Circle Packings and Conformal Geometry

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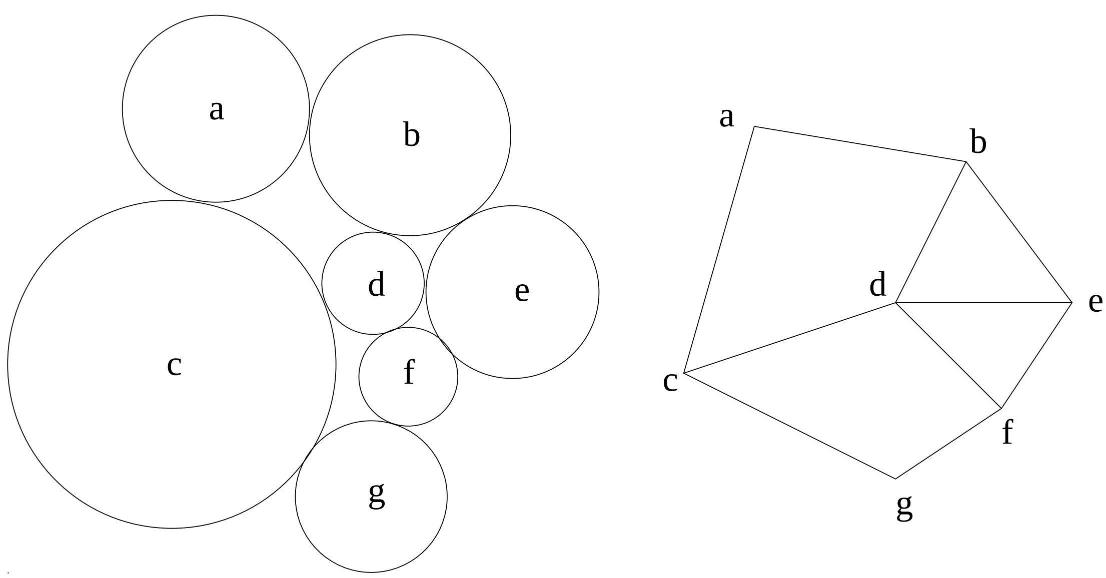
The Circle Packing Theorem

Circle Packing Theorem (Koebe 1934). For every finite planar graph G, there is a disk packing $P = (P_v : v \in V(G))$, whose tangency graph is G.

That P is a packing means that the interiors of the disks in P are disjoint. That G is the tangency graph means that the disks of P are in 1-1 correspondence with the vertices of G, $v \leftrightarrow P_v$, and $\forall v, u \in V(G)$,

$$P_v \cap P_u \neq \emptyset \text{ iff } [v, u] \in E(G).$$



Proofs and generalizations of the CPT

Koebe proved his CPT by appealing to his theorem that every finitely connected planar domain is conformal to a circle domain, and taking limits.

Later proofs and generalizations have been given by Andreev ('70), Thurston ('85), Rivin ('86), Colin de Verdière ('89), Beardon & Stephenson ('90), Schramm ('90), Garrett ('92), Brägger ('92), Hodgson ('92), Bowers ('93), Brightwell & Scheinerman ('93), He & Schramm ('95), Dubejko ('95).

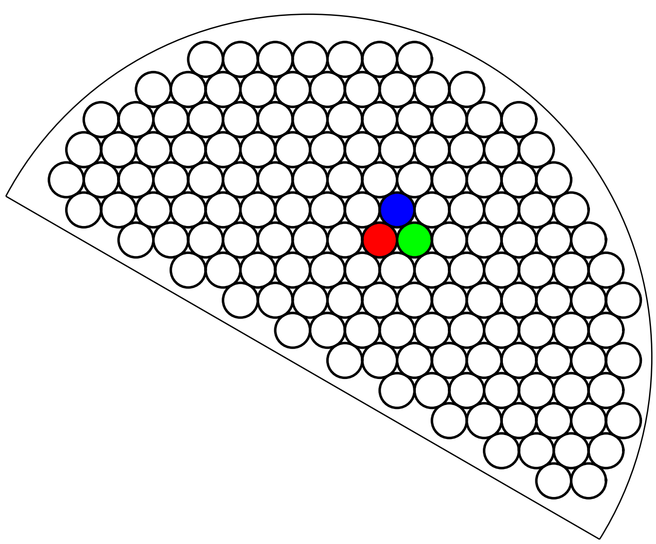
The methods of proof can be grouped into types: topological arguments, Perron method, optimization.

