and collinear quadruples (these are random sets, throughout the paper we use bold face to denote random variables). It is obvious that

$$\mathbb{E}\,(|\boldsymbol{\mathcal{P}}|) = pN = \frac{n (\log n)^{2/3}}{100 (\log \log n)^{2/3}},$$

$$\mathbb{E}\,(|\boldsymbol{\mathcal{H}}_3|) = p^3 |\mathcal{H}_3| \leq \frac{2n (\log n)^2}{1000 \log \log n},$$

$$\mathbb{E}\,(|\boldsymbol{\mathcal{H}}_4|) = p^4 |\mathcal{H}_4| \leq \frac{n (\log n)^{2/3}}{100000 (\log \log n)^{2/3}}.$$

We have chosen the probability so that the number of the surviving edges of $\mathcal{H}_4$ will be negligible compared to the number of surviving vertices.

Using elementary probability theory the following events will hold at the same time with high probability:

$$\frac{n (\log n)^{2/3}}{1000 (\log \log n)^{2/3}} < |\boldsymbol{\mathcal{P}}| < \frac{n (\log n)^{2/3}}{10 (\log \log n)^{2/3}},$$

$$|\boldsymbol{\mathcal{H}}_4| \leq \frac{n (\log n)^{2/3}}{10000 (\log \log n)^{2/3}},$$

$$|\boldsymbol{\mathcal{H}}_3| < \frac{n(\log n)^2}{100 \log\log n}.$$

Hence $\overline{d}(\boldsymbol{\mathcal{H}}_3) < \frac{30(\log n)^{4/3}}{(\log\log n)^{1/3}}$, where $\overline{d}$ denotes the average degree.

Now choose one of the good outcomes of the above probabilistic process such that all the above events hold. Let $\mathcal{H}_3^{(0)}$, and $\mathcal{H}_4^{(0)}$ the corresponding 3- and 4-uniform hypergraphs.

Consider $\mathcal{H}_3^{(0)}$, and throw away all points of surviving quadruples of $\mathcal{H}_4^{(0)}$, and throw away each point that has degree higher than $\frac{100(\log n)^{4/3}}{(\log\log n)^{1/3}}$. Let $\mathcal{L}$ denote the "leftover" points with the "leftover" triples.

We are still left with at least one third of the points. Hence the leftover hypergraph has at least $\frac{n(\log n)^{2/3}}{3000(\log\log n)^{2/3}}$ many vertices. By throwing away the high degree vertices the maximal degree of $\mathcal{L}$ at most is $\frac{100(\log n)^{4/3}}{(\log\log n)^{1/3}}$. Furthermore $\mathcal{L}$ is very "uncrowded": we cannot have 2-cycles in $\mathcal{L}$. Indeed, two edges along the same pair of vertices would give a quadruple that is an edges in $\mathcal{H}_4$. Our process eliminated all of them, hence there are no 2-cycles in $\mathcal{L}$. The conditions in Theorem 2 are satisfied with $v = \Theta(\frac{n(\log n)^{2/3}}{(\log\log n)^{2/3}})$, $t = \frac{10(\log n)^{2/3}}{(\log\log n)^{1/6}}$, and $\log t = \Theta(\log\log n)$.

The rest of the proof is the application of the Theorem 2 and simple arithmetic:

$$\alpha(\mathcal{H}_3) \geq \alpha(\mathcal{L}) \geq c\frac{v}{t}\sqrt{\log t} \geq c'n,$$

where c is the constant in Theorem 2 and c' can be determined from c based on our previuos calculation.

Theorem 3 can be easily deduced from this: Let $\nu = \min\{n, c'n\}$. Our point set contains $\Theta(\frac{\nu^2 \log \nu}{\log\log \nu})$ points and our argument guarantees that it contains ν points on a line or ν independent points. The proof is complete.

3. The proof of Theorem 4

Let $S := \{(x,y) \in \mathbb{R}^2 : |x|, |y| \leq 1/2\}$ be a unit square on the plane. Choose N (a parameter that will be chosen later) random points (independently with uniform distribution) from $(1/2)S = \{(x/2, y/2); (x,y) \in S\}$. Let $\boldsymbol{\mathcal{P}}$ be the point set $\{P_1, P_2, \ldots, P_N\}$ we obtain this way. $\boldsymbol{\mathcal{P}}$ is a random point set. The reason we place our points into $(1/2)S$ is technical. This way we know that any connecting line of two points from $\mathcal{P}$ has an intersection with S of length $\Theta(1)$, furthermore any distance determined by points of $\boldsymbol{\mathcal{P}}$ is smaller than 0.9.

