Topological Graph Theory

A Survey

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1 Introduction

Graphs can be represented in many different ways: by lists of edges, by incidence relations, by adjacency matrices, and by other similar structures. These representations are well suited to computer algorithms. Historically, however, graphs are geometric objects. The vertices are points in space and the edges are line segments joining select pairs of these points. For example, the points may be the vertices and edges of a polyhedron. Or they may be the intersections and traffic routes of a map. More recently, they can represent computer processors and communication channels. These pictures of graphs are visually appealing and can convey structural information easily. They reflect graph theory's childhood in "the slums of topology."

Topological graph theory deals with ways to represent the geometric realization of graphs. Typically, this involves starting with a graph and depicting it on various types of drawing boards: 3-space, the plane, surfaces, books, etc. The field uses topology to study graphs. For example, planar graphs have many special properties. The field also uses graphs to study topology. For example, the graph theoretic proofs of the Jordan Curve Theorem, or the theory of voltage graphs depicting branched coverings of surfaces, provide an intuitively appealing and easily checked combinatorial interpretation of subtle topological concepts.

In this paper we give a survey of the topics and results in topological graph theory. We offer neither breadth, as there are numerous areas left unexamined, nor depth, as no area is completely explored. Nevertheless, we do offer some of the favorite topics of the author and attempt to place them in context.

We begin with some background material in Section 2. Section 3 covers map colorings, and Section 4 contains other classical results. Section 5 examines several variations on the basic theme, including different drawing boards and restrictions. Section 6 looks at locally planar embeddings on surfaces. Chapter 7 gives a brief introduction to graph minors, Chapter 8 to random topological graph theory, and Chapter 9 to symmetrical maps. Chapter 10 contains some open problems, and Chapter 11 is the conclusion.

2 Background Material

In this section we introduce some of the basic terms and concepts of topological graph theory. The reader seeking additional graph-theoretic definitions should consult the book by Bondy and Murty [44]. A more detailed treatment of embeddings is in the book by Gross and Tucker [103]. We examine in turn the basic terms, surfaces, Euler's formula and its consequences, the maximum and minimum genus, combinatorial descriptions of embeddings, and partial orders.

2.1 Basic Terms

A *graph* G is a finite collection of *vertices* and *edges*. Each edge has two vertices as *ends*. An edge with both endpoints the same is called a *loop*. Two edges with the same pair of endpoints are *parallel*. In some applications it is common to require that graphs are *simple*, that is, have no loops or parallel edges. In topological graph theory it is common to allow both.

Each graph G corresponds to a topological space called the *geometric realization*. In this space the vertices are distinct points and the edges are subspaces homeomorphic to $[0, 1]$ joining their ends. Two edges meet only at their common endpoints. An *embedding* of G into some topological space X is a homeomorphism between the geometric realization of G and a subspace of X. For convenience, we freely confuse a vertex in the graph, the point in its geometric realization, and the corresponding point when embedded in X.

Where should we embed a graph? Perhaps the most natural space to consider is the real plane $\mathcal{R}^2$. A graph embedded in the plane, $G \subset \mathcal{R}^2$, is called a *plane graph*; a graph admitting such an embedding is *planar*. In a connected plane graph each component of $\mathcal{R}^{\in} - \mathcal{G}$ is homeomorphic to an open 2-cell. However, as shown by an embedding of the graph with a single vertex and two loops in the plane, it may be that the closure of this open 2-cell is not a closed disk. Instead, there may be repeated points along the boundary.

2.2 Surfaces

As we will show, not every graph embeds in the plane. How then can we picture it? Keeping the space locally planar, we can try to embed graphs in *surfaces*; that is, compact Hausdorf topological spaces which are locally homeomorphic to $\mathcal{R}^2$. There are two ways to construct such surfaces: take a sphere and attach n handles, or take a sphere and attach m crosscaps. We denote these surfaces by S_n and $\tilde{S}_m$ respectively. By a theorem of Brahana [59] any surface falls in one of these two infinite classes (see [103] for details). In particular, the surface obtained by adding in n handles and m crosscaps ($m \geq 1$) is homeomorphic to $\tilde{S}_{2n+m}$. A surface S_n is *orientable*, that is, it is possible to assign a local sense of clockwise and anticlockwise so that along any path between any two points in the surface the local sense is consistent. However, $\tilde{S}_m$ is *nonorientable*, a consistent assignment of sense is impossible.

It is easily shown that any graph embeds in some surface: draw it in the plane with crossings and use a handle to "jump over" each crossing. We wish the graph to carry a reasonable amount of information about the surface in which it's embedded. In particular, if the surface has a handle or crosscap, then we want the graph to use that feature. For example, a single loop embedded in a small local neighborhood of a point in a torus does not use the handle. An embedding is *cellular* if each component of $X - G$ (i.e., each *face*) is homeomorphic to an open 2-cell. In a cellular embedding any curve in the surface is homotopic to a walk in the graph. Note that only connected graphs have cellular embeddings. Henceforth we declare that all graphs are connected and all embeddings are cellular. If an embedding has the additional property that the closure of each face is homeomorphic to a closed disk, then the embedding is *circular* or *closed 2-cell* (CTC).

Given an embedded *primal* graph there is a natural way to form an embedded *geometric dual graph.* We place a vertex of the dual in the interior of each face of the primal embedding. Whenever two faces of the primal share a common edge, add an edge of the dual from the middle of one face, through the middle of the common edge, to the middle of the other face. This dual is embedded in the surface in a natural manner. The duality operator swaps the 0-dimensional points with the 2-dimensional faces, leaving the 1-dimensional edges fixed. Observe that the dual of the dual is the (embedded) primal graph.

A cycle C in a surface S may be *contractible*, that is, homotopic to a point. A noncontractible cycle is called *essential*. An essential cycle may still be *separating*, that is, $S - C$ may be disconnected. Noncontractible separating cycles are homologically but not homotopically null.

2.3 The Euler Characteristic

Let G be a graph (cellularly) embedded in a surface S. Suppose that $\#V$ is the number of vertices of G, $\#E$ is the number of edges, and $\#F$ is the number of faces in the embedding. The *Euler Characteristic* of the embedding is $\chi(G) = \#V - \#E + \#F$. It is well known [150] that the Euler Characteristic of the embedding depends only on the surface and not on the embedding. If the surface is the sphere with n handles attached, then the Euler Characteristic is $2 - 2n$, and if it is the sphere with m crosscaps, then the Euler Characteristic is $2 - m$. We call the quantity $\bar{\gamma} = 2 - \chi$ the *Euler genus* of the surface. This parameter has also been called the *generalized genus* and the *complexity* of the surface. Each handle contributes two to the Euler genus and each crosscap contributes one.

Euler's formula can be used in combination with other inequalities to derive some interesting bounds. We begin with the observation that an embedding of a connected graph which is not a tree has the length of each face bounded below by the girth g. Since the sum of the face lengths is $2\#E$, this gives $g\#F \leq 2\#E$. In combination with Euler's formula this gives:

$$\#E \leq (\#V + \bar{\gamma} - 2)g/(g - 2).$$

Roughly speaking, for girth $g = 3$ and fixed $\#V$, each crosscap (increasing $\bar{\gamma}$ by one) can carry up to three edges and each handle (increasing $\bar{\gamma}$ by two) can carry six edges. When $g = 4$ these numbers drop to two edges and four edges respectively.

Inequalities of this type are used to show the nonexistence of embeddings. For example, suppose by way of contradiction that K_5 has a planar embedding. Using Euler genus $\bar{\gamma} = 0$ for the sphere, and girth $g = 3$, $\#V = 5$, and $\#E = 10$ for K_5, we violate the preceding inequality. This contradiction shows no such planar embedding exists. A similar argument works for $K_{3,3}$.

2.4 The Maximum and Minimum Genus

A graph can have many possible embeddings on many different surfaces. Naturally, the extremal embeddings are of interest. Define the (minimal orientable) *genus* of G, $\gamma(G)$, to be the smallest n such that G embeds on the sphere with n handles. Likewise define the *nonorientable genus*, $\tilde{\gamma}(G)$, as the smallest m such that G embeds on the sphere with m crosscaps. We consider a planar graph to be of nonorientable genus zero, although some authors say it is of nonorientable genus one. The *Euler genus* $\bar{\gamma}(G) = min\{2\gamma(G), \tilde{\gamma}(G)\}$. Define the *maximum genus*, $\gamma_M(G)$, the *maximum nonorientable genus*, $\tilde{\gamma}_M(G)$, and the *maximum Euler genus*, $\bar{\gamma}_M(G)$, in a similar manner.

The maximum and minimum genus completely determine the orientable surfaces on which a connected graph cellularly embeds. This follows from the interpolation theorem of Duke [79], which states that if a graph embeds on a sphere with n handles and on one with m handles, then it embeds on all intermediate surfaces. The proof uses the concept of rotations (defined in the following section) and the observation that moving a single edge end to a different location in a rotation changes the genus of the resulting embedding by at most one. A similar interpolation theorem for nonorientable surfaces is due to Stahl [214].

Bounds on these maximum and minimum orientable and nonorientable genus are given in the following lemma.

Lemma 2.1 *Let G be a graph with $\#V$ vertices, $\#E$ edges, and girth g. Then:*

$$(g-2)\#E/(2g) - \#V/2 + 1 \le \gamma(G) \le \gamma_M(G) \le (\#E - \#V + 1)/2,$$

$$(g-2)\#E/g - \#V + 2 \le \tilde{\gamma}(G) \le \tilde{\gamma}_M(G) = \#E - \#V + 1, \quad \text{and}$$

$$\tilde{\gamma}(G) \le 2\gamma(G) + 1.$$

The two lower bounds are proved using Euler's formula with an upper bound on the number of faces, similar to the argument that K_5 is nonplanar.

The two upper bounds are also proved using Euler's formula and a lower bound of one face. Note that in the orientable case the number of faces is determined by the graph up to parity, so that the lower bound is either one or two faces. Finally, note that in the nonorientable case there is always an embedding with just a single face (I usually cite [214], but I have also heard the result credited to Edmonds). We examine the maximum genus parameter more closely in Section 4.2.

The third inequality shows that the nonorientable genus cannot be too large compared to the orientable genus. However, conversely, there are graphs of nonorientable genus one and orientable genus n [22], so there is no such bound in the other direction.

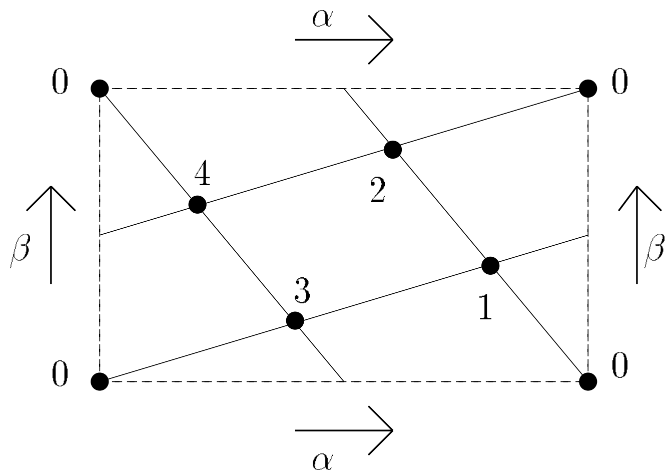
Graphs in which equality holds in the third equation are called *orientably simple*. For example, K_7 embeds in the torus but not in Klein's bottle and so is orientably simple. To the author's knowledge, there is no detailed study of orientably simple graphs.

2.5 Combinatorial Descriptions of Embeddings

We need a convenient combinatorial way to describe an embedding. It is easiest to begin with an orientable surface. The following was implicit in the work of Heffter [112] with Edmonds [80] and Youngs [264] usually credited with being the first to (respectively) dualize and formalize the process.