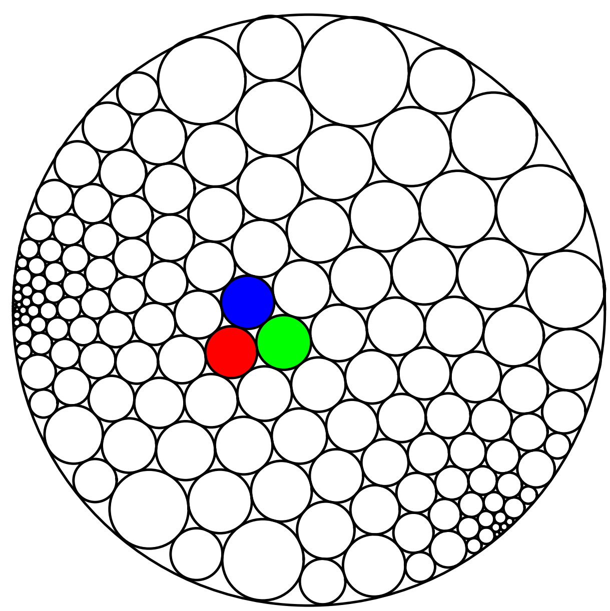
Generalizations include: packings on surfaces, infinite graphs, circle patterns with specified intersection angles, packings of more general shapes, branched packings.

Lacking: higher dimensional theory. Some recent progress on this has perhaps been made by Lovász.

Convergence to conformal maps

Take a simply connected domain $D \subset \mathbb{C}$. Inside the domain take a fine hexagonal circle packing P. By the CPT, there is a circle packing P' inside the unit disk, with the same combinatorics, and with the boundary circles tangent to the unit circle. Thurston conjectured, and Rodin-Sullivan proved ('87) that when P' is appropriately normalized, the correspondence $P_v \to P'_v$ tends to the conformal map from D to U, as the radii of the circles of P goes to zero.





Generalizations and extensions of the Rodin-Sullivan theorem

Z.-X. He showed that in fact, the first derivative, when appropriately defined, converges as well.

Doyle, He and Rodin showed that the second derivative converges as well, and obtained estimates for the rate of convergence.

Stephenson found a probabilistic proof of convergence, based on a random walk.

It has been observed that the Rodin-Sullivan proof works for packings that are based on more general combinatorics, but their method is limited to bounded degree packings.

In joint work with He, we found an entirely different proof of convergence, that does not use QC maps, and therefore is not restricted to bounded degree packings. It also gives the convergence of the first two derivatives. It is based on the argument principle.

C^∞ convergence

Theorem (He & S). For hexagonal disk packings the convergence in the Rodin-Sullivan Theorem is C^∞.

To define what this means, consider a triangular grid with mesh ϵ. Let f_ϵ be defined at the centers of the disk of P, and map each center to the center of the corresponding disk in P', where the disks in P have diameter ϵ. Define the discrete derivative $\nabla_\epsilon g$ of a function g by

$$\nabla_\epsilon g(z) = \frac{g(z+\epsilon) - g(z)}{\epsilon}.$$

The above theorem means that for each k,

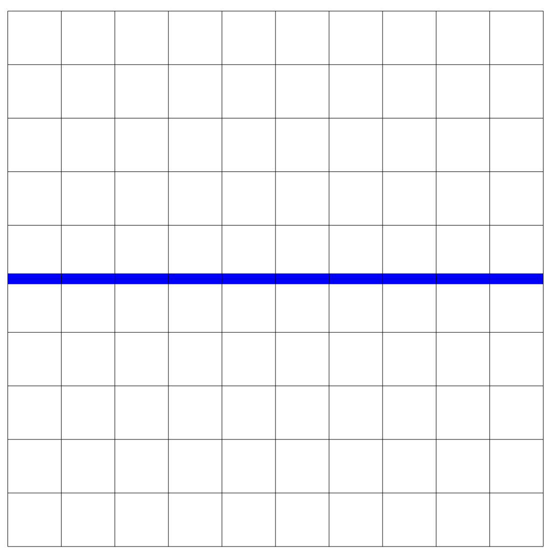
$$\lim_{\epsilon \longrightarrow 0} (\nabla_\epsilon)^k f_\epsilon = f^{(k)},$$

locally uniformly in D, where f is the appropriately normalized Riemann map.

An application of CP: planar separators

Theorem (Lipton & Tarjan 1979). Let G be a finite planar graph. Then there is a $V_0 \subset V(G)$ with $|V_0| \leqslant C\sqrt{|V(G)|}$, such that each connected component of $G - V_0$ has less than $|V(G)|/2$ vertices, and C is an absolute constant.

Example:



Miller & Thurston found a proof of this based on the CPT. It is a gem, so we will sketch it now.

The Miller-Thurston proof

By the CPT, there is a disk packing $P = (P_v : v \in V(G))$ with tangency graph G. By taking a stereographic projection, take P to be on the unit sphere $S^2 := \{|p| = 1 : p \in \mathbb{R}^3\}$. For each $v \in V(G)$, let p_v be some point in P_v; $p_v \in S^2$.

A well known result in convexity (a consequence of Helly's Theorem) states that given any finite set of points $X \subset \mathbb{R}^d$, there is some point $p \in \mathbb{R}^d$ so that every half space that contains p must contain at least $|X|/(d+1)$ points of X. So there is a $p \in \mathbb{R}^3$ such that every half space containing p contains at least $|V(G)|/4$ of the points $\{p_v : v \in G(V)\}$.

