Independent sets, perfect graphs

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Independent vertex sets

Reminder

We call a set $F \subseteq V(G)$ an independent vertex set if every edge of G has at most one endpoint in F, i.e., if the vertices of F are not connected by an edge.

This concept is already well-known from our previous studies.

Using it, we define the following polytope.

Definition, vertex packing polytope

$$\mathcal{PP}(G) := \text{conv}\,\{\chi_F : F \subseteq V(G) \text{ independent vertex set}\} \subseteq \mathbb{R}^V,$$

where $\chi_F \in \{0,1\}^V$ is the characteristic vector of the vertex set F.

Simple inequalities

Observation

We can write a simple containment for this set:

$$\mathcal{PP}(G) \subseteq \{x \in \mathbb{R}^V : x \succeq 0, \text{ and for every } e = uv \text{ edge}, \\ x_u + x_v \leq 1\} =: \mathcal{PP}_0(G).$$

The set on the right-hand side consists of non-negative vectors for which the sum of two components is less than or equal to 1 whenever there is an edge between the corresponding vertices in G.

To establish the containment, we only need to verify that the characteristic vectors of independent vertex sets are contained in the intersection of the half-spaces described on the right-hand side (which obviously defines a convex vertex set).

Case of Bipartite Graphs

The following theorem, based on what we learned about totally unimodular matrices, can be easily proven (we omit the formal proof).

Theorem

Let G be a bipartite graph without isolated vertices. Then

$$\mathcal{PP}(G) = \mathcal{PP}_0(G).$$

So, bipartiteness is sufficient for our obvious upper bound described with inequalities to coincide with our combinatorially defined vertex set.

The inequalities beyond the sign conditions in the description of $\mathcal{PP}_0(G)$ are called edge conditions.

In the general case, further conditions are necessary.

Clique Conditions

Among the missing conditions, perhaps the simplest ones are given below.

Definition, clique conditions

$$\widehat{\mathcal{P}\mathcal{P}}(G) := \{x \in \mathbb{R}^V : x \succeq 0, \text{ and for every } K \text{ clique}, \sum_{u \in K} x_u \leq 1\}.$$

Simple Containments

• A pair of vertices connected by an edge actually forms a clique of size two. Thus, the edge conditions are special clique conditions. Therefore,

$$\widehat{\mathcal{PP}}(G) \subset \mathcal{PP}_0(G).$$

• A clique and an independent vertex set can have at most one common vertex (since if there were two vertices, they would have to be connected, which they are not).

• When considering the components corresponding to the elements of a clique in the characteristic vectors of independent vertex sets, only one common 1 is allowed.

• Thus,

$$\mathcal{PP}(G) \subseteq \widehat{\mathcal{PP}}(G).$$

• Summarizing,

$$\mathcal{PP}(G) \subseteq \widehat{\mathcal{PP}}(G) \subseteq \mathcal{PP}_0(G)$$

Example

Example

Now we show that, for example, in the case of the five-cycle graph, $\widehat{\mathcal{PP}}$ is strictly larger than $\mathcal{PP}$.

Since C_5 has only one- and two-element cliques, it is sufficient for elements of $\widehat{\mathcal{PP}}(C_5)$ to satisfy the edge conditions (the sum must be less than one for adjacent vertices).

For example, $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \in \widehat{\mathcal{PP}}(C_5)$.

Note that $\alpha(C_5) = 2$, so the independent vertex sets have at most two elements.

Thus, in their characteristic vectors, the sum of components does not exceed 2. The same holds true for any element of their convex hull.

However, the vector $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ does not fit this condition.

Another Example

Example

For C_3, not only the edge conditions appear, but also the inequality $x_1 + x_2 + x_3 \leq 1$.

- It's easy to verify that

$$\{x \in \mathbb{R}^3 : x \succeq 0, x_1 + x_2 + x_3 \leq 1\} = \mathrm{conv}(0, e_1, e_2, e_3),$$

where e_i are the standard basis, i.e., one-element subsets (which are exactly the nonempty matchings in K_3) characteristic vectors.