Fix a consistent orientation at each point on the surface, say anticlockwise. By looking at the a neighborhood of a point, this orientation determines a cyclic permutation of the edges with ends at a vertex v, or more precisely in the case of loops, of the edge ends at v. We call such a cyclic permutation a *local rotation* at v. A *rotation* on a graph G is a collection of local rotations, one at each vertex. (Some authors prefer the term *rotation scheme*.) As we have shown, an oriented embedding determines a rotation.

Conversely, suppose that we are given a rotation. We will show how to construct an embedding into an orientable surface which determines this particular rotation. First use the rotation to trace out the facial walks. An *arc* is defined by fixing one of the two possible directions on an edge in a graph. Begin by walking along an arc in a graph with a rotation. Upon reaching the other end of the arc at the vertex v, the local rotation at v leads us to another edge end. Continue the walk along the arc on that edge rooted at v. Proceeding in this manner trace out a walk in the graph. This



$$
\begin{array}{l}
0 : 2\ 4\ 3\ 1 \\
1 : 3\ 0\ 4\ 2 \\
2 : 4\ 1\ 0\ 3 \\
3 : 0\ 2\ 1\ 4 \\
4 : 1\ 3\ 2\ 0
\end{array}
$$

Figure 1: A rotation embedding K_5 in the torus

walk traverses each directed edge at most once and is independent of the starting arc. Doing this for each possible arc, determine a set F of walks which traverse each arc exactly once. Next, for each $f \in F$ of length n, take a convex n-gon together with its interior in the plane and label the sides as in the walk of length n. This will serve as one face in the embedding. Finally, glue the $\#F$ polygons together using the labeling determined by the graph. The result is a surface, since each edge lies in two walks, and around a vertex the polygons line up in the cyclic ordering given by the local rotation. Moreover, this surface is orientable, and so is homeomorphic to some S_g.

We illustrate this with an example. Let K_5 be the complete graph on the vertex set the integers modulo five. Around vertex 0 we label the edges 1,2,3,4 depending on their other endpoint. The local rotation at vertex 0 is $(2, 4, 3, 1)$. The local rotations at vertices 1,2,3, and 4 are given in Figure 1. Tracing the faces of this embedding yields five quadrilaterals (the cyclic symmetry helps simplify the calculation). These fit together to form a torus as shown in Figure 1. In this figure the top of the rectangle is identified with the bottom and the left with the right to recover the torus.

There are two possible ways to consistently orient a surface: clockwise and anticlockwise. Each graph embedded on the surface will lead to exactly two different rotations depending on the sense of the local rotations. That is, the rotations are in 1-1 correspondence with embeddings of the graph into *oriented* surfaces, and in 2-1 correspondence with embeddings into orientable surfaces. There are exactly $\Pi_{v \in V}(deg_G(v)-1)!$ rotations, so this is the number of different cellular embeddings of G into oriented surfaces.

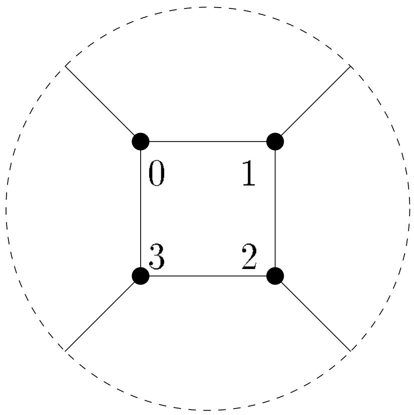
How do we describe a nonorientable embedding? We use a combinatorial structure called a signed rotation. This consists of a rotation and a *signature*, an assignment of a plus or minus on each edge.

We first describe how to get a signed rotation from an embedded graph. Let G be embedded on a (possibly nonorientable) surface. Fix a local orientation at each vertex. If the surface is nonorientable this cannot be done consistently. As before, this local orientation determines a local rotation at each vertex. To get the signature label an edge with a plus if the two orientations at the ends agree, and label it with a minus if they disagree. This gives a signed rotation.

Conversely, we can take a signed rotation and construct an embedding. As before, we use the signed rotation to first trace out the facial walks. This time we keep track of the current state of a walk, either plus or minus. We start out at one end of an edge in a plus state. We walk along that edge. If the edge is plus, we keep the current state; if it is minus, we toggle the current state. When we reach the other end, if our current state is plus we use the local rotation; if our current state is minus we use the inverse of the local rotation. We continue this way until we return to the same edge-end in the same state. This algorithm traces out walks which contain every edge exactly twice. The toggling between states when we traverse along a negative edge corresponds to the fact that the local rotations disagree at the ends. Instead of using the local sense we last used, we must use the opposite sense. Once we have traced the facial walks we identify them with the edge of n-gons as before and use the labelings to reconstruct the surface.

We illustrate this procedure with an example embedding K_4 in the projective plane. We take as the vertices of K_4 the integers modulo 4, again identifying the edges incident with a vertex by their other endpoint. The local rotation at 0 is $(1, 2, 3)$; the other local rotations are given in Figure 2. The edges signed minus are 02 and 13. The resulting embedding is depicted in Figure 2. There the projective plane is recovered by identifying each point x on the boundary circle with its antipodal point $-x$.

Two different signed rotations may lead to the same embedding. For example, we can switch the local rotation ρ_v to the inverse ρ_v^{-1} while simultaneously toggling the sign on each edge incident with v. The resulting



0 : 1 2 3
1 : 2 3 0
2 : 3 0 1
3 : 0 1 2

negative edges : 02, 13
positive edges : 01, 12, 23, 30

Figure 2: A signed rotation embedding K_4 in the projective plane

signed rotation is different, but the embedding described is the same. We call this a *local switch of sense*. Any two signed rotations leading to the same nonorientable embedding are related by a sequence of local switches of sense. Hence, any embedding can be described by $2^{\#V}$ different signed rotations. Counting the number of signed rotations, it follows that there are $2^{\#E-\#V}\Pi_{v\in V}(deg(v)-1)!$ different cellular embeddings of G into surfaces.

Knowing the total number of embeddings and the number of orientable embeddings, we can calculate the number of nonorientable embeddings. These counts reveal that "most" embeddings are in nonorientable surfaces. An embedding described by a signed rotation is in an orientable surface if and only if it is equivalent under a sequence of local switches to a signed embedding in which each edge is positive. As this is unlikely, it confirms that the number of nonorientable embeddings exceed orientable ones.

2.6 Partial Orders on Graphs and Embeddings

In many cases it is convenient to place a partial order on the set of graphs or embedded graphs. We mention four such orders in particular.

The first is the subgraph ordering $H \subset G$. Observe that if G embeds in a surface the H does as well, although the latter embedding may not be cellular.

The second ordering is the topological ordering. A graph H is an *elementary subdivision* of G if it is formed from G by deleting an edge uv and replacing it with a path uwv where w is not a vertex of G. In this case we say that G is formed from H by supressing the degree two vertex w. Two graphs H and G are *homeomorphic* if they are related by a sequence of elementary subdivisions and supressing degree two vertices. The name arises because G and H are homeomorphic as graphs if and only if they are homeomorphic as topological spaces. It follows that embedding properties are determined by the homeomorphism class of a graph. The *topological order* is defined by $H \leq G$ if and only if H is homeomorphic to a subgraph of G. Again, if G embeds in a surface then so does any $H \leq G$, but the latter embedding may not be cellular.

The third order is the *minor order*. The elementary operations defining $H \leq G$ are of three types. The first is the deletion of isolated vertices in G. The second is the deletion of edges, $H = G \setminus e$. The third is the *contraction* of an edge e, $H = G/e$, defined by first deleting e and then identifying its endpoints. If H can be formed from G by a sequence of these operations, then H is a *minor* of G. The edge deletion and edge contraction (of a non-loop) can be done in a host surface (where by convention contracting an essential loop is equivalent to deleting it), so that if G embeds in a surface, then so does any minor H. Note that for embedded graphs edge contraction and edge deletion are dual operations, that is, contracting an edge in the primal graph corresponds to deleting the corresponding edge in the dual graph. Equivalently, deleting a primal edge corresponds to contracting the dual edge.

The fourth ordering includes the three minor operations and the $Y\Delta$ *operation*. In this operation a degree 3 vertex w adjacent in G with vertices x, y, z is deleted and edges xy, yz, zx are added. As with the minor operations, the $Y\Delta$ operation can be done to a graph embedded in the surface. Hence if G embeds on a surface and $H \leq G$, then H also embeds on that surface.

We close by mentioning that occasionally one considers the ΔY operation (the inverse to $Y\Delta$) and the class of graphs equivalent under $Y\Delta$ and ΔY operations. Finally, a slight extension of the last partial order involves first subdividing an edge joining two degree 3 vertices, then performing a $Y\Delta$ operation on each of the vertices. (The resulting subgraph looks like a "bow-tie" $K_5 - C_4$.)

3 Map Colorings

In 1852 Francis Guthrie was coloring a map of England. Each region was to get a color and when two regions shared a boundary line they were to be colored differently. In a flash of insight he asked what was the fewest number of colors needed, not just for this map, but for any map [156]. He conjectured that four colors suffice, but a (correct) proof of this four-color conjecture was many years coming. In this section we examine such map colorings and related problems.

A *map* is an embedded graph. A *coloring* of the map is an assignment of colors to the faces. The coloring is *proper* if whenever two faces share a common edge they receive different colors. Colorings herein will be proper unless otherwise stated. A map has a proper coloring if and only if each edge lies on the boundary of two distinct faces. A coloring of a map is equivalent to a vertex coloring of its dual, that is, assigning a color to each vertex so that adjacent vertices receive distinct colors.

3.1 Planar Graphs

Kainen and Saaty [203] write "One of the many surprising aspects of the four-color-conjecture is that a number of the most important contributions to the subject were originally made with the belief that they were solutions." One of the first of these was by Kempe [130] who introduced the recoloring methods now known as Kempe chains. Heawood [110] pointed out the error in Kempe's argument, but was able to modify it to give a correct proof that every planar graph was 5-colorable. Tait [227] introduced the relation with edge-coloring cubic plane graphs. He too thought he had solved the 4-color problem; his mistake was believing that every cubic graph was Hamiltonian. Petersen [172] clarified the relation with edge-colorings and introduced his famous graph (see [113]).

The Four-Color Theorem was proved by Appel and Haken [3, 4] in 1977. The proof was at first controversial, in part because of the reliance on long computer calculations. However, the result has been proven several times independently, most recently by Robertson, Sanders, Seymour, and Thomas [189]. All proofs to date rely on the same basic technique of finding an unavoidable set of reducible configurations. We refer the reader to [203] for a description of these methods.

We explore the relation between face and edge colorings of plane graphs first introduced by Tait [227]. A *cubic* graph is one which every vertex is incident with exactly three edges. A *Tait coloring* of a cubic graph is a 3-coloring of the edges such that at each vertex each color appears exactly once.

If a planar graph is face 4-colored, then it can be edge 3-colored. This can be seen by using the elements of the Klein 4-group $\mathcal{Z}_2 \times \mathcal{Z}_2$ as face colors; the edge coloring then assigns each edge the sum of the colors on either side. The converse is also true: if a planar cubic graph can be Tait colored, then it can be face-4-colored. So the claim that every (bridgeless) planar cubic graph can be Tait colored is equivalent to the claim that every (loopless) planar graph can be 4-colored.

The relationship with the four-color conjecture led to a search for cubic graphs which could not be Tait colored. Since the graphs are bridgeless and exactly three edges meet at each vertex, there do not exist four pairwise adjacent edges. That is, there is no immediate obstacle to Tait colorings. However, there are several other "trivial" reasons a graph cannot be Tait colored. For example, a graph with a loop has no proper edge colorings. Likewise, any graph with a cut-edge cannot be Tait colored. (To see this, observe that the union of any two color classes in a Tait coloring forms a subgraph which is regular of degree two. Hence every edge is in a cycle, and the graph must be 2-edge-connected.)