Consider the following 4-uniform hypergraph $\boldsymbol{\mathcal{Q}}$ on the vertex set $\boldsymbol{\mathcal{P}}$: A point set of size 4, $\{P, Q, R, S\}$ forms an edge iff $Area(PQRS) < \tau$, where $Area(PQRS)$ is the area of the convex hull of $\{P, Q, R, S\}$, and τ is a threshold, to be determined later. (Similarly $Area(PQR)$ is the area of the PQR triangle.) $\boldsymbol{\mathcal{Q}}$ is a random 4-uniform hypergraph.

The major part of the proof is bounding the expected values of combinatorial parameters of $\mathcal{Q}$.

Let $A, B \in \mathcal{P}$ be two different points and

$$\mathbf{deg}(A,B) = |\{\{C,D\} : \{A,B,C,D\} \in \mathcal{Q}\}| \leq |\{(C,D) : \{A,B,C,D\} \in \mathcal{Q}\}| \qquad (1)$$

i.e. denotes the number of edges of $\mathcal{Q}$, that contains both A and B. The upper bound counts ordered pairs (hence it is an overcounting by a factor of 2). Our goal is to give an upper bound for this parameter. We will count how many ordered pairs of points C, D are considered when $deg(A,B)$ is determined.

Let $strip(AB,w)$ denote the set of points from S, that are in the strip of width w with midline AB (see Figure 1). I.e. $strip(AB,w)$ contains those points of S that have distance at most $w/2$ from the line AB.

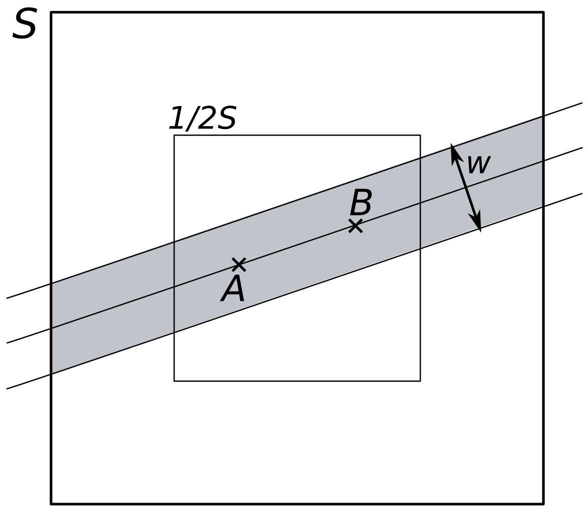

Figure 1: The shaded region is $strip(AB,w)$. Its area is $\Theta(w)$.

Fix A and B, and let $d = dist(A,B)(< 1)$. $deg(A,B)$ counts certain C, D pairs of points, see (1). We distinguish cases according to the position of C, an arbitrary point from $\mathcal{P} - \{A,B\}$ and we bound the possible positions of the D's that contributes to $deg(A,B)$ with the current C.

Case 1: $C \notin strip(AB, 4\tau/d)$.

In this case the area of $ABC\triangle$ is at least τ, hence this C doesn't contributes to $deg(A,B)$.

Case 2: $C \in strip(AB, 4\tau/\sqrt{d})$. Note that $strip(AB, 4\tau/\sqrt{d})$ has area $\Theta(\tau/\sqrt{d})$. We distinguish two subcases:

Case 2a: $D \notin strip(AB, 4\tau/d)$. Similarly to Case 1 no D contributes to $deg(A,B)$.

Case 2b: $D \in strip(AB, 4\tau/d)$. Note that this strip has area $\Theta(\tau/d)$.

Case 3: $C \in strip(AB, 4\tau/d) - strip(AB, 4\tau/\sqrt{d})$ (note that $d < \sqrt{d} < 1$). $strip(AB, 4\tau/d) - strip(AB, 4\tau/\sqrt{d})$ has area $\Theta(\tau/d)$.