Note that the group of projective transformations of $\mathbb{R}^3$ that take S^2 onto itself acts transitively on the unit ball. (This is the same as the group of isometries of $\mathbb{H}^3$ in the Klein model.) Elements of this group take disks on S^2 to disks. Hence assume WLOG that $p = 0$.

Proof, continued

Let $u \in S^2$ be some unit vector, and let $V_0(u)$ be the set of $v \in V(G)$ such that P_v intersects $u^\perp$. Then each component of $G - V_0(u)$ has at most $(3/4)|V(G)|$ vertices. Now, the minimum of $|V_0(u)|$, where $u \in S^2$, is at most the average, which is

$$\begin{aligned}
(4\pi)^{-1} \int_{u \in S^2} |V_0(u)| &= (4\pi)^{-1} \sum_v \text{area} \left(\bigcup_{z \in P_v} z^\perp \right) \\
&= (4\pi)^{-1} \sum_v 2\pi \text{diam}(P_v) \\
&\leqslant \left(|V(G)| \sum_v \text{diam}(P_v)^2 \right)^{1/2} \\
&\leqslant C\sqrt{|V(G)|}\,.
\end{aligned}$$

To get that each component of $G - V_0(u)$ has at most $|V(G)|/2$ vertices, just iterate a few times. ∎

And the moral of the story is...

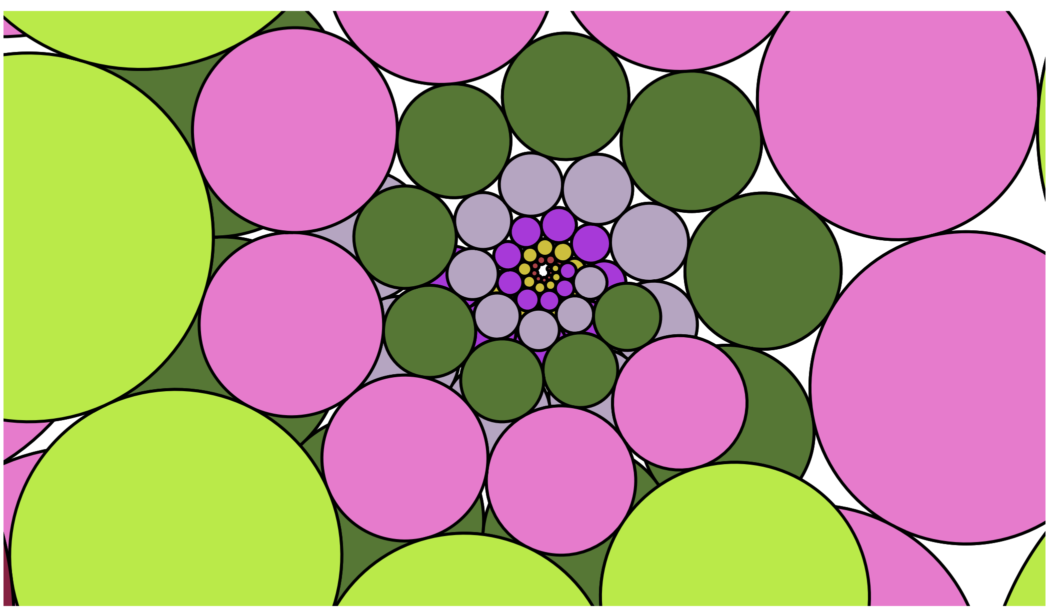
The CPT gives a “good” geometric representation of the graph.

There are numerous applications of the CP theory to other fields. For example, to random walks on planar graphs.

Circle Packing Immersions

There are several ways to generalize beyond circle packings. There are branched circle packings, which are analogous to analytic maps which are not locally 1-1. These have been studied by Bowers, Dubejko, Garrett, Stephenson, and others.

A more restricted class is the class of circle packing immersions. These are analogous to analytic maps that are locally 1-1.



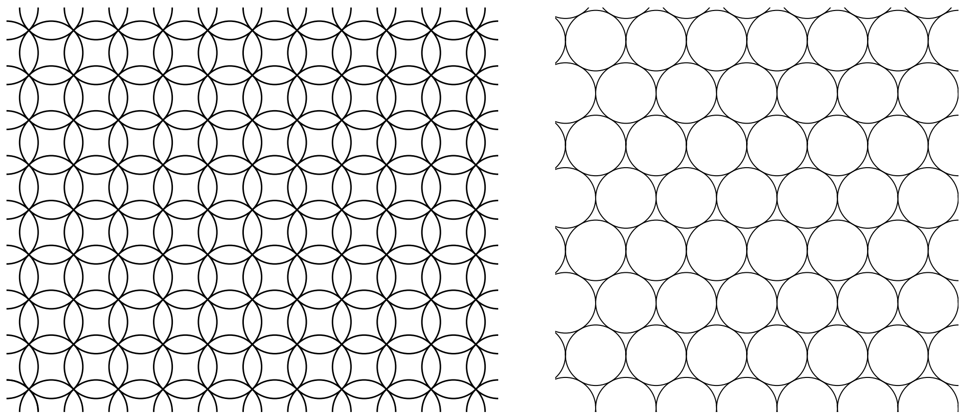
A Doyle spiral

Other immersions?