In 1973 Martin Gardner [91] called nontrivial non Tait colorable graphs *snarks*, after the mythical creature from a poem by Lewis Carroll [63]. By nontrivial he meant that a snark must be cyclically 4-edge-connected and of girth at least 5. The idea was to avoid graphs which were not Tait colorable but which contained easy reductions to smaller non Tait colorable graphs. We refer the reader to Swart [64] and to Issacs [118] for interesting discussions about what constitutes a trivial (or petty) snark and should be exiled.

The most famous snark is the Petersen graph [172], discovered in 1891. It was over fifty years later before another example was found. In 1946 Danilo Blanuša found [40] a non Tait colorable graph on 18 vertices (the same techniques lead to a second such graph of the same order). In 1948 Blanche Descartes found [78] an example on 210 vertices. It was not until 1973 that a fourth example was found by Szekeres [226]. At this time several powerful construction techniques were developed by Issacs [118], yielding two infinite classes of snarks. One of these classes, the flower snarks, had been discovered independently three years earlier by Grinberg in some unpublished work. Recently, Brinkman (personal communication) has found all snarks on 28 or fewer vertices. For other recent results see [14, 62].

Four colors suffice to vertex color any planar graph, but only if they are the same four colors available at each vertex! Suppose that we have a list of colors available at each vertex, but the lists may be different. A *list coloring* assigns each vertex one of the colors in its list; as usual, adjacent vertices receive different colors. At first glance this should be easier than standard colorings where all of the lists are the same. After all, if the lists on the ends of an edge are different, then it is more likely that the colors on the ends are distinct. Define the *list chromatic number* of G, $\chi_l(G)$ as the smallest k such that every assignment of lists of size k to the vertices has a list coloring. The list chromatic number of the plane is the maximum $\chi_l(G)$ over all planar G.

Theorem 3.1 *The list chromatic number of the plane is five.*

Thomassen [236] shows sufficiency with an elegant proof that every planar graph is list 5-colorable. Voigt [251] gives an example of a planar graph and a list assignment of four colors which can not be list colored.

Four colors suffice for planar graphs. When do three? We mention two famous 3-color theorems. In 1898 Heawood [111] proved that a plane triangulation is vertex 3-colorable if and only if every vertex was of even degree. Grötzsch [104] proved that every triangle-free planar graph was vertex 3-colorable. (My favorite proof of this theorem is due to Thomassen [237].) Thomassen [241] gives a list version of Grötzsch's Theorem. Król [139, 140] noted that a plane graph is 3-colorable if and only if it is a subgraph of an Eulerian triangulation. It would be interesting to find either a common extension of or relationship between Heawood and Grötzsch's theorems. No good characterization is to be expected, since Garey, Johnson and Stockmeyer [93] showed that in general the problem is NP-complete. We refer the reader to Steinberg [222, 223] for a recent survey of this three color problem.

The 2-color problem is easy. A plane graph is 2-colorable if and only if every face is bounded by a walk of even length [133, 134]. The 1-color problem is not quite pointless, but it is edgeless.

Coloring the faces of a plane graph is equivalent to coloring the vertices of the dual. What if we try to color the vertices and faces simultaneously? Specifically, a *coupled coloring* of an embedded G is a coloring of the vertices and faces so that adjacent or incident elements receive distinct colors. Ringel [180] conjectured that every plane graph has a coupled 6-coloring. The theorem was first shown for cubic graphs [180] and then for triangle-free graphs [7]. Borodin [46, 47, 48] proved the even stronger result that every graph which embeds in the plane so that each edge crosses at most one other has chromatic number at most six.

A *total coloring* of a graph colors vertices and edges so that adjacent or incident elements receive distinct colors. The *total chromatic number* $\chi''(G)$ is the minimum number of colors needed. It is bounded below by $\Delta + 1$ where Δ is the maximum degree. Independently Behzad and Vizing [39, 248] conjectured that for simple graphs it is bounded above by $\Delta + 2$. This conjecture is known to be true for several classes of graphs. Various general upper bounds are also known. We refer the reader to [123] for a discussion of the known results. The conjecture is true for plane graphs except for the cases $\Delta = 6, 7$. The low degree cases are due to Rosenfeld [202] ($\Delta = 3$) and Kostochka [136, 137, 138] ($\Delta = 4, 5$). These cases do not use planarity. The high degree cases are due to Borodin [56] ($\Delta \geq 9$) and Andersen [2] ($\Delta = 8$) and do use planarity.

An *entire coloring* of a plane graph colors the vertices, faces, and edges simultaneously so that adjacent or incident elements receive distinct colors. Kronk and Mitchem [141, 142] conjectured that the entire chromatic number of a graph was equal to $\Delta + 4$, where Δ is the maximum degree. This is true for plane cubic graphs (by M. Neuberger, as reported by Izbicki [119]), and for plane graphs with Δ sufficiently large by Borodin [49, 50, 51, 52]. The conjecture is still open for $\Delta = 4, 5, 6$.

Along these lines we also mention Vizing's Conjecture [249, 250] that every planar graph with $\Delta \geq 6$ can be properly edge-colored in Δ colors. This conjecture is known to be true for $\Delta \geq 8$. It is false if extended to $\Delta \leq 5$.

What if you color the faces of an embedded graph so that faces that share a vertex or edge receive different colors? In the dual form this requires a vertex coloring so that around each face no color is repeated. These are called *cyclic colorings* by Ore and Plummer [170]. The bound depends on the size of the largest face, Δ^*. They showed that the cyclic chromatic number was at most $2\Delta^*$. This was improved to $\lfloor 9\Delta^*/5 \rfloor$ by Borodin, Sanders, and Zhao [57]. Plummer and Toft [173] proved that for 3-connected graphs the cyclic chromatic number is at most $\Delta^* + 9$. They also give a lower bound of

$\lfloor 3\Delta^*/2 \rfloor$ for the general problem.

Finally, we mention the problem of coloring the faces of a plane cubic map so that around each edge each four (or fewer) faces receive distinct colors. Bouchet, Fouquet, Jolivet, and Riviere [58] showed that this can always be done in 12 colors. Borodin [53] improved the bound to 11; Sanders and Zhao [205] further improved this to 10. The conjectured correct answer is 9 colors.

3.2 Maps on Other Surfaces

We turn our attention to coloring maps on surfaces. Define the chromatic number of a surface as the maximum chromatic number of all graphs that embed on that surface. The dual question is to ask for the minimum number of colors needed to properly color the faces of every map on that surface. For example, the plane has chromatic number four. The following Map Color Theorem settled this question for all surfaces other than the plane.

Theorem 3.2 *The chromatic number of S_g is $\lfloor (7+\sqrt{1+48g})/2 \rfloor$ for all $g \geq 1$. The chromatic number of $\tilde{S}_h$ is $\lfloor (7+\sqrt{1+24h})/2 \rfloor$ for all $h \neq 0, 2$. The chromatic number of Klein's bottle is 6.*

Surprisingly, this was proved before the 4-color theorem. Also surprisingly, the easy part of the 4-color theorem is hard in the map color theorem, but the hard part of the 4-color theorem is easy in the map color theorem. We elaborate on these two halves of the proof in turn.

It is easy to show that 4 colors are needed for some planar maps; you only need to draw K_4 in the plane. In the map color theorem the required number of colors is the largest n such that K_n embeds on the surface. However, the difficult part was to show that K_n (or some other n-chromatic graph) did in fact embed on the smallest surface allowed by Euler's formula (with the exception of K_7 in Klein's bottle). That task, broken into 24 cases by the residue of n modulo 12 and the orientablility or nonorientability of the surface, was completed by Ringel and Youngs [186] in 1968 (see esp. [265] for the nonorientable case). A nice account of the proof is given in Ringel [181].

Conversely, it is hard in the plane to prove that four colors suffice for all maps. However, on other surfaces (except the projective plane) it is easy to show that the conjectured number of colors n is sufficient. One need merely use an Euler characteristic argument to show that every graph on the surface has a vertex of degree at most $n-1$ and use induction.

There are variations of coloring graphs in surfaces similar to those mentioned for the plane. These include improper colorings [74] (where the number of adjacent vertices of the same color is bounded), generalizations of Grötzch's theorem [237, 86] (coloring triangle-free graphs), simultaneously coloring vertices and faces [182, 135], and acyclic colorings [54, 55] (where any two color classes induce a forest).

We define the *list chromatic number* of a surface analogously to the list chromatic number of the plane. Archdeacon and Širáň (unpublished work) have shown that for any surface other than the plane, the list chromatic number is equal to the chromatic number.

The relationship between edge colorings and face colorings is not as clear in other surfaces as in the plane. Nevertheless, we mention an intriguing conjecture due to Grünbaum [105]. This conjecture states that every simple graph that triangulates an orientable surface has an edge coloring such that each color appears (necessarily exactly once) around each triangle. In the dual form, this states that if a cubic graph (the dual of triangulation) is embedded on an orientable surface such that any two faces share at most a single edge (the dual of simple), then the graph can be Tait colored. This conjecture is strictly stronger than the 4-color theorem, because any planar graph can be placed in a single face of a triangulation of a surface, and the coloring of that triangulation induces a coloring of the planar part as well. It is equivalent to asserting that any orientable embedding of a non-Tait colorable graph must have two faces sharing more than one edge. Note that the orientable condition is necessary as evidenced by K_6 on the projective plane. This embedded graph has dual the Petersen graph which is not Tait colorable, hence K_6 does not have the desired coloring.

Although maps on other surfaces may have large chromatic number, it is the case that if the embedding is locally planar for large neighborhoods of a vertex, then the chromatic number can be bounded. We investigate this in Section 6.

We close with the wonderful quote from Tutte [245], "The Four-Colour Theorem is the tip of the iceberg, the thin end of the wedge and the first cuckoo of spring."

4 Classical Results

In this section we state some of the classical results of topological graph theory. As noted previously, the most natural space for depicting graphs is the plane. In Section 4.1 we state some theorems characterizing planar graphs. Section 4.2 gives theorems relating to the maximum and minimum genus of graphs. Section 4.3 studies graphs which embed on a fixed surface.

4.1 Characterizing Planar Graphs

An important early question is to characterize those graphs which embed on the plane. One such characterization was given by Kuratowski [145] in 1930. The same theorem was proven independently and roughly concurrently by Frink and Smith [87], who never published their paper after hearing of Kuratowski's proof. This theorem is very important, and is the most cited research paper in graph theory [37].

Theorem 4.1 *A graph is planar if and only if it does not contain a subgraph homeomorphic to K_5 or to $K_{3,3}$.*

We earlier proved that the two graphs in question were not planar, from which it follows that any graph containing a topological copy of these graphs is nonplanar. The beauty of the theorem lies in the proof that these are the only two topologically minimal nonplanar graphs.

The theorem can be rephrased slightly using the minor ordering, where it was first stated by Wagner [253].

Theorem 4.2 *A graph is planar if and only if it does not contain K_5 or $K_{3,3}$ as a minor.*

To state the next theorem we need some algebraic terminology. In this context a *cycle* in a graph is a set of edges which are incident with each vertex an even number of times. Equivalently, it is a subgraph in which each component is Eulerian. A *cocycle* is a minimal edge-cut in the graph. Note that each cycle intersects each cocycle in an even number of edges. Consider subsets of edges as vectors over the integers modulo two where addition is defined as the symmetric difference of sets. Then the collection of cycles is a subspace $Z(G)$, as is the collection of cocycles $B(G)$. These subspaces are orthogonal.

A graph G^* is an *algebraic dual* of a graph G if there is a function $\phi : E(G) \to E(G^*)$ such that C is a cycle of G if and only if $\phi(C)$ is a cocycle of G^*. The following is due to Whitney [258, 259].