The contributing D's must come from $strip(AB, 4\tau/d) \cap strip(AC, 4\tau/dist(A, C))$.

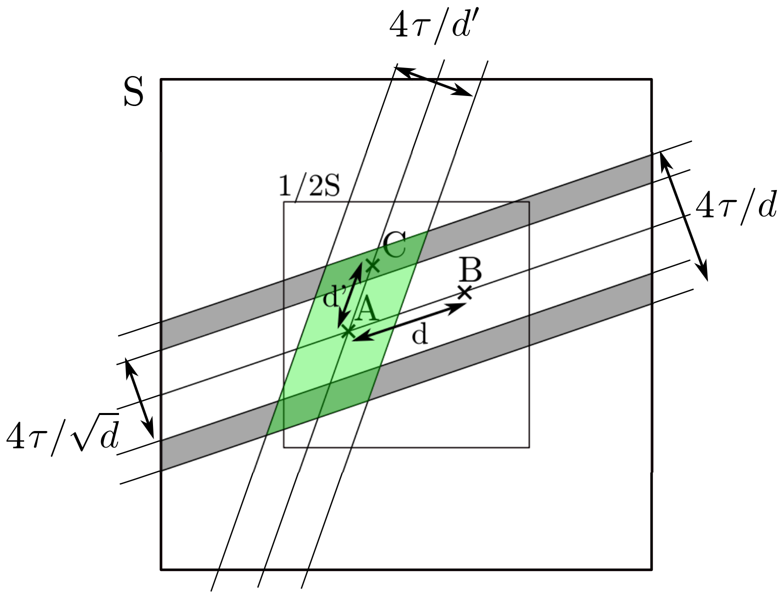

Figure 2: The shaded region is the space for those C's where Case 3 applies. The green region contains those D's, that can form an edge of $\mathcal{Q}$ with A, B and C.

Elementary geometry gives that the above region has area $\Theta(\tau^2/Area(ABC\triangle))$ (see the green parallelogram on Figure 2), bounding the possible positions of contributing D's. Since we are in Case 3 we have $Area(ABC\triangle) = \Omega(d \cdot \tau/\sqrt{d}) = \Omega(\tau\sqrt{d})$, hence the parallelogram has area $\mathcal{O}(\tau/\sqrt{d})$.

The expected value of $\mathbf{deg}(A, B)$ can be bounded easily. The contributing C,D's are covered by Case 2b and Case 3. In both cases the contributing C's and D's are coming from a restricted domain with known area. Since the choice of C and D are independent the number of contributing (C, D)'s in expectation is a product of the two expectations, that we can bound. In each of the three cases this product is $\mathcal{O}(\tau^2 d^{-3/2} N^2)$. Hence

$$\mathbb{E}\left(\mathbf{deg}(A, B)\right) = \mathcal{O}(\tau^2 (dist(A, B))^{-\frac{3}{2}} N^2).$$

With a similar argument we obtain bound on the number of 2-cycles through $A, B \in \mathcal{P}$. In a 4-uniform hypergraph there are two types of 2-cycles: (I): $\{A, B, C, D\}$, $\{A, B, C', D'\}$ and (II): $\{A, B, C, D\}$, $\{A, B, C, D'\}$ (now different symbols denote different points). Let $\mathcal{C}_I(A, B)$, resp. $\mathcal{C}_{II}(A, B)$ denote the number of 2-cyles of type (I), resp. type (II) for given points, A and B. Bounding the expected value of $\mathcal{C}_I(A,B)$ is easy, based on the previous calculation

$$\mathbb{E}\ (\mathcal{C}_I(A,B)) = \mathcal{O}(\tau^4(dist(A,B))^{-3}N^4).$$

Bounding the expected value of $\mathcal{C}_{II}(A,B)$ is a little bit more technical. We distinguish the contribution of C's that satisfy Case 2 and those that satisfy Case 3:

$$\mathbb{E}\ (\mathcal{C}_{II}(A,B)) = \mathcal{O}((N\tau/\sqrt{d})(N\tau/d)^2 + (N\tau/d)(N\tau/\sqrt{d})^2) = \mathcal{O}(\tau^3 N^3 d^{-2.5}).$$

Now we sparsify our point set a little bit in order to have a lower bound on the minimal distance determined by our points.