It is not known if there are any further entire hexagonal CP immersions, except for the regular hexagonal packing and the Doyle spirals (which have two degrees of freedom, up to similarities).

However, there are more entire CP immersions based on the combinatorics of the square grid.

SG immersions

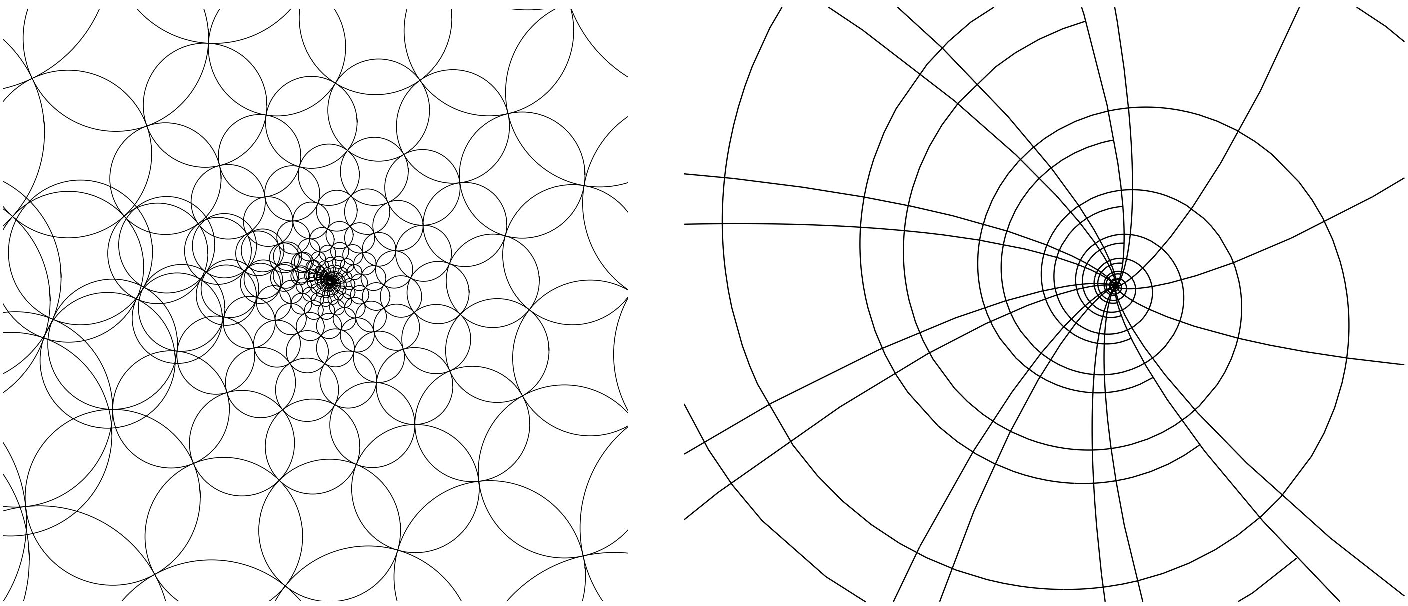


The regular SG pattern and hexagonal pattern

The SG theory is analogous to the theory of the hexagonal circle packings. However, there are distinct advantages to the SG theory. One hint that the SG theory is better is that for each circle in an SG pattern you see one Möbius invariant, the cross ratio. Also, in the hexagonal theory there are interstices; which play a different role.

One big question is whether many of the results on the SG patterns carry over to the hexagonal setting.

SG Doyle Spirals



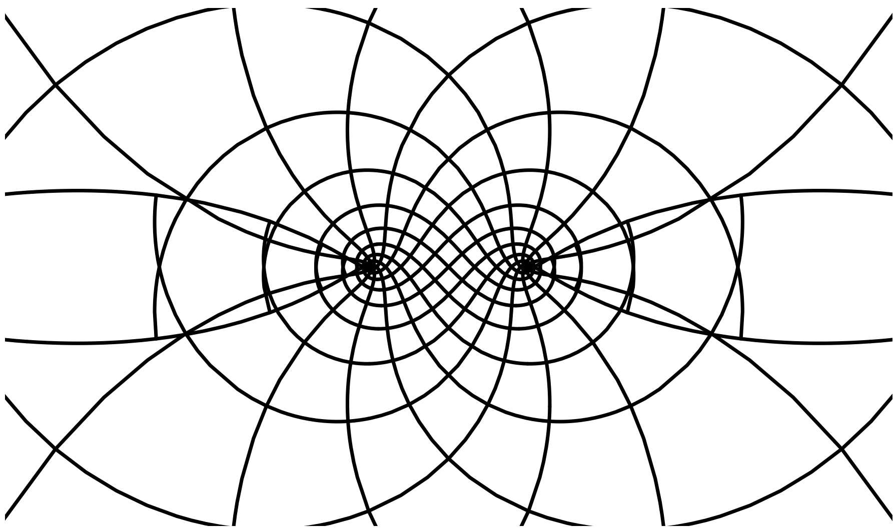
An SG Doyle spiral and the exponential map