Theorem 4.3 *A graph is planar if and only if it has an algebraic dual.*

In fact, if G and G^* are algebraic duals, then there exists an plane embedding of G so that G^* is the geometric dual.

Let G be a 2-connected plane graph with face set F. Each face boundary is a simple closed walk whose edges form a cycle. The collection of any $\#F-1$ of these cycles forms a basis for the cycle space. Moreover, no edge appears in more than two members of this collection. MacLanes Theorem [152] gives the converse.

Theorem 4.4 *A graph is planar if and only if there is a collection of cycles which generate the cycle space together with one additional cycle such that every edge is in exactly two of these cycles.*

Other authors [13, 151] have given similar algebraic characterizations of planar graphs.

4.2 The Genera of Important Graphs

Recall that the hard part of the Map Color theorem was establishing the minimum orientable and nonorientable genus of the complete graph K_n. Lower bounds on these genera are given by Euler's formula. However, constructing embeddings achieving these bounds can be difficult. Similarly, what are the minimum genera of complete bipartite graphs, regular complete tripartite or quadripartite graphs, the cubes Q_n, and the general octahedra $K_{n(2)} = K_{2n} - nK_2$? The answers are given in the following table.

Graph	γ	$\tilde{\gamma}$	Comments
K_n	$\lceil (n-3)(n-4)/12 \rceil$	$\lceil (n-3)(n-4)/6 \rceil$	$n \geq 3$, $\tilde{\gamma}(K_7) = 3$
$K_{n,m}$	$\lceil (n-2)(m-2)/4 \rceil$	$\lceil (n-2)(m-2)/2 \rceil$	$n, m \geq 2$
$K_{n,n,n}$	$(n-1)(n-2)/2$	??	
$K_{4(n)}$	$(n-1)^2$	??	$\gamma(K_{4(3)}) = 5$
Q_n	$(n-4)2^{n-3}+1$	$(n-4)2^{n-2}+2$	$n \geq 6$
$K_{n(2)}$	$(n-3)(n-1)/2$	??	$n \not\equiv 2 \pmod 3$

The results for K_n form the Map Color Theorem [186, 181, 264]. The orientable and nonorientable genus of complete bipartite graphs were found by Ringel [183, 184]. The orientable genus of complete regular tripartite graphs was found by Ringel and Youngs [187] and by White [256]; for a particularly nice proof see Stahl and White [221]. The orientable genus of regular quadripartite graphs is due to Garmen [94] and Jungerman [125] (the special case $K_{4(3)}$ is due to White [255]). The orientable genus of the cube has been found by several authors [185, 33, 100]. The nonorientable genus is due to Jungerman [126]. For low dimensional cubes we note that the formula for the orientable genus holds for $n \geq 2$ while $\tilde{\gamma}(Q_4) = 3$ and $\tilde{\gamma}(Q_5) = 11$. The genus of the octahedron is found in [95, 99, 127]. Three of these table entries are question marks. To the author's knowledge these values are unknown, although my literature search may have been incomplete. The genera undoubtedly equal the lower bound given by Euler's formula except possible for some small values of n.

If a graph breaks into pieces along a small set of vertices, then it might be possible to relate the genus of the graph with the genus of the pieces. The first theorem along these lines is due to Battle, Harary, Kodama and Youngs [31].

Theorem 4.5 *The genus of a graph is the sum of the genus of its blocks.*

The Euler genus is also additive over the blocks of the graph. The nonorientable genus is not additive; counterexamples are the one-point union of two orientably simple graphs. Stahl and Beineke [220] show that the nonorientable genus of the one-point union of two graphs differs from the sum of their nonorientable genera by at most one.

The orientable, nonorientable, and Euler genus are all nearly additive over 2-point unions [76, 8], that is, the genus of a 2-point union differs from the sum of the genera of the components by at most a constant. However, these parameters behave quite differently when amalgamating over three or more vertices. Archdeacon has shown [8] that the nonorientable and Euler genus are both almost additive over k-point unions (the constant depends on k), but there exist [9] graphs G_n and G'_n and a 3-point union $G_n \cup G'_n$ such that $\gamma(G_n \cup G'_n) - \gamma(G_n) - \gamma(G'_n) = n$.

The maximum genus of a graph turn out to be an easier parameter to calculate than the minimum genus. We begin with the maximum orientable genus. A large genus surface has a small number of faces. A graph is *upper embeddable* if it has an embedding with 1 or 2 faces (the parity is determined by the graph's Betti number). This embedding necessarily achieves the maximal genus. Xuong [262] gave a remarkable theorem determining the maximum genus of a graph. To state the theorem, let $C_o(H)$ denote the number of components of H with an odd number of edges.

Theorem 4.6 *Let $\mathcal{T}$ be the set of all spanning trees of a graph G. Then*

$$\gamma_M(G) = max_{T \in \mathcal{T}} \lfloor (\#E - \#V + 1 - C_o(G-T))/2 \rfloor .$$

The constructive portion of Xuong's Theorem is especially nice. A $\vee$ is a pair of edges with a common endpoint. Xuong shows that if $G - \vee$ has an embedding with either one or two faces, then so does G. The construction of the desired maximal embedding proceeds by first embedding a spanning T achieving the above maximum with a single face, then successively adding as many $\vee$'s as possible creating a upper embedded subgraph, and finally adding in the remaining $C_o(G-T)$ edges each increasing $\#F$ by one.

Xuong's theorem implies that if a graph has two disjoint spanning trees, then it is upper embeddable. By a result of Kundu [144], any 4-edge-connected graph has two disjoint spanning trees. Hence any 4-edge-connected graph is upper embeddable. These include the complete multipartite graphs and cubes listed in the table for minimum genus.

Xuong's theorem gives an easily applied certificate to verify that a large genus embedding exists. The following theorem of Nebesky's [165] gives an easily applied certificate to verify that no embedding exists in a higher surface. These two theorems are very powerful used in concert. Let $c(H)$ denote the number of components of a graph H and let $o(H)$ denote the number of components with odd Betti number.

Theorem 4.7 *The minimum number of faces in an embedding of G is*

$$\max_{A \subset E(G)} \{c(G-A) + o(G-A) - \#A\}.$$

The maximum nonorientable genus is very easy to calculate. Every graph has an embedding into a nonorientable surface with just a single face! This theorem was first noted by Edmonds [81]. Stahl [214] gave a proof which includes the nonorientable version of Duke's interpolation theorem. Geometrically, begin with a minimum genus embedding of G with two or more faces. Find an edge e lying on two distinct faces. Sew a crosscap in the surface in the middle of this edge. Then the resulting embedding has one fewer face. Continue in this way until only one face remains.

The maximum orientable genus is not additive over k-connected components, but it is nearly additive. The maximum nonorientable genus is additive over all unions.

One topic of interest is to find the smallest possible value for the maximum genus of graphs in a certain class. For example, the maximum genus of G is at least $\beta(G)/4$ for simplicial graphs [70] and this bound is tight. Chen has shown that $\beta(G)/3$ is a tight lower bound for 3-connected graphs [66] and for simplicial 2-connected graphs [129]. Chen, Archdeacon, and Gross give tight lower bounds for k-connected and for k-edge-connected graphs [67].

We close our discussion on the minimum and maximum genus of a graph with the computational aspects.

Theorem 4.8 *Determining the minimum orientable genus of a graph is NP-complete (Thomassen [230]). There is a polynomial-time algorithm to find the maximum orientable genus of a graph (Furst, Gross and McGeoch [88]).*

The techniques of [230] extend easily to show that determining the minimum nonorientable genus is also NP-complete. Since every graph embeds in a nonorientable surface with a single face, determining the maximum nonorientable genus is trivial. Similarly, the determining the minimum Euler genus is NP-complete while the maximum is trivial.

4.3 Graphs on a Fixed Surface

To date we have focused on the graph used in the embedding; namely, given a graph, what surfaces does that graph embed in? We now ask a similar question focusing on the surface: given a surface, what graphs embed on that surface? The first such characterization of this type was Kuratowski's Theorem (and the closely related Wagner's theorem). These characterize planarity by excluding topological subgraphs (or minors). Various sources cite König and Erdös for the origin of the conjecture that there are similar theorems for other surfaces. Specifically, that for each surface S there exists a finite set $I(S)$ whose topological (or minor) exclusion characterizes embedding in S. Graphs in $I(S)$ are called *irreducible* since they do not embed in the surface but any proper subgraph (or minor) does embed.

The first result for a surface other than the plane is due to Archdeacon [5, 6].

Theorem 4.9 *There are exactly 103 graphs topological irreducible graphs for the projective plane.*

The graphs were originally found by Glover, Huneke, and Wang [77] (Neil Robertson found the 103^{rd} graph). Archdeacon proved that the list was complete. Vollmerhaus [252] independently verified this completeness, unaware of Archdeacon's work. These 103 topological graphs correspond to a set of 35 excluded minors. This is implicit in [5] and first stated explicitly in [153].

The projective plane is the only other surface for which a complete list of irreducible graphs is known. The author has performed calculations giving hope that a complete list for the torus may be assembled using suitable computer programs. Phil Huneke estimates that there may be 10,000 topologically irreducible graphs.

The Erdös-König finiteness conjecture has been solved in the affirmative for all other surfaces.

Theorem 4.10 *For each surface there is a finite list of excluded topological subgraphs which characterize embeddability on that surface. Similarly, there is a finite list of excluded minors.*

Several teams of researchers worked independently to establish this result. Archdeacon and Huneke [18] proved the result for nonorientable surfaces. They also proved a similar result for graphs which are minimal with respect to Euler genus. Their proof was constructive, providing a method in theory at least for finding the graphs. Bodendiek and Wagner gave a similar result for orientable surfaces. Their proof was assembled in [42].

Robertson and Seymour approached the problem entirely differently, examining a conjecture by Wagner that any infinite set of graphs contains one which is a minor of another. This much stronger conjecture implies the Erdös-König conjecture since no two irreducible graphs are comparable. In [194] they gave a proof of this for graphs of bounded genus, which implies both the orientable and nonorientable cases. They have since given a proof of Wagner's conjecture in general [196]. Their proofs are nonconstructive, but apply in much broader scope and have many other important applications. We briefly survey their results in Section 7.

We close by noting the plausible conjecture by Glover and independently by Vollmerhaus (personal communications) that any graph which is irreducible for a surface has every edge in a topological K_5 or $K_{3,3}$. This conjecture was disproved by Brunet, Richter, and Širáň [61] with a clever toroidal example. It is true however, for nonorientable surfaces and for 3-connected graphs.

5 Variations on the Theme

In this section we examine some variations on the classical theories mentioned above. We begin with restricting our attention to the plane, but allowing edges to cross one another. We then examine other planarity restrictions. Next we describe ways of embedding graphs in which some of the information, either parts of the local rotation or the signature, have been pre-ordained. Finally, we consider pseudosurfaces.

5.1 Drawings in the Plane

Suppose that we want to depict K_5 in the plane. We cannot do it without edge crossings, but we can do it if we allow edges to cross. A drawing is like an embedding of the geometric graph in the surface, except that we do not require the function to be 1-1. We put the following restrictions on crossings in our drawings. First, all of the vertices must be distinct points. Second, the interior of an edge may not pass through any vertex point. Third, any two edges share at most a finite number of points in common.