- $\mathcal{PP}(C_3) = \widehat{\mathcal{PP}}(C_3)$ holds.
- $$\widehat{\mathcal{PP}}(C_3) \subsetneq \mathcal{PP}_0(C_3) \ni (1/2, 1/2, 1/2)$$

- Under what conditions on G will $\mathcal{PP}(G) = \widehat{\mathcal{PP}}(G)$ hold? We explore this question further, but it requires a small digression into graph theory.

Break

Reminder

Let c be a proper coloring, and K be a clique of G. We know that a clique can only be colored by assigning different colors to each of its vertices. Therefore, the number of required colors is at least $|K|$.
We obtain the graph parameter $\chi(G)$ by considering the minimal number of colors needed, and $\omega(G)$ by considering the maximal clique size. However, the inequality holds true even between these optima, so $\chi(G) \geq \omega(G)$.

Definition

G graph is nice if $\chi(G) = \omega(G)$.

The Main Character

Definition

G graph is perfect if every induced subgraph R of G is nice.

Usually, we only mention the authors of major theorems and proofs, but in some cases, it's worth mentioning the inventors of important definitions as well. Perfect graphs were first defined by Claude Berge, a French mathematician, in 1962.

Examples I.

Example

The complete graph with n vertices, K_n, and the empty graph with n vertices, $\overline{K_n} = E_n$, are both perfect.

- In complete graphs, $\chi(K_n) = \omega(K_n) = n$.
- In empty graphs, $\chi(E_n) = \omega(E_n) = 1$ for all n.
- This is only a small part of the definition.
- But also note that any induced subgraph of a complete graph or an empty graph remains complete or empty, respectively.
- Thus, both are perfect.

Examples II.

Example

If G is bipartite, then it's also perfect. Here we provide an entire class of examples. As we know, any induced subgraph of a bipartite graph is also bipartite, so it suffices to prove the niceness of bipartite graphs. Suppose G has an edge. Then it can be colored with at most two colors, and its maximal cliques are the edges, so $\chi(G) = \omega(G) = 2$.

Example

If G is bipartite, then $\overline{G}$ is perfect.
In the complement of bipartite graphs, the lower and upper vertex sets form cliques, and the remaining edges connect vertices between these sets. The niceness of this is nontrivial, equivalent to Kőnig's theorem. We leave the verification to the interested student.

Examples III.

Example

Consider an arbitrary partially ordered set $(P, <)$. The partial ordering means that $<$ is a relation, but we don't require that any two elements of P be comparable.
From this set, we can define a graph by taking the vertex set as P and connecting two points only if they are comparable as elements of the partially ordered set. This is called the comparability graph of $(P, <)$.
Analogously, we can define the incomparability graph, and we can observe that these two concepts are graph theoretically related by complementation.
The interested student can also verify that both are perfect graphs.

Examples IV.

Example

C_{2k+1} ($k \geq 2$), an odd cycle of length $2k+1$, is not perfect and not even nice, as $\chi(C_{2k+1}) = 3 \neq 2 = \omega(C_{2k+1})$.
This does not apply to the triangle graph, so we cannot generally speak of odd cycles, only those with more than five vertices.
We leave it to the interested student to verify that $\overline{C_{2k+1}}$ ($k \geq 2$) is also not nice.

A Note

Notation

$G \sqsupseteq R$ means R is an induced subgraph of G.

Observation

After the definition of perfection and the examples, the following are trivial:

(i) If G is perfect and $G \sqsupseteq R$, then R is also perfect.

(ii) If G is not perfect and $S \sqsupseteq G$, then S is also not perfect.

(iii) If $G \sqsupseteq C_{2k+1}$ or $G \sqsupseteq \overline{C_{2k+1}}$ for some $k \geq 2$, then G is not perfect.

For all known non-perfect graphs, non-perfection follows from observation (iii). Based on this, Berge first formulated in 1962 as a conjecture that observation (iii) can be reversed (strong perfect graph conjecture).