The erf function

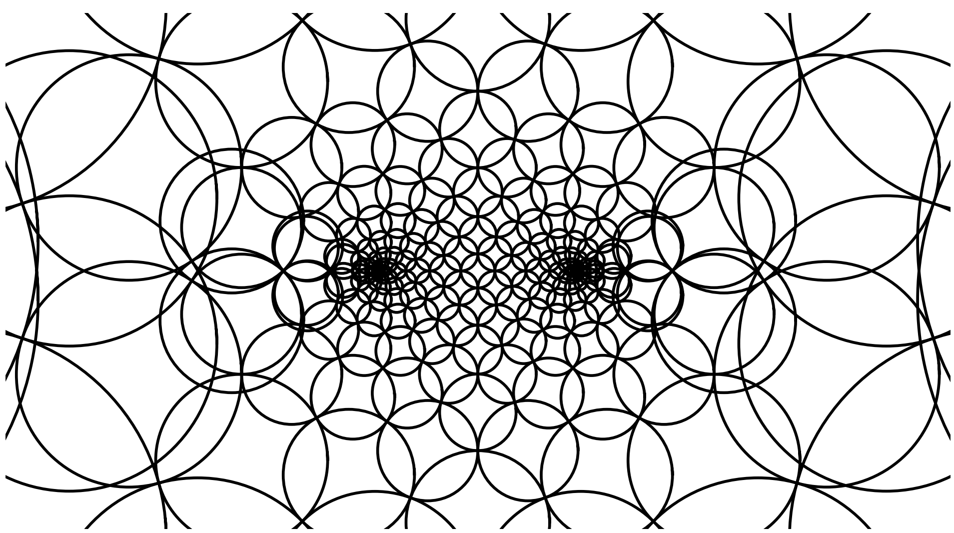
$$\mathsf{erf}(z) = \frac{2}{\pi} \int^{z} \exp(-w^2)dw\,.$$

This is an entire analytic locally univalent function. In a way, the simplest one beside the exponentials and the similarities.

The erf SG pattern



The erf-image of the square grid, erf($\sqrt{i}SG$)



An $\sqrt{i}$SG erf

What is the advantage with the SG pattern?

The equations are simpler. The necessary and sufficient equation for the radius of a circle and the four surrounding it is

$$r^2 = \frac{r_1^{-1} + r_2^{-1} + r_3^{-1} + r_4^{-1}}{r_1 + r_2 + r_3 + r_4} r_1 r_2 r_3 r_4 .$$

Note that this is symmetric about permutations of r_1, r_2, r_3, r_4.

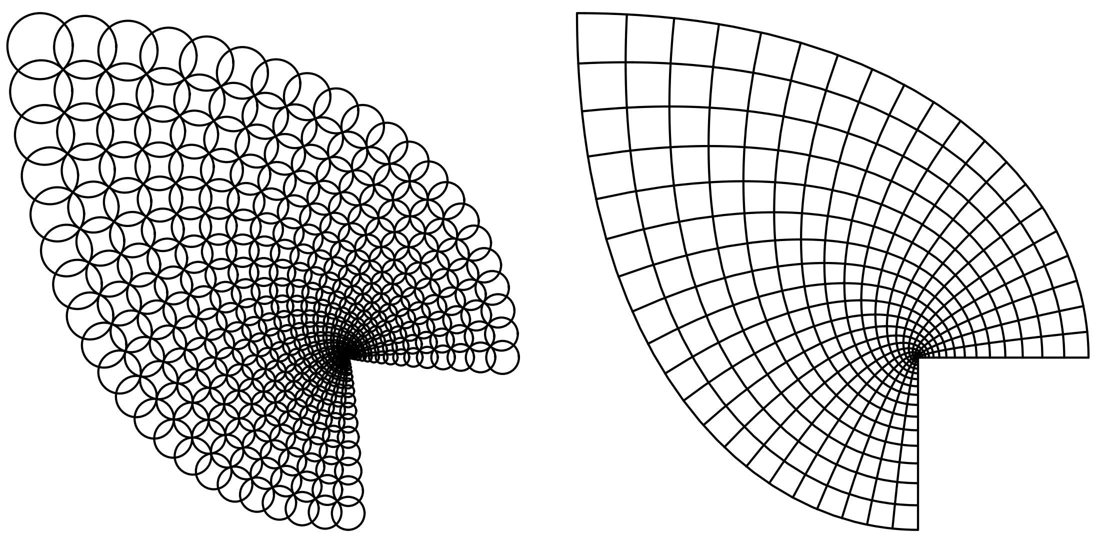
The radius function for the erf SG-pattern is just

$$r(x + iy) = \exp(xy) .$$

No other entire planar SG patterns are known, except for $\exp$, erf, the regular pattern, and their compositions with similarities.

SG Polynomials

Due to the simplicity of the SG-rad equation, in joint work with R. Kenyon, we have been able to find exact formula for some SG-polynomials.