With these restrictions we can depict any graph in the plane (for example, place the vertices of K_n on the vertices of a convex n-gon and connect them

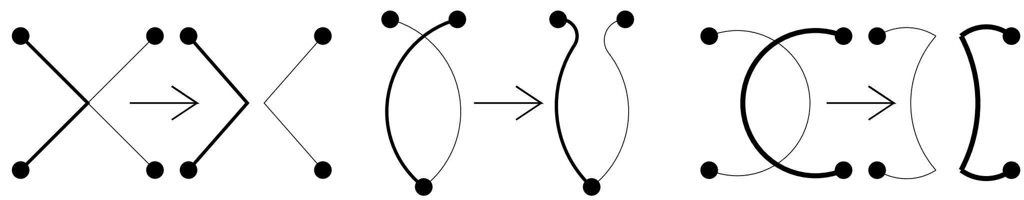


Figure 3: Modifying drawings to reduce crossings

in pairs with straight line segments). However, we have the "cost" of having edge crossings. A frugal mathematician will try to find a drawing of a graph G with the minimal number of crossings. This parameter is called the *crossing number* of G, denoted $cr(G)$.

We make several elementary observations about crossings. First, when two edges meet at a point they should cross instead of meeting tangentially. When they meet tangentially, then the drawing can be modified in a local manner to reduce the total number of crossings. Secondly, no pair of edges with a common endpoint cross. For if this occurs, then the drawing can be modified to remove that crossing and hence decrease the number of crossings. Thirdly, no pair of edges can meet in more than one point. Again, if this occurs there is a similar drawing with fewer crossings. A drawing satisfying these restrictions is called *good*. The modifications transforming any drawing into a good drawing are illustrated in Figure 3. A drawing with the minimum number of crossings is necessarily good. Henceforth all drawings will be good unless otherwise specified.

The minimum number of crossings in a drawing of G where all edges are straight line segments is called the *rectilinear crossing numbers* $\bar{cr}(G)$. Clearly $cr(G) \leq \bar{cr}(G)$. Equality need not hold in general.

What is known about the crossing numbers of various classes of graphs? Not much. In fact, the crossing number of the complete graph K_n is unknown in general, as is the crossing number of $K_{n,m}$. Another class of interest is the product of cycles $C_n \times C_m$. Known values are summarized in the table below.

Graph	$cr(G)$	$\bar{cr}(G)$	Comments
K_n	$\frac{1}{4}\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor\lfloor\frac{n-2}{2}\rfloor\lfloor\frac{n-3}{2}\rfloor$	$\geq cr(K_n)$	$n \geq 3$
$K_{n,m}$	$\leq \lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor$	$\geq cr(K_{n,m})$	$cr = \bar{cr}$?
$C_m \times C_n$	$\leq n(m-2)$	$\leq n(m-2)$	$m \leq n$

We note that $cr(K_n) = \bar{cr}(K_n)$ only for $n \leq 7$ and $n = 9$. In all other cases strict inequality holds.

Most of the bounds given are upper bounds. These are in general demonstrated by exhibiting specific drawings with these number of crossings. Much more difficult is the problem of demonstrating lower bounds, of stating that every drawing must have a particular number of crossings. The bound on $cr(K_n)$ is known to be exact for $n \leq 12$ (see, e.g., [106]). The bound on $cr(K_{n,m})$ is exact for $n \leq 6$ [132] and for $n = 7$, $m \leq 10$ [260]. The bound on $cr(C_n \times C_m)$ is exact for $n = 3$ [109, 175, 178], for $n = 4$ [82, 35], and for $n = 5$ [175, 224]. In each case the first reference (or two) gives the proof for $m = n$, while the last reference uses an induction argument to extend this for general m. Note that K_{10} is the smallest complete graph for which the rectilinear crossing number is not known.

The problem of finding the crossing number of $K_{n,m}$ is sometimes known as Turan's brickyard problem. The idea is that each of n kilns is connected to each of m shipping centers by rail tracks. The carts used to transport the bricks from kiln to shipping center are likely to derail where rails cross. Hence it is convenient to minimize the number of crossings.

Can anything be said about the crossing number of complete or complete bipartite graphs? In each case, the bound above is conjectured to be exact. Perhaps some partial progress could be made by proving that the bounds were asymptotically exact, that is, by proving e.g. $\lim cr(K_n)/n^4 = 1/64$. In an optimal drawing of K_n each of the n induced K_{n-1} subgraphs occurs with at least $cr(K_{n-1})$ crossings. Accounting for multiplicities, this shows that $cr(K_n)/n^4$ is increasing and so this limit exists. The best known bound currently is due to Kleitman [132] who shows that the limit is at least $3/240$. For $K_{n,n}$ the conjectured equality shows that $\lim cr(K_{n,n})/n^4$ should be $1/64$. If this limit holds, then so does the one for the complete graphs.

For the rectilinear crossing number, Jensen [122] has demonstrated an upper bound on $\bar{cr}(K_n)/n^4$ that is asymptotically $7/432$. He has since (private communication) reduced this to $1/63$. It is conjectured that the correct value is also $1/64$, in particular, that $\lim cr(K_n)/\bar{cr}(K_n) = 1$. Finally, we note that it is conjectured that the rectilinear crossing number equals the crossing number for $K_{n,m}$

We turn our attention to the maximum number of crossings in a drawing of G, $cr_M(G)$. This maximum exists since our drawings are good, so the number of crossings is bounded above by the number of pairs of nonadjacent edges. The maximum crossing number is easily calculated for complete and complete bipartite graphs. In particular, $cr_M(K_n) = \binom{n}{4}$ (each induced K_4 can have at most one crossing, and choosing the vertices on a circle achieves this). Likewise, $cr_M(K_{n,m}) = \binom{n}{2}\binom{m}{2}$ (each induced C_4 can have at most one crossing, and placing the vertex parts on two parallel lines achieves this).

The maximum rectilinear number appears not to have been widely studied except for [261]. In general, this should be much less than the maximum crossing number.

When can a graph have a drawing so that every pair of nonadjacent edges cross? John Conway calls a graph *thrackled* if there exists such a drawing. The curious word also refers to a tangle of fishing line, which such drawings often resemble.

Conjecture 5.1 *If a connected graph can be thrackled, then $\#E \leq \#V$.*

A connected graph with the same number of vertices as edges necessarily has a single cycle. Such graphs are sometimes called *unicycles*.

Partial results towards the thrackle conjecture are sparse. As noted above, the graph cannot have an induced C_4. Woodall [261] showed that any cycle of length greater than 4 can be thrackled. Likewise, any tree can be thrackled. If a graph can be thrackled, then so can its subgraphs. With some extra work it can be shown that the thrackle conjecture reduces to showing that the one-point union of two even cycles cannot be thrackled. It is known [225] that if a graph can be thrackled, then $\#E \leq 2\#V - 3$.

A systematic study of the maximum crossing number $cr_M(G)$ is hampered by the fact that the parameter is not known to be monotone. That is, if H is a subgraph of G, then is $cr_M(H) \leq cr_M(G)$? One would expect this to be true. However, as shown in [179], a drawing of H does not always extend to a drawing of G, which negates the obvious way to proceed.

Various people have investigated the crossing number of other surfaces. Kainen [128] worked out a lower bound using Euler's formula. Specifically, a graph with girth g on a surface of Euler genus $\bar{\gamma}$ has crossing number at least $\#E - (\#V - \bar{\gamma})g/(g-2)$. Upper bounds for a few graphs [97, 107, 128] are given by exhibiting drawings.

We close by noting that it would be interesting to find the total number of non-isomorphic drawings of a graph. However, aside form some results of Harbourth on small graphs (personal communication) not much is known in this area.

5.2 Thickness

Not every graph is planar. What is the smallest n such that G can be written as the union of n subgraphs? This number is called the *thickness* of G, denoted $\theta(G)$. One application of the thickness is when the graph represents a computer circuit to be laid out on a stack of circuit boards. The thickness represents the number of boards needed.

A planar graph has the number of edges bounded by a function of the number of vertices. This leads to the following Lemma.

Lemma 5.1 *For a simple graph G, $\theta(G) \geq \lceil \#E/(3\#V-6) \rceil$. Moreover, if G is triangle-free, then $\theta(G) \geq \lceil \#E/(2\#V-4) \rceil$.*

Can this bound be achieved? Not in general. However, it is true for several interesting classes of graphs. The known results are summarized in the following table.

Graph	Thickness	Comments
K_n	$\lfloor (n+7)/6 \rfloor$	$n \neq 9, 10$
$K_{n,m}$	$\lceil nm/(2n+2m-4) \rceil$	see exceptions below
Q_n	$\lfloor n/4 \rfloor + 1$	see [131]
$K_{n(2)}$	$\lceil n/3 \rceil$	

The thickness of K_n for $n \not\equiv 4 \pmod 6$ was proven by Beineke and Harary [32]. The remaining congruence class was shown independently by Alekseev and Gonchakon [1] and by Vasak [246]. Some small individual cases were done by hand, including $n = 16$ by Mayer [157].

The thickness of $K_{n,m}$ is unknown in the case $n \leq m$, both are odd, and there is an integer k with $(n+5)/4 \leq k \leq (n-3)/2$ with $m = \lfloor 2k(n-2)/(n-2k) \rfloor$. The smallest such case is $K_{19,29}$. Beineke, Harary, and Moon [34] did the known cases.

The thickness of the octahedral graphs follows quickly from that of the complete graphs [32].

Several authors have investigated thickness on other surfaces, that is, partitioning the edge-set of graphs into subgraphs of a certain genus. We refer the reader to [257] for a survey of these results.

5.3 Book Embeddings

Books provide another drawing board on which to depict a graph. A *page* is closed half-plane. A *book* is a collection of pages identified along the boundary of the half-planes. This common boundary is called the *spine*. How many pages are needed to depict a graph?

Theorem 5.1 *Any graph embeds in a book with three pages.*

The following proof is due to Babai: draw the graph in the plane so that all crossings involve only two edges and these crossings all lie on a common line. This plane forms two pages of the book. Sew the third page along this line. The crossings can easily be removed using this extra page.

A more common form of book embeddings is to require that the vertices lie along the spine of the book and that edges lie entirely in one page. These restrictions are useful in the applications to VLSI layouts, where the pages can represent circuit boards, or queues used in scheduling tasks [171].

Define the *page-number* of a graph $pn(G)$ as the minimum number of pages need to draw G in this manner.

Since two pages form $\mathcal{R}^2$, one might guess that every planar graph has page-number 2. However, if this were so, then one could always add edges along the spine so that the given graph was a spanning subgraph of a Hamiltonian graph. This cannot always be done (although it is true for triangle-free graphs). The best bound is due to Yannakakis [263] who showed that any planar graph can be embedded in a book with 4 pages, and that 4 pages were sometimes necessary.

The page-number of K_n is $\lceil n/2 \rceil$. The lower bound can be easily seen since in any ordering of the vertices along the spine there is a set of $\lceil n/2 \rceil$ edges which cross pairwise and hence must lie on different pages. The upper bound is established by example.

The page-number of the complete bipartite graph is surprisingly a much harder question and is still unknown. The best known bound [164] is $pn(K_{n,m}) \leq \lceil (2n+m)/4 \rceil$.

We refer the reader to the seminal works by Chung, Leighton, and Rosenberg [72] and Bernhart and Kainen [38] for further discussions on the subject.

5.4 Relative Embeddings

As noted in Section 2.5 any embedding can be represented in terms of a rotation scheme and a signature. In this section we examine embeddings in which a portion of the rotation has been pre-ordained. We introduce this topic with nonorientable embeddings.

Suppose that we are given a graph G and edges $a = uv, b = uw$. What can we say about embeddings in which a is adjacent to b in the local rotation at u? To answer this question, create a related graph G' by adding in a new edge vw which lies alongside a and b so that vuw is a face of the embedding. Any embedding of G' where vuw is a face corresponds to an embedding of G with a adjacent to b in the local rotation. Thus the local rotation constraint on G is equivalent to the fixed-face constraint on G'.