Let $\delta = \frac{1}{100}N^{-1/2}$. We count the number of pairs in $\mathcal{P}$ that are closer than δ. Let $C(\mathcal{P})$ be the set of these pairs (this is a random set). Let $C_A(\mathcal{P}) = \mathcal{P} \cap Disc(A;\delta)$, where $Disc(A;\delta)$ denote the disc of radius δ centered at A. It is clear that $|C(\mathcal{P})| = 1/2\sum_{A\in\mathcal{P}}|C_A(\mathcal{P})|$ and $\mathbb{E}\ (|C_A(\mathcal{P})|) \le (N-1)Area(Disc(A;\delta)) = 1/2\pi\cdot\delta^2 N < 1/1000$. Hence

$$\mathbb{E}\ (|C(\mathcal{P})|) \le N/1000.$$

With high probability $|C(\mathcal{P})| \le N/4$. After deleting these pairs we obtain $\mathcal{P}_0$, our new point set. $\mathcal{P}_0$ has size at least $N/2$ with high probability, and the distance of any two points of it is at least δ.

Let $\mathcal{Q}_0$ be the restriction of $\mathcal{Q}$ to $\mathcal{P}_0$. From now on we will work with $\mathcal{Q}_0$.

Lemma 5. *Let $\mathcal{M}$ be a set of M points from S so that the minimal distance among them is at least δ. Let $P \in S$. Let $Ann_i(P,\delta)$ be the annulus*

$$Ann_i(P;\delta) = \{X \in \mathbb{R}^2 : (i-1)\delta < dist(P,X) \le i\delta\}.$$

$Ann_1(P;\delta)$, $Ann_2(P;\delta)$, …, $Ann_{\mathcal{O}(\delta^{-1})}(P;\delta)$ are disjoint and cover S (hence they cover our point set). Furthermore at most $\mathcal{O}(i)$ of our M points can be covered by $Ann_i(P,\delta)$.

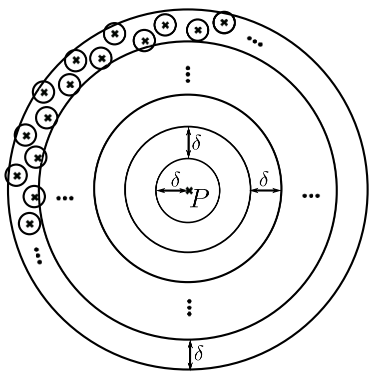

Figure 3: The annuli around P, and the elementary volume argument in the proof.

Proof. The covering property is obvious. The bound on the number of points in the annulus is a simple volume argument: Draw $Disc(A, \delta/3)$ for all points $A \in \mathcal{M} \cap Ann_i(P; \delta)$. These discs are disjoint subsets of $\{X \in \mathbb{R}^2 : (i - 4/3)\delta < dist(P, X) \le (i + 1/3)\delta\}$. The claim follows immediately. □

Using Lemma 5 and the earlier estimation for $deg(A, B)$'s it is easy to bound the expected number of edges, in $\boldsymbol{\mathcal{Q}}_0$:

$$\begin{aligned}
\mathbb{E}\ (deg_{\boldsymbol{\mathcal{Q}}_0}(A)) &\le \sum_{B\in\boldsymbol{\mathcal{P}}_0} \mathbb{E}\ (\mathbf{deg}(A,B)) = \sum_{i=1}^{\mathcal{O}(N^{1/2})} \sum_{B\in Ann_i(A,\delta)\cap\boldsymbol{\mathcal{P}}_0} \mathbb{E}\ (\mathbf{deg}(A,B)) \\
&\le \sum_{i=1}^{\mathcal{O}(N^{1/2})} \sum_{B\in Ann_i(A,\delta)\cap\boldsymbol{\mathcal{P}}_0} \mathcal{O}(\tau^2 N^2 (i/\sqrt{N})^{-3/2}) \\
&\le \sum_{i=1}^{\mathcal{O}(N^{1/2})} i\cdot \mathcal{O}(\tau^2 N^{2.75} i^{-3/2}) = \mathcal{O}(\tau^2 N^{2.75}) \sum_{i=1}^{\mathcal{O}(N^{1/2})} i \cdot i^{-3/2} \\
&= \mathcal{O}(\tau^2 N^{2.75})\mathcal{O}(N^{0.25}) = \mathcal{O}(\tau^2 N^3).
\end{aligned}$$