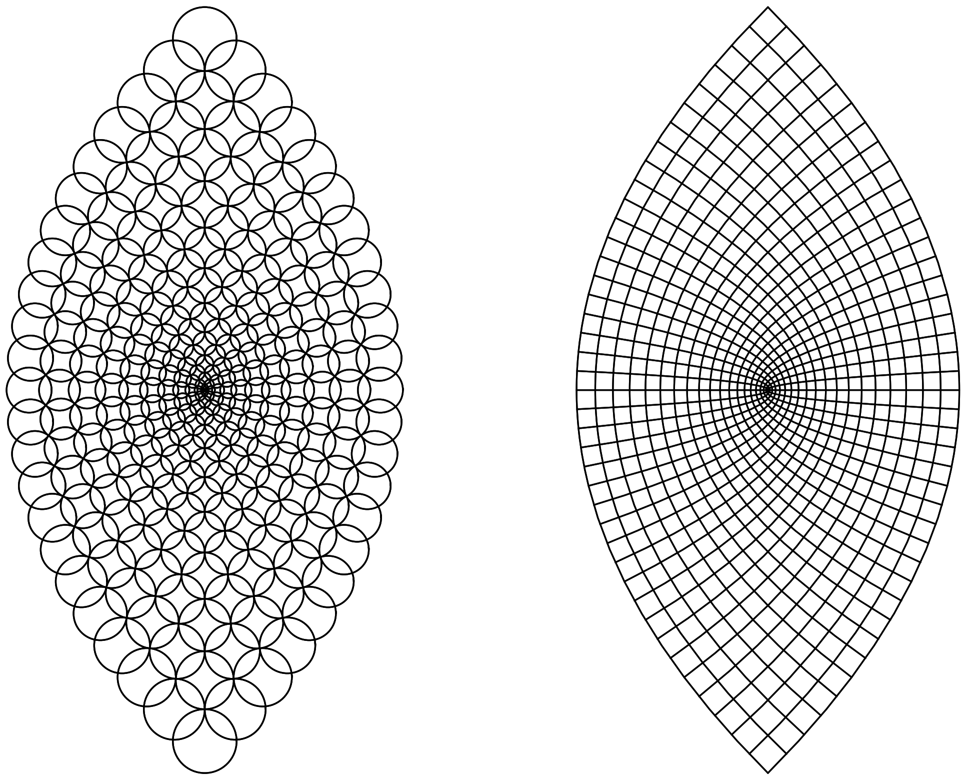
SG z^3 and z^3

Getting SG Polynomials

This is based on the fact that if $r(z)$ is a solution, then $1/r(z)$ is also a solution. Moreover, if we have an SG-pattern, then we can invert it to get another SG-pattern. Let's call the first operation J, and the second I. What do I and J correspond to in the world of meromorphic functions?

Well, clearly $I(f) = 1/f$. Moreover, $|J(f)'| = 1/|f'|$. So $J(f)' = 1/f'$. Set $p_k(z) = z^k$. Then $(J \circ I(p_k))(z) = z^{k+2}$ (ignoring constants), except for $k = 0$. That's how you can get from p_k to p_{k+2}. On the other hand, $(I \circ J(p_2))(z) = \log z$.

SG z^2



SG z^2 and z^2

Getting the SG z^2

The SG z^2 is only **conjectured** to work. That is, there is a recursive formula for the radii, but we cannot prove that it stays nonnegative.

It is obtained as follows. For the z^2 SG pattern, we guess

$$r(z) = |z|, \qquad \text{when } |\mathrm{Re}(z)| = |\mathrm{Im}(z)|\,,$$

and set

$$r(1) = 2/\pi\,.$$

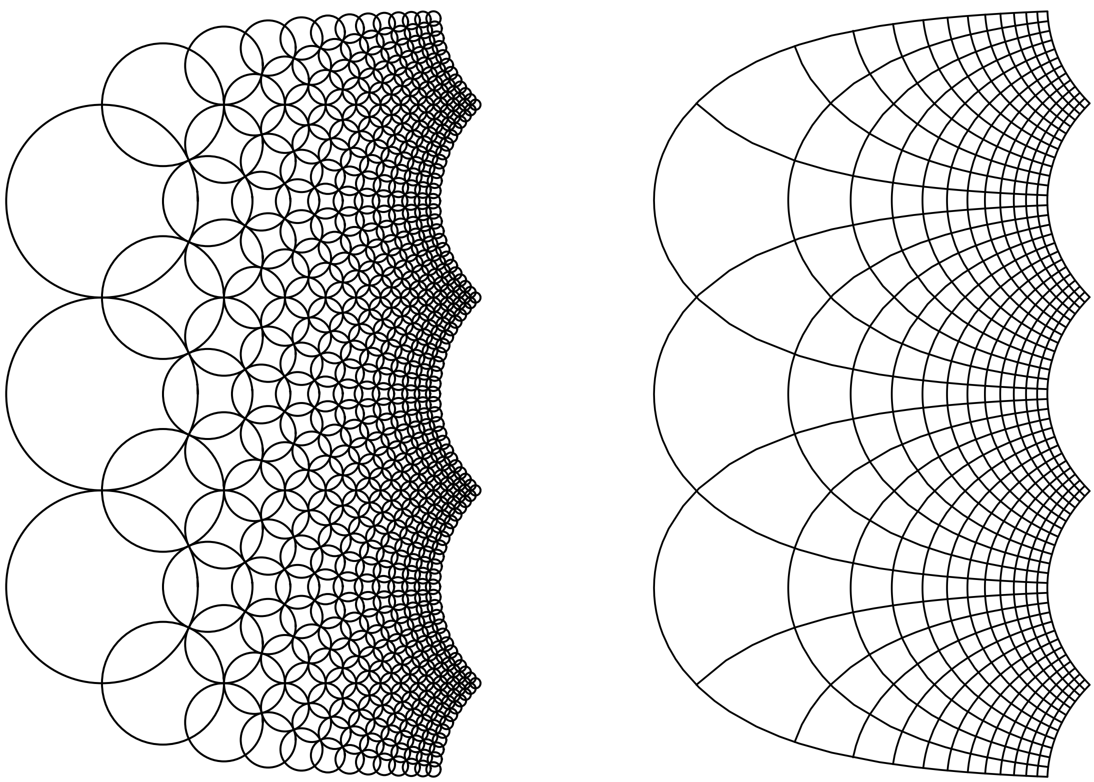
The rest is determined by requiring the obvious symmetry $r(iz) = r(z)$ and using the rad-SG equation. The value $2/\pi$ for $r(1)$ is the only one that makes the values on the sub-diagonal $\mathrm{Im}(z) = \mathrm{Re}(z) - 1$ to behave reasonably.