If we are given a set of restrictions on the local rotations, then we can iterate the above process to form a related graph encoding all of the restrictions. If an edge is unaffected by the restrictions, then we can replace it by a digon which forms one of the distinguished walks. Thus we can assume that every edge is in exactly one of the distinguished walks.

This motivates the following definition. A *relative graph* is a graph G' together with a collection of walks which partition the edge set. A *relative embedding* of G' is an embedding of the underlying graph such that the distinguished walks are face boundaries. Each edge lies on two faces, exactly one a distinguished walk. A relative graph G' is a *fat graph* if each of the fixed faces are of length two. In this case we can recover the underlying graph G by replacing each digon with a single edge. Relative embeddings of G' are equivalent to (the usual) embeddings of G.

In the orientable case we require the underlying relative graph to have a fixed direction on each edge. The distinguished walks must respect these directions. The embeddings are those in which the directed walks are faces in the oriented surface.

Relative embeddings have proven useful in studying the amalgamations of graphs [8, 9, 215, 216], the distribution of embeddings [219], and in re-embedding theorems [10]. Bonnington [45] has shown the relative analogue of Xuong's theorem, and Archdeacon, Bonnington and Siran [15] have shown the relative version of Nebesky's Theorem.

5.5 Signed Embeddings

Recall that an embedding in a nonorientable surface is characterized by a signature and a rotation. What can be said about embeddings if the signature is pre-ordained? Define a signature on a closed walk as the product of the signatures on its edges. In an embedding using this signature, a walk is orientation reversing if and only if it is signed minus. So an embedding of a signed graph can be interpreted as preordaining the orientation preserving orientation reversing walks.

For the general theory of signed embeddings we refer the reader to the works of Zaslavsky [268, 269, 270]. We mention three particular results of interest to the author. Zaslavsky [266] has determined the maximum Euler genus among all signed graphs on n vertices where every edge is signed negatively. This is given by $\lceil n(n-3)/4 \rceil + 2$ (where $n \geq 6$). He has also [267] characterized by forbidden minors the projective planar signed graphs. Širáň [212] has investigated the spectrum of the signed genus and notes that no interpolation theorem like Duke's holds in this setting.

5.6 Embeddings in 3-space

To date our focus has been to depict graphs in topological spaces which are for the most part locally 2-dimensional. This is due in part to the following theorem.

Theorem 5.2 *Any simplicial graph embeds in 3-space with all edges straight lines.*

We do not require arbitrary embeddings to be rectilinear, but to avoid pathologies we do require embeddings to be piecewise linear.

The above theorem shows existence, but there is much that can be said about embeddings in 3-space. One question is when can the graph be embedded so that the cycles are all "nice". There are various different notions of "nice"; we give three. An embedding is *linkless* if no two disjoint cycles are linked in 3-space (that is, if they can be pulled apart). An embedding is *knotless* if each cycle is unknoted (that is, if the fundamental group of the complement is free). An embedding is *flat* if each cycle bounds a disk disjoint from the rest of the graph. Note that flat implies linkless and knotless.

When does a graph have a linkless, knotless, or flat embedding? Sachs [204] and Conway and Gordon [73] showed the following.

Theorem 5.3 *Any embedding of K_6 in 3-space contains a pair of disjoint cycles which are linked.*

Robertson, Seymour, and Thomas [197, 198] proved that a graph admits a flat embedding if and only if it admits a linkless embedding, settling a conjecture of Böhme [43]. So while the two concepts are different for specific embeddings, the class of graphs so embeddable are equivalent. More strongly, they developed a theory of spatial embeddings in which they proved that if two flat embeddings are not ambient isotopic, then they differ on a subdivision of K_5 or of $K_{3,3}$. They also showed that any flat embedding can be transformed to any other flat embedding by a series of 3-switches, similar to Whitney 2-switches for planar graphs. Finally, they showed that a graph admits a flat embedding if and only if it does not contain as a minor one of the seven graphs. These seven graphs are the $\Delta Y \Delta$-equivalents of the Petersen graph.

Negami has given a type of Ramsey theorem for knots [169].

Theorem 5.4 *Let N be a knot. There exists an $f(N)$ such that any drawing of $K_{f(N)}$ with straight line segments contains a cycle which is equivalent to N.*

Notice here that rectilinear embeddings are necessary. Otherwise, each edge could be embedded in a very knotty fashion so that every cycle was a very complicated knot type. A similar theorem for complete bipartite graphs is given by Miyauchi [158].

5.7 Pseudosurfaces

A *pseudosurface* is a topological space formed from a (not necessarily connected) 2-manifold by taking a finite number of points, partitioning these points into parts, then topologically identifying the points within each part. The points which are not locally $\mathcal{R}^2$ are called *pinch points*. For example, the *spindle surface* is formed by identifying the north and south poles on a sphere. Equivalently, it is formed from the torus by identifying a noncontractible cycle to a point. The *banana surface* is formed from two spheres by identifying their north poles to a single point and their south poles to a second point. (The surface looks like two bananas joined at their bottoms as well as at their tops.)

Embeddings of graphs into pseudosurfaces usually carry the restriction that the pinch points are vertices of the embedded graph. They can be described combinatorially using local rotations that are simply permutations of the incident edge ends, not necessarily cyclic permutations. The number of cycles in the permutation is the number of disks meeting at that point.

Many of the questions asked of surfaces have also been asked of pseudosurfaces. For example, does there exist a finite set of graphs whose topological exclusion characterizes embedding of a particular pseudosurface? Bodendiek and Wagner [42] have shown that the answer is yes for the spindle surface. Širáň et al. [41, 213] showed that the answer is no for the banana surface, finding an infinite class of graphs which do not embed on the banana surface but such that each proper subgraph does so embed (however there are only 84 irreducible graphs of connectivity 2). In this context, note that embedding on a pseudosurface is not hereditary under minors because of the restriction that vertices must lie on pinch points. An edge joining two pinch points cannot be contracted in the pseudosurface. Hence Siran's result does not contradict the Robertson and Seymour proof of Wagner's conjecture.

Coloring graphs embedded on psuedosurfaces or families of pseudosurfaces has a rich history. We first examine pseudosurfaces formed from the sphere by identifying points (*pinched spheres*). An *M-pire* is a graph embedded on a pinched sphere where each pinch point corresponds to at most M spherical points. What is the maximum chromatic number of all M-pires? The name arises from the dual map coloring problem where each country (empire) may have as many as M components, all of which must receive the same color.

Heawood [110] showed an upper bound (not surprisingly based on Euler Characteristic) of $6M$ for $M \geq 2$ and gave a map with 24 regions broken into 12 M-pires for $M = 2$. The upper bound is exact as shown by the constructions of Jackson and Ringel [120]. Note that the bound is wrong for $M = 1$ by the 4-color-theorem.

We next examine graphs formed from two spheres by identifying pairwise points in the first with points in the second. We express the appropriate coloring problem in dual form. Consider two planar graphs, G_E on the "earth" and G_M on the "moon", together with a bijection $\phi : F(G_E) \rightarrow$

$F(G_M)$. An *earth-moon* coloring assigns colors to the faces of both graphs such that in both graphs adjacent faces receive distinct colors but f and $\phi(f)$ receive the same color. The idea is that each country on the earth has a colony on the moon and a country and its colony should receive the same color. What is the maximum chromatic number of all earth-moon graphs? The problem is equivalent to the dual problem of determining the maximum chromatic number of all graphs of thickness two. An Euler argument shows that 12 colors suffice; an example due to Sulanke (cf [92]) needs 9 colors. The bound has not been tightened further.

We refer the reader to [92, 117] for expository articles on earth-moon colorings and M-pires.

6 Locally Planar Embeddings

The class of planar graphs is one of the most important in graph theory. Some graphs are not planar, but they are if you look closely enough. For example, consider $C_{100} \times C_{100}$ embedded in the torus in a natural manner. The graph is not planar; however, if you fix a vertex and look at a local neighborhood the embedding looks planar. In this example the subgraph induced by all vertices of distance at most 49 from a fixed vertex is planar. The local planarity of the surface is reflected by the fact that such large local neighborhoods of vertices are planar. It is hoped that such locally planar graphs share properties with planar ones.

We develop three different measures of the local planarity of a graph G embedded in a surface S. The *edge-width* of the embedding, $ew(G)$, is the length of the shortest walk in the graph which is non-homotopically null in the surface. Such a walk is necessarily a simple cycle. In fact, it is the shortest simple cycle which does not bound a disk in the surface. This parameter was first introduced by Thomassen [231]. The dual-width, $dw(G)$, is the edge-width of the dual embedding. The *face-width* $fw(G)$ is the minimum $n = C \cap G$ taken over all noncontractible C in the surface. A cycle C achieving this minimum can be chosen to be simple and intersecting only vertices of the graph. The face-width of the graph is also known as the *representativity* of the embedding. The idea behind representativity is that the parameter measures how well the embedded graph represents the surface, where the idea behind the width is that the embedded graph measures how wide the handles are (in terms of the graph). A third point of view is that these parameters measure the density of the graph embedded on the surface. Finally, this parameter measures the local planarity of the embedding. Observe that the face-width of an embedding is equal to the face-width of the dual.

The face-width was first introduced by Robertson and Seymour [193]. The first extensive study of this parameter was by Robertson and Vitray [200]. Several special cases have been commonly used. For example, an embedding of a connected G is of $fw \geq 1$ if and only if it is cellular. Likewise, embeddings of 2-connected graphs with $fw \geq 2$ have been called *circular* (each face is bounded by a simple cycle), or *closed 2-cell*, *CTC* (the closure of each face is a closed 2-cell). Embeddings of 3-connected graphs with $fw \geq 3$ are called *polyhedral*. In a polyhedral embedding the face boundaries are all induced nonseparating cycles.

In the following subsections we discuss embedded graphs which are minimal with respect to these width parameters, what these parameters tell us about other embeddings, coloring locally planar graphs, and how to find cycles of a special homotopy type.

6.1 Minimal Embeddings

Let G be an embedding of a graph with face-width k. This embedding is $(fw = k)$-*minimal* if every embedded minor of G has $fw < k$. A $(fw = k)$-minimal embedding is connected and has the property that the deletion or contraction of any edge will lower the face-width. It follows from Robertson and Seymour's proof of Wagner's Conjecture (discussed in Section 7) that for each fixed surface S there are only finitely many $(fw = k)$-minimal embeddings. Malnič and Mohar [154] proved this directly when $k = 2$. The general case was also shown directly by Malnič and Nedela [155] and by Gao, Richter, and Seymour [90].

Barnette [24] and independently Vitray [247] found the k-minimal graphs for the projective plane. Barnette [25, 26, 27] has also discussed various ways to generate triangulations, polyhedral, and closed 2-cell maps in simple surfaces.

Randby (cf [200]) has shown that any $(fw = k)$-minimal embeddings in the projective plane can be obtained from a certain $k \times k$ projective grid by a sequence of $Y\Delta$-transformations. Schrijver [206] has shown that for the torus there are exactly $(k^3 + 5k)/6$ (k odd) classes of $k - minimal$ embeddings up to $Y\Delta$ and ΔY transformations. The corresponding number for k even is $(k^8 + 8k)/6$.

Hutchinson [114, 115] has shown that any triangulation with face-width k of the surface with genus g has at least $ck^2 g/\log^2 g$ vertices. Przytycka and Przytycki [174] gave such constructions with $ck^2 g/\log g$ vertices (the constants are different).

Schrijver examined the following refinement of face-width. A *kernel* is an embedded graph such that any edge deletion or contraction decreases the face-width in some homotopy class, that is, the global face-width may not be decreased but for some curve C the minimum $G \cap C'$ over all C' homotopic to C decreases. The name comes from doing deletions and contractions which change the width of no homotopic class until arriving at a "core" minor where no further such operations are possible. Schrijver [207] observed that taking the dual and doing ΔY and $Y\Delta$ transformations do not change the width in any homotopy classes. Conversely, he showed [207] that any two embedded kernels with the same width in each homotopy class were equivalent up to duality, ΔY and $Y\Delta$ transformations.