Hence

$$\mathbb{E}\ (|\boldsymbol{\mathcal{Q}}_0|) = \mathcal{O}(\tau^2 N^4).$$

The bound of the expected number of 2-cycles ($\boldsymbol{\mathcal{C}} = \boldsymbol{\mathcal{C}}_I + \boldsymbol{\mathcal{C}}_{II}$) is similar:

$$\begin{aligned}
\mathbb{E}\ (\boldsymbol{\mathcal{C}}_I) &\le \sum_{A,B\in\boldsymbol{\mathcal{P}}_0} \mathbb{E}\ (\boldsymbol{\mathcal{C}}_I(A,B)) = \sum_{A\in\boldsymbol{\mathcal{P}}_0} \sum_{i=1}^{\mathcal{O}(N^{1/2})} \sum_{B\in Ann_i(A,\delta)\cap\boldsymbol{\mathcal{P}}_0} \mathbb{E}\ (\boldsymbol{\mathcal{C}}_I(A,B)) \\
&= \sum_{A\in\boldsymbol{\mathcal{P}}_0} \sum_{i=1}^{\mathcal{O}(N^{1/2})} \sum_{B\in Ann_i(A,\delta)\cap\boldsymbol{\mathcal{P}}_0} \mathcal{O}(\tau^4 \cdot i^{-3} N^{1.5} \cdot N^4) \\
&= \sum_{A\in\boldsymbol{\mathcal{P}}_0} \sum_{i=1}^{\mathcal{O}(N^{1/2})} \mathcal{O}(\tau^4 \cdot i^{-2} \cdot N^{5.5}) = \sum_{A\in\boldsymbol{\mathcal{P}}_0} \mathcal{O}(\tau^4 N^{5.5}) \sum_{i=1}^{\mathcal{O}(N^{1/2})} i^{-2} = \\
&= \mathcal{O}(\tau^4 N^{6.5}).
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}\ (\boldsymbol{\mathcal{C}}_{II}) &\le \sum_{A,B\in\boldsymbol{\mathcal{P}}_0} \mathbb{E}\ (\boldsymbol{\mathcal{C}}_{II}(A,B)) = \sum_{A\in\boldsymbol{\mathcal{P}}_0} \sum_{i=1}^{\mathcal{O}(N^{1/2})} \sum_{B\in Ann_i(A,\delta)\cap\boldsymbol{\mathcal{P}}_0} \mathbb{E}\ (\boldsymbol{\mathcal{C}}_{II}(A,B)) \\
&= \sum_{A\in\boldsymbol{\mathcal{P}}_0} \sum_{i=1}^{\mathcal{O}(N^{1/2})} \sum_{B\in Ann_i(A,\delta)\cap\boldsymbol{\mathcal{P}}_0} \mathcal{O}(\tau^3 \cdot i^{-2.5} N^{1.25} \cdot N^3) \\
&= \sum_{A\in\boldsymbol{\mathcal{P}}_0} \sum_{i=1}^{\mathcal{O}(N^{1/2})} \mathcal{O}(\tau^3 \cdot i^{-1.5} \cdot N^{4.25}) = \sum_{A\in\boldsymbol{\mathcal{P}}_0} \mathcal{O}(\tau^3 N^{4.25}) \sum_{i=1}^{\mathcal{O}(N^{1/2})} i^{-1.5} = \\
&= \mathcal{O}(\tau^3 N^{5.25}).
\end{aligned}$$

$$\mathbb{E}\,(\boldsymbol{\mathcal{C}}) = \mathbb{E}\,(\boldsymbol{\mathcal{C}}_I + \boldsymbol{\mathcal{C}}_{II}) = \mathcal{O}(\tau^4 N^{6.5}) + \mathcal{O}(\tau^3 N^{5.25}).$$

Now choose one of the good outcomes of the above probabilistic process so that $\boldsymbol{\mathcal{P}}_0$ and $\boldsymbol{\mathcal{Q}}_0$ satisfies the following properties:: the number of points is $N/2$, the number of quadruples is $\mathcal{O}(\tau^2 N^4)$, the number of 2-cycles is $\mathcal{O}(\tau^4 N^{6.5})+\mathcal{O}(\tau^3 N^{5.25})$ (the two terms correspond to the two types of 2-cycles: the first term to cycles on six points, the second term to cycles on five points). Let $\mathcal{Q}_1$ be the 4-uniform hypergraph we obtained this way.