The radii for z^2

$\frac{16-3\pi^2}{-16+\pi^2}$	$\frac{32-2\pi^2}{-8\pi+\pi^3}$	$\frac{16-3\pi^2}{-16+\pi^2}$	$\frac{8}{\pi}$	3
$\frac{\pi}{2}$	$\frac{8}{-4+\pi^2}$	$\frac{\pi}{2}$	2	$\frac{8}{\pi}$
1	$\frac{2}{\pi}$	1	$\frac{\pi}{2}$	$\frac{16-3\pi^2}{-16+\pi^2}$
$\frac{2}{\pi}$	0	$\frac{2}{\pi}$	$\frac{8}{-4+\pi^2}$	$\frac{32-2\pi^2}{-8\pi+\pi^3}$
1	$\frac{2}{\pi}$	1	$\frac{\pi}{2}$	$\frac{16-3\pi^2}{-16+\pi^2}$

SG $\log$

The SG $\log$ can be obtained from the conjectured SG z^2 by using the relation $(I \circ J(p_2))(z) = \log z$.



SG $\log$ and $\log$

Immersions in the sphere

As we mentioned, except for mild variations, the only entire SG immersions known are the regular pattern, the Doyle spirals (exponentials) and the SG-erf. However, one can show

Theorem (S). The space of entire SG immersions on the sphere is infinite dimensional

For the study of immersions on the sphere, the radius function $r(z)$ is inappropriate, and one uses Möbius-invariants, rather than isometry-invariants.

The σ and τ invariants

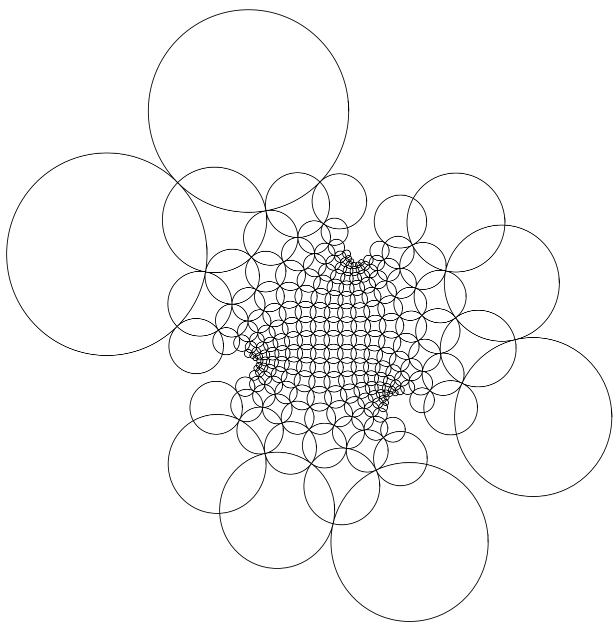
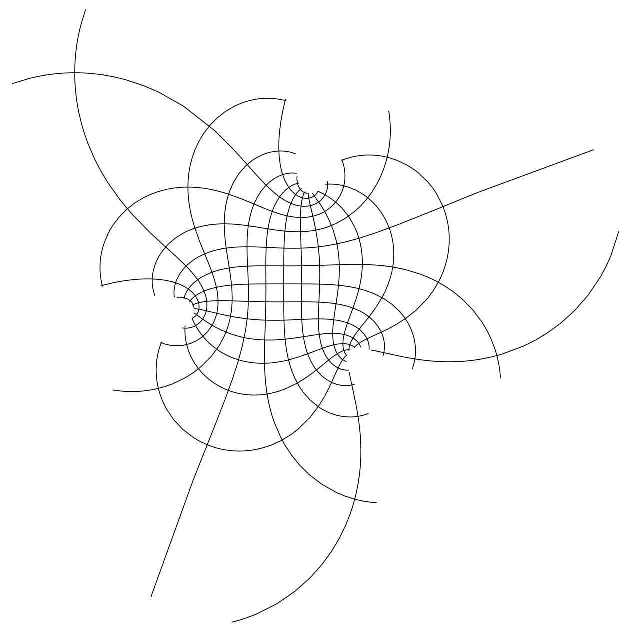
$\tau(z)$, $\sigma(w)$, are two Möbius invariants of SG-patterns. Here, z is a vertex of the square grid, and w is a vertex of the dual grid. They are defined in terms of cross ratios of intersection points of the circles. See Duke97 for the precise definition. They turn out to be analogous to the real and imaginary part of the Schwarzian derivative.