6.2 Re-embedding Theorems

A 3-connected graph has at most one planar embedding. Are embeddings which are locally planar necessarily unique? Are they necessarily minimum genus?

In one sense the answer to the two questions above is yes. Robertson and Vitray [200] proved that a 3-connected graph embedded in a surface of genus g with face-width greater than $2g + 2$ is necessarily a unique minimal genus embedding. Seymour and Thomas (personal communication) have improved this to $O(\log g)$, and then [211] to $100 \log g/\log\log g$. Mohar [159] has similar results.

In another sense the answer to the two questions above is no. The bounds above depend on the surface. Robertson and Vitray [200] conjectured that a constant bound sufficed for all surfaces, in particular, 3. However, Thomassen [231] and independently Barnette and Riskin [30] found simple counterexamples involving toroidal graphs with nontoroidal embeddings of face-width 4. (E.g., in a toroidal $C_4 \times C_n$ take $2n$ nonadjacent faces and n homotopic disjoint essential 4-cycles as the $3n$ face boundaries in a face-width 4 nontoroidal embedding.) Robertson and Vitray then raised the conjectured bound to

10^{10}. However, Archdeacon [10] constructed n-connected graphs G_n with two different embeddings of face-width n. The surfaces involved can either be the same (violating uniqueness) or different (violating genus).

Robertson and Vitray [200] and independently Thomassen [231] observed that any nonplanar embedding of a planar graph has face-width at most two. Mohar, Robertson and Vitray [163] characterized embeddings of planar graphs in the projective plane. This was extend to other surfaces in [162]. Mohar [160] showed that *apex* graphs (graphs G with $G - v$ planar for some v) have no nonorientable face-width three embeddings, but they do have orientable ones.

Various authors [146, 147, 166, 167, 168, 188, 28] have examined the uniqueness of embeddings on the projective plane, Klein bottle, and torus. In particular, we note that Lawrenchenko and Negami [148] have found all graphs which triangulate both the torus and the Klein bottle.

The following amazing theorem of Fiedler, Huneke, Richter, and Robertson [84] gives the orientable genus of all projective planar graphs.

Theorem 6.1 *Let G be a projective planar graph embedded with face-width $n > 2$. Then the orientable genus of G is $\lfloor n/2 \rfloor$*

In particular, note that any two embeddings of a projective planar graph must have face-width that differs by at most one. Robertson and Thomas [199] have a similar result giving the orientable genus of graphs on the Klein bottle.

Thomassen [231] has examined *large-edge-width*, or *LEW*-embeddings. These are ones in which the edge-width exceeds the length of the longest face. He has shown that these embeddings are genus embeddings, are unique if the graph is 3-connected, have only Whitney-type switches if the graph is 2-connected, and gives a polynomial-time algorithm for finding such embeddings if they exist.

6.3 Coloring Locally Planar Graphs

Planar graphs can be 4-colored. Can locally planar graphs be 4-colored? No, Fisk [85] constructed graphs of arbitrarily large face-width with chromatic number five. But four is close, as shown by Thomassen [235].

Theorem 6.2 *A graph embedded on the orientable surface of genus g with edge-width at least 2^{14g+6} is 5-colorable.*

The proof involves cutting along a set of cycles to reduce to a planar graph, then invoking a special version of the 5-color theorem. In a similar manner Hutchinson [116] has shown that a graph embedded with large face-width and all faces of even length is 3-colorable.

6.4 Finding Cycles in Embedded Graphs

One use of large face-width embeddings is to guarantee cycles of a certain homotopy type. We examine some results of this type.

When does an embedded graph contain a noncontractible separating cycle? Clearly, the surface involved must be of genus at least two. Barnette conjectured that any simple triangulation has such a cycle. Independently and more generally, Zha conjectured that face-width 3 suffices. The best known result is due to Zha and Zhao [271], who showed that face-width 6 suffices (see also Richter and Vitray [176] and Brunet, Mohar, and Richter [60]).

Brunet, Mohar and Richter [60] have shown that an embedded graph with face width w contains at least $\lfloor (w-1)/2 \rfloor$ disjoint noncontractible homotopic cycles. Schrijver [208] improved this to $\lfloor 3w/4 \rfloor$ for the torus.

Thomassen [235] showed that any graph embedded on the surface of genus g with face-width at least $16(2^g - 1)$ has a set of g disjoint cycles which can be cut along to form a planar graph. Graaf and Schrijver [96] proved that every face-width $w \geq 5$ toroidal graph contained a $C_{\lfloor 2w/3 \rfloor} \times C_{\lfloor 2w/3 \rfloor}$ minor.

We mention the following on cycles in graphs unrelated to homotopy. Whitney proved that every 4-connected plane triangulation is Hamiltonian. A number of authors [17, 29, 83, 229, 240, 243, 244, 228] have investigated generalizations of this concept, relaxing the connectivity, replacing planarity with locally planar on other surfaces, and replacing Hamiltonian with various types of walks. We refer the reader to the survey by Ellingham in this volume.

7 Graph Minors

Recall that a graph H formed from G by a sequence of deleting isolated vertices and deleting or contracting edges is called a *minor*. The theory of graph minors has recently played a central role in topological graph theory. The prominence is due in most part to the recent proof of Wagner's Conjecture by Robertson and Seymour [196].

Theorem 7.1 *In any infinite set of graphs one is a minor of another.*

The theoretical and algorithmic implications of this result are enormous. We illustrate this by rephrasing the Robertson-Seymour theorem in terms of graph properties.

A property is *hereditary* with respect to an order $\leq$ if whenever G has the property, then any $H \leq G$ has that property. For example, the property of embedding on a fixed surface is hereditary under the minor or topological orderings. In an order without infinite descending chains, any graph G not possessing a hereditary property contains an $H \leq G$ which is *minimal* without that property, i.e., H does not have the property but every $K < H$ does have the property. For example, we can talk about graphs that are minor minimal with respect to not embedding on a fixed surface. The set of these minimal elements are pairwise noncomparable. Hence Robertson-Seymour's Theorem implies the following.

Theorem 7.2 *For any hereditary property $\mathcal{P}$ there is a set $M(\mathcal{P})$ such that G has property $\mathcal{P}$ if and only if it has no minor in $M(\mathcal{P})$.*

As a corollary we obtain the powerful generalization of Kuratowski's Theorem mention in Section 4.3.

Corollary 7.1 *For each surface S, a graph G embeds in S if and only if it has no subgraph homeomorphic to one in a finite collection $M(S)$.*

The Robertson-Seymour theorem is sometimes stated in terms of orders. A partial order $\leq$ is a *well-quasi-order* if for any sequence $G_1, G_2, \ldots$ there is an $i < j$ such that $G_i \leq G_j$. A well-quasi-order has no infinite strictly decreasing sequence. If a partial order has no strictly decreasing sequence, then being well-quasi-ordered is equivalent to having no infinite set of noncomparable elements. Since graphs under the minor order have no infinite decreasing chains, the Robertson-Seymour Theorem asserts that this is a well-quasi-ordering.

Trees and tree-like graphs play an important role in the theory of minors. The algorithmic implications of the Robertson-Seymour Theorem are very important. We examine these two aspects in the following subsections.

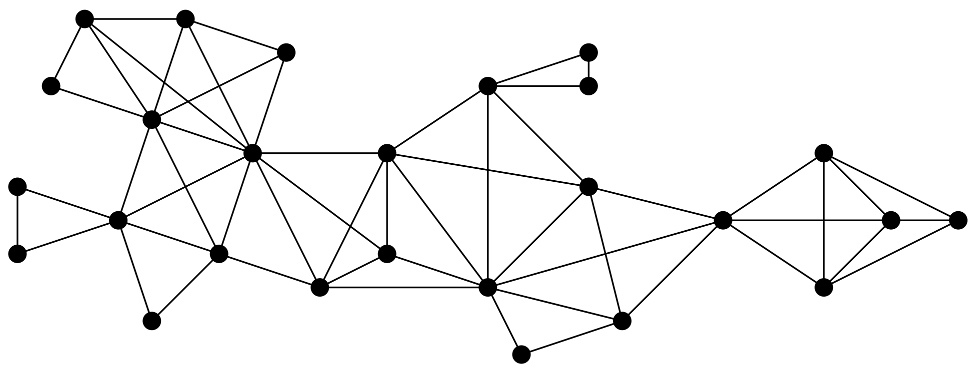


Figure 4: A graph of tree-width three

7.1 Trees and Tree-Width

Kruskal [143] proved in 1960 that rooted finite trees were well-quasi-ordered under the topological containment order. It follows that there are also no infinite antichains in this collection under the minor order. Wagner [253] pointed out that the collection of all graphs does contain an infinite antichain in the topological order. Namely, for each $n \geq 3$ let C_n^2 be the graph obtained from the n cycle by replacing each edge with two edges in parallel. No C_n^2 is topologically contained in another, so these are an antichain. This collection does not violate the Robertson-Seymour Theorem since C_m^2 is a minor of C_n^2 whenever $m \leq n$.

If trees are well-quasi-ordered, maybe so are tree-like graphs. We use the following measure of how closely a graph resembles a tree. A graph G is a *K_n-cockade* if there is a sequence of subgraphs $G_1, \ldots, G_m$ such that each G_j is a complete graph on n vertices and $G_j \cap (G_1 \cup \ldots \cup G_{j-1})$ is contained in some G_i for $i < j$. In other words, a K_n-cockade is formed by repeatedly adding in copies of a fixed complete graph on n vertices by identifying some of the vertices in the j^{th} copy with those in an earlier i^{th} copy. The *tree-width* of a graph H, $tw(H)$, is the smallest n such that H is a subgraph of some K_{n+1}-cockade. Note that a graph is a tree if and only if it is of tree-width 1. If H is a minor of G, then $tw(H) \leq tw(G)$.

The *n-grid* is the planar graph $P_n \times P_n$. For every planar graph G there is an $n = n(G)$ such that G is a minor of the n-grid. The assumption of planarity is necessary, since every minor of a planar graph is planar. The proof of this result is easy conceptually: imagine the graph drawn in the plane with disks for the vertices. Place a very fine grid on the paper. Deform the line segments until they follow horizontal and vertical lines on the grid. A subdivision of the graph is now a minor of the grid after contracting all edges within the vertex disks.

The n-grid is related to the width of the graph. The n-grid has large tree-width [191]. Hence any graph which contains the n-grid as a minor also has large tree-width. The Tree-Width Theorem asserts the converse.

Theorem 7.3 *[190] There exists a function f such that a graph G has $tw(G) \geq f(k)$ if and only if it has a k-grid minor.*

As a partial result toward's Wagner's minor conjecture, Robertson and Seymour [192] were able to show that the class of graphs of bounded tree-width were well-quasi-ordered under minors. Using that result and a structure theorem they were able to prove the general case [196].

7.2 Algorithmic Implications

The graph minors project has produced some fascinating results on structural properties of graphs and proofs of some far-reaching fundamental theoretical results. At the same time it has important algorithmic implications in theoretical computer science. We briefly discuss these implications in this section.

A fundamental quest in computer science is to find efficient algorithms for problems. Many problems are thought to be hard in the sense that there are no algorithms to solve the problem which run in a time bounded by a polynomial in the size of the input. In fact, the existence of polynomial-time algorithms for a large class of problems is known to be equivalent—these problems are called NP-complete. Since the general belief (not proven) is that these algorithms are hard in general, it is interesting to find classes of graphs which do have such polynomial-time algorithms.

A number of problems which are NP-complete in general are polynomial for graphs of bounded tree-width. Here the algorithm is able to exploit the "tree-like" structure of the graph. For example Arnborg [21] has found polynomial time algorithms for k-coloring and Hamiltonicity of graphs of bounded tree width.