In order to get rid of the 2-cycles we need a random sparsification (as in the previous section): with probability p keep a point and with probability $1-p$ throw it away, and do this independently for all points. Let $\boldsymbol{\mathcal{Q}}_1$ be the random 4-uniform hypergraph we obtain this way. Its parameters can easily be bounded:

$$\mathbb{E}\,(|V(\boldsymbol{\mathcal{Q}}_1)|) = \Theta(pN),$$

$$\mathbb{E}\,(|\boldsymbol{\mathcal{Q}}_1|) = \mathcal{O}(p^4\tau^2 N^4),$$

$$\mathbb{E}\,(\boldsymbol{\mathcal{C}}) = \mathcal{O}(p^6\tau^4 N^{6.5}) + \mathcal{O}(p^5\tau^3 N^{5.25}).$$

The end of the proof is straightforward: We choose p so that

$$\boldsymbol{\mathcal{C}} \ll |V(\boldsymbol{\mathcal{Q}}_1)|. \tag{2}$$

Choose one of the good outcomes of the above probabilistic process such that we obtain a 4-uniform hypergraph with the property that after deleting the points of the 2-cycles we obtain a leftover hypergraph with $\Theta(pN)$ points, and $\mathcal{O}(p^4\tau^2N^4)$ edges, and without 2-cycles. Let $\overline{d}$ denote the average degree. Throw away the points with degree at least $10\overline{d}$. The leftover hypergraph (without 2-cycles) is denoted by $\mathcal{L}$ and its parameters are:

$$|V(\mathcal{L})| = \Theta(pN),$$

$$|\mathcal{L}| = \mathcal{O}(p^4\tau^2N^4),$$

$$\Delta(\mathcal{L}) = \mathcal{O}(p^3\tau^2N^3).$$

Now we can apply Theorem 2. We choose N, τ such that $\alpha(\mathcal{L}) \geq n$ will hold. The n points forming an independent set will prove Theorem 4.

Set the parameters as follows:

$$p := n^{-0.001}, \quad N := n^{1.01}, \quad \tau := n^{-3/2}\sqrt{\log n}.$$

Now we are going to check that with this choice of parameters (2) is satisfied:

$$\begin{aligned}\mathbb{E}\,(\boldsymbol{\mathcal{C}}) &= \mathcal{O}(p^6\tau^4N^{6.5}) + \mathcal{O}(p^5\tau^3N^{5.25})\\ &= \mathcal{O}(n^{0.006}(n^{-6}\log^2 n)\cdot n^{6.565}) + \mathcal{O}(n^{0.005}(n^{-4.5}\log n)n^{5.3025}) = o(n),\end{aligned}$$

and at the same time

$$\mathbb{E}\,(|V(\boldsymbol{\mathcal{Q}}_1)|) = \Theta(pN) = \Theta(n^{1.009}).$$

Hence getting rid of 2-cycles is easy.

In order to apply Theorem 2 we introduce a parameter t, such that $\Delta(\mathcal{L}) \leq t^3$. Based on our previous estimate $\Delta(\mathcal{L}) = \mathcal{O}(p^3\tau^2N^3)$, the right choice for t is

$$t = \Theta(p\tau^{2/3}N) = \Theta(n^{0.001}(n^{-1}\log^{1/3} n)n^{1.01}) = \Theta(n^{0.009}\log^{1/3} n).$$

Hence Theorem 2 is applicable and it provides the following bound:

$$\alpha(\mathcal{L}) \geq \frac{\Omega(pN)}{t}\log^{1/3} t = \Omega(n).$$

Thus, $\alpha(\mathcal{L}) \geq cn$ for certain constant c, and the theorem is proven by the same scaling argument as Theorem 3.

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