τ and σ together satisfy a nonlinear discrete system of equations that is analogous to the Cauchy-Riemann equations. σ can be eliminated from these equations, giving the following nonlinear discrete analogue of the Laplace equation:

$$\tau(z)^2 = \frac{(\tau(z+1)+1)(\tau(z-1)+1)}{(\tau(z+i)^{-1}+1)(\tau(z-i)^{-1}+1)}\,.$$

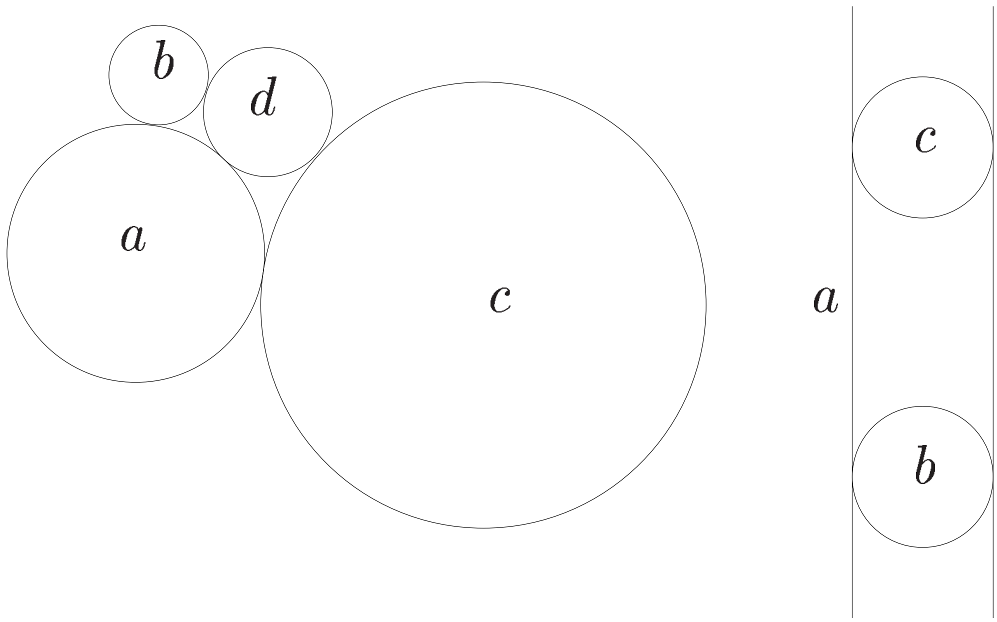
This equation permits guessing explicit solutions, analogous to some known meromorphic functions.

Some explicit immersions

About the proof of C^∞ convergence

It turns out that similar Möbius invariants where instrumental in the proof of C^∞ convergence of hexagonal circle packings. In the hexagonal setting, the Möbius invariants are denoted $s_k(v)$. Here, v is a vertex in the triangular grid, and $k \in \{0, 1, \ldots, 5\}$. These numbers correspond to tangencies in the circle pattern.



$s([a, d])$ is defined as the aspect ratio of this rectangle divided by $\sqrt{3}$.

The basic equation satisfied by these invariants is:

$$s_k + s_{k+2} + s_{k+4} = 3 s_k s_{k+1} s_{k+2}\,.$$

In the Rodin-Sullivan setup with mesh ϵ, set

$$h_k = \epsilon^{-2}(s_k - 1)\,.$$

Theorem (Z.-X. He and S).

$$\lim_{\epsilon\to 0} h_k = \frac{1}{6}\mathrm{Re}(\omega^{2k}\mathcal{S}(f)) \,,$$

where f is the Riemann map, $\mathcal{S}$ is the Schwarzian derivative, and $\omega = (-1)^{1/3}$. In fact, the convergence is C^∞.

The proof of C^∞ convergence runs as follows. To show that h_k are uniformly bounded on compacts as $\epsilon \to 0$, one uses Z.-X. He's "bad area" estimate. Playing around with the equation for the s_k, one finds that Δh_k is a polynomial in the h_j's and their translates. Then one needs to make a discrete analogue of the compactness principle for elliptic equations.

Summary

It is often worthwhile to look at the Möbius invariants.