A fundamental result due to Robertson and Seymour [195] is the following solution to the k-path problem.

Theorem 7.4 *For each fixed k there exists a polynomial time algorithm for deciding if a graph G with vertices $x_1, y_1, \ldots, x_k, y_k$ has k disjoint paths P_i each joining x_i to y_i.*

As a corollary to the above is the existence of a polynomial-time algorithm for testing if a fixed graph H is a minor of an input graph G. A modification of the algorithm also tells if H is a topological subgraph of G in polynomial time.

A combination of the k-path algorithm and the solution of Wagner's conjecture leads to a very powerful result. There are a finite number of minor-minimal graphs whose exclusion determines when a graph has a hereditary property. Testing for each of these H as a minor of G can be done in polynomial time. Hence we get the following result.

Theorem 7.5 *For any property hereditary under the minor order there is a polynomial-time algorithm to test if a graph has this property.*

This implies, among other things, a (fixed degree) polynomial-time algorithm for testing embeddability into a fixed surface and a polynomial-time algorithm for testing if the tree-width of a graph is less than a fixed constant.

8 Random Topological Graph Theory

To date much attention has been focused on the minimum and maximum genus of a graph. But what does a typical embedding look like? Because we describe our embeddings combinatorially in terms of rotations and signatures, it is possible to pick an embedding at random. What should we expect the drawing to look like?

White [254] has described five models for random topological graph theory. In the first model you fix the graph G and select a rotation uniformly at random (here the embeddings are all orientable). One goal is to study the distribution over all embeddings of the genus of the surface. This embedding distribution was first introduced by Gross and Furst [101]. The complete embedding distribution is known only for a few small graphs and for a few infinite classes. The latter include bouquets [102] (see also [177, 217]), closed ended ladders [89], and cobblestone paths [89].

It is interesting to note that all known embedding distributions are unimodal, in fact, strongly unimodal. It is conjectured that this is always the case [102].

Short of calculating the entire embedding distribution, the next parameter of interest would be the expected value of the genus, also known as the *average genus*. Stahl [218] has given upper bounds on the expected number of regions. These translate to lower bounds on the average genus. Using this Lee [149] has shown that if the number of edges is asymptotic to $cn^{1+\epsilon}$ (where n is the number of vertices), then the average genus is asymptotic to the maximum. In particular, this holds for many classes such as complete and complete bipartite graphs. Stahl [219] does some calculations which show that the average genus is roughly linear for "linear families" of graphs. These are graphs made up of a chain of components arranged in a path-like manner where each component shares only a few vertices with its neighbors. Both Lee and Stahl's results indicate that the average genus is linear in the number of edges. This was shown for simplicial graphs by Chen and Gross [68, 69]. Their work also shows that the set of values of the average genus for 3-connected or for 2-connected simplicial graphs has no limit points. This work leads to a linear-time algorithm for testing isomorphism of graphs from a class with bounded average genus [65].

A second model includes nonorientable embeddings. Here we fix the graph and select a rotation and a signature at random. White [254] shows that in this model the probability that an embedding is orientable is $2^{-\beta(G)}$. In other words, for most families of graphs including complete and complete bipartite graphs almost all embeddings are nonorientable.

A different approach would be to consider the sample space of all labelled graphs on n vertices with edges occurring independently with probability $p(n)$. The first goal here is to find the expected value of the minimum genus of the graph. This was first done by Archdeacon and Grable [16] who showed that for sufficiently large $p(n)$ (including constant probabilities) the minimum genus was roughly $pn^2/12$, the bound given by Euler's formula for triangular embeddings. This was also proven by Rödl and Thomas [201] who refined the bounds on the edge-probability.

We close by noting that Bender, Gao, and Richmond [36] have shown that for each fixed surface almost all rooted embeddings of graphs with m edges have face-width $O(\log m)$.

9 Symmetrical Maps

In the study of mathematics highly symmetric objects are widely studied. Indeed, their symmetries are considered beautiful. The same is true in topological graph theory. Let G be a connected graph embedded on an oriented surface, or an oriented map for short. An *automorphism* of the map G is a function from vertices to vertices and edges to edges that preserves incidence and respects the rotation. That is, it is an automorphism of the graph which extends to an (orientation-preserving) automorphism of the surface. Equivalently, it is a function which preserves oriented region boundaries.

How many automorphisms can an oriented map have? We claim that it can have as many as $2\#E$. In particular, fix a directed edge (x, y). Suppose that arc is carried to the edge (u, v). This target (u, v) determines the whole automorphism. The idea behind the proof is that preserving the local rotation determines where each edge adjacent to (x, y) maps, and this is then extended to the whole graph by connectivity. A oriented map is called *regular* if the order of it's automorphism group is $2\#E$, since acting on arcs the group has order equal to the degree.

A *reflection* of an oriented map is an isomorphism between that oriented map and it's mirror image (the embedded graph with the opposite orientation on the surface). The *extended automorphism group* allows reflections as well as orientation-preserving automorphisms. It is known that the automorphism group of the map is a subgroup of index one or two in the extended automorphism group. It follows that if a regular map admits a reflection, then its extended automorphism group is of order $4\#E$. Maps achieving this bound are called *reflexible*.

In the nonorientable case a map is regular if and only if its automorphism group is of order $4\#E$. There are two automorphisms fixing a directed edge, the identity and a second swapping the faces on either side of that edge. There is some confusion in terminology in the literature regarding when a map is regular. Some authors, especially those working with both orientable and nonorientable surfaces, prefer to call an orientable map regular only if the extended automorphism group is as large as possible, reserving the words *rotary* or *orientably regular* for what I've called here regular non-reflexible maps. The terminology here is the same as that of Coxeter and Moser [75] and Jones and Singerman [124].

There are three main ways to approach regular maps, 1) fix the graph involved, 2) fix the automorphism group involved, or 3) fix the surface involved.

Construction techniques for the first two frequently involve Cayley graphs. In a Cayley graph the vertex set is the set of elements in some group Γ and the edges are (g, gh) for $g \in \Gamma$ and $h \in \Delta^* = \Delta \cup \Delta^{-1}$ for some set Δ. This Δ is a *generating set* if it generates the group, or equivalently, if the resulting Cayley graph is connected. In a Cayley graph the group acts transitively on the vertices. An embedding of a Cayley graph commonly uses a fixed cyclic permutation on Δ^* to define the local rotation. Depending on properties of the group, generating set, and the cyclic rotation, the embedding may be regular.

We do not examine Cayley graphs further in this section, but instead refer the reader to White's survey in this volume. We do, however, mention that James and Jones [121] have completely classified the orientable maps based on complete graphs. We also mention that several papers [108, 20] have examined lifting automorphisms from an embedded voltage graph to its derived covering map leading to some nice constructions. This technique involves some but not all of the algebraic power of Cayley graphs.

We turn our attention to regular maps on a fixed surface. Hurwitz examined finite sets of homeomorphisms of a surface with itself. He showed the following.

Theorem 9.1 *Every finite homeomorphism group of* S_g $(g \geq 2)$ *with itself has order at most* $168(g-1)$.

Tucker [242] related the general problem of finite homeomorphism groups to graphs with the following theorem.

Theorem 9.2 *For any finite group* H *of homeomorphisms of* S_g*, there exists a Cayley graph* G *on the vertex set* H *embedded on* S_g *such that each isomorphism of* G *extends to a homeomorphism of* S_g*.*

The preceding two theorems do not imply that there are only finitely many Cayley graphs of each genus $g \geq 2$. Instead, Wormald (see [103]) constructs infinitely many Cayley graphs of genus 2, despite the fact that this surface has only finitely many homeomorphisms groups.

Babai [23], and independently Thomassen [233] have shown that this behavior is an anomaly of S_2. In particular, they showed that there are only finitely many vertex-transitive graphs of given genus $g \geq 3$.

Yet another kind of symmetry for an embedded graph is self-duality, that is, the embedded map is isomorphic to the dual map. A slightly weaker version is to require that G and G^* be isomorphic as graphs, but allows their embeddings to be different. Some interesting results on self-dual planar graphs are given by Archdeacon and Richter [19] and by Servatius and Servatius [209, 210]. We refer the reader to the survey by Archdeacon [11].

Combining these types of symmetry, Archdeacon, Širáň, and Škoviera [20] have constructed classes of self-dual regular maps. Payley maps (cf [255]) have the remarkable property of being regular, self-dual, and self-complementary! They also give rise to self-dual partially balanced incomplete block designs.

10 Ten Problems

Despite the plethora of results presented, there remain many interesting questions. We offer the following unsolved problems.

Problem 10.1 *Find an easy (i.e., noncomputer) proof of the 4-Color-Theorem.*

Problem 10.2 *Show that every 2-edge-connected graph has a set of simple cycles which together contain every edge exactly twice.*

Problem 10.3 *Show that every 2-edge-connected cubic graph has a set of six perfect matchings which together cover every edge exactly twice.*

Problem 10.4 *Show that every simple triangulation of an orientable surface can be edge-3-colored so that each color appears on each face.*

Problem 10.5 *Find the earth-moon coloring number.*

Problem 10.6 *Find the crossing number of $K_{n,m}$.*

Problem 10.7 *Find the crossing number of K_n.*

Problem 10.8 *Find the genus of $K_{p,q,r}$.*

Problem 10.9 *Show that a graph has a planar cover if and only if it embeds in the projective plane.*

Problem 10.10 *Find a regular map for each nonorientable surface.*

Problem 10.2 is called the *Cycle Double Cover Conjecture*. It was independently posed by Seymour and by Tutte. It is equivalent to claiming that every graph has an embedding such that the dual is loopless. Problem 10.3, *Fulkerson's Conjecture*, is a type of dual to Problem 2.

David Craft has found many of the embeddings needed for Problem 10.8. Note that if $p \geq q \geq r$, then the number of triangles is at most $2qr$. Assuming all other faces are quadrilaterals gives the conjectured bound through Euler's formula.

Regarding Problem 10.9, a graph G covers H if there is a graph map from G to H which is an isomorphism on the neighborhood of each vertex. A graph embedding in the projective plane has a 2-fold planar cover. Since having a planar cover is preserved under minors, it suffices to show that the 35 minor-minimal non-projective-planar graphs do not have planar covers. There are two remaining cases, $K_7 - 3K_2$ and $K_{4,4} - K_2$ whose proof would complete the general result.

11 Conclusion

I must express my regret at the vast areas of topological theory which I was not able to cover in this survey. Several areas in particular are worthy of inclusion in any survey.

I direct the reader to Carsten Thomassen's wonderful work using graph theory to prove results in topology. Among his results are graph theoretic proofs of the Jordon Curve Theorem [234], a deeper understanding of the relationship between the Jordon Curve Theorem and Kuratowski's Theorem [232], and a nice proof that every surface admits a triangulation [234].

Another area not included is the dual theories of current and voltage graphs. This techniques uses quotient structures under group actions to describe embeddings. This allows an economical description of embeddings. It proved essential to the proof of the Map Color Theorem. The reader is referred to the seminal articles by Gross and Alpert [98, 99, 100] (see also [12]). A more leisurely introduction is the book by Gross and Tucker [103].

Other surveys on topological graph theory are presented by White and Beineke [257] and Carsten Thomassen [238, 239]. The reader is referred to [71, 62] for additional results on snarks. Joan Hutchinson [117] gives a nice exposition on map colorings, empires, and the earth-moon problem. In [161], Mohar gives a survey of results on local planarity. Robertson, Seymour and Thomas survey linkless embeddings in [197].

The book by Jensen and Toft [123] on graph colorings cannot be recommended highly enough.

I learned a lot writing this paper. I hope that the reader has also found it informative.

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