Strong Perfect Graph Conjecture/Theorem

It was only proven in 2006 by the quartet of Chudnovsky—Seymour—Robertson—Thomas. Their paper is over 100 pages long, and the proof of the theorem alone would fit into a semester-long PhD course.

Theorem (Strong Perfect Graph Theorem, Chudnovsky—Seymour—Robertson—Thomas, 2006)

G is not perfect if and only if it does not contain C_{2k+1} and $\overline{C_{2k+1}}$ graphs as induced subgraphs for $k \geq 2$.

Let G be perfect. We know it cannot contain any odd cycle of length at least five or its complement as induced subgraphs. This property naturally holds for its complement. Thus, if the strong perfect graph conjecture is true, then $\overline{G}$ is also perfect. Consequently, the perfection of G and $\overline{G}$ go hand in hand.

Weak Perfect Graph Theorem

Berge also saw this line of thought. He couldn't prove it (it might be possible with the proof of the strong conjecture absent). He referred to this assertion as the weak perfect graph conjecture. It remained unsolved *only* for ten years.

Theorem (Weak Perfect Graph Theorem, Lovász, 1972)

G is perfect if and only if $\overline{G}$ is perfect.

Without the strong version, this theorem is not a trivial statement either, but it can be proven in a single lecture of an MSc course. In the rest of the lecture, our aim will be to prove this theorem.

Break

Perfect Graphs and Polytopes

Theorem (Fulkerson—Chvatal Theorem, 1973)

The following are equivalent:

(i) $\mathcal{PP}(G) = \widehat{\mathcal{PP}}(G)$

(ii) The vertices of $\widehat{\mathcal{PP}}(G)$ are integers

(iii) G is perfect

The equivalence $(i) \iff (ii)$ is based on a simpler statement, which we do not formally describe. After the previous ideas and reasoning, the interested student can easily understand it.

Two Sides of the Fulkerson—Chvatal Theorem

The proof of $(ii) \iff (iii)$ relies on the following assertions:

$$G \text{ is perfect} \Rightarrow_{(\mathrm{A})} \text{The vertices of } \widehat{\mathcal{PP}}(G) \text{ are integers} \Rightarrow_{(\mathrm{B})} \overline{G} \text{ is perfect.}$$

Repeating the above inferences for $\overline{G}$, we obtain the complete proof of the Fulkerson—Chvatal Theorem and, along with it, the weak perfect graph theorem.

It also follows that for non-perfect graphs, beyond the clique conditions, additional inequalities are necessary for describing $\mathcal{PP}$. This research direction remains active and important to this day.

Graph Theory Preliminaries, 1st Lemma

To prove (A), we will first establish two lemmas.

Lemma

G is perfect if and only if every spanning subgraph satisfies

$(\star)$: there exists an independent set intersecting every optimal (maximum size) clique.

Proof of 1st Lemma ($\Longleftarrow$)

Since the property ($\star$) is inherited by spanning subgraphs, it suffices to prove the niceness of G.

Let F_1 be an independent set in G satisfying ($\star$).

Let $k := \omega(G)$. Then $\omega(G - F_1) = k - 1$, as F_1 cuts at least one vertex from each maximal clique, and it cannot cut more because F_1 is an independent set and can have at most one common vertex with a clique.

We can apply property ($\star$) again to $G - F_1$, obtaining an independent set F_2 for which $\omega(G - F_1 - F_2) = k - 2$.

Repeating this process $(k - 1)$ times, we reach the graph $H := G - F_1 - F_2 - \ldots - F_{k-1}$, where $\omega(H) = 1$. This means that H itself is an independent set.

Thus, we have obtained k independent sets, a k-coloring.

Proof of 1st Lemma ($\Longrightarrow$)

We will prove from the niceness of G that it satisfies ($\star$).

Indeed, let $\chi(G) = \omega(G) = k$ and consider one of its optimal colorings with k color classes.

These classes are independent sets, so if we take an optimal clique, it can intersect each color class at most once, but due to the niceness of G, it is necessary that it also contains one vertex from each class.

Thus, by choosing any color class, we exhibit a set satisfying ($\star$).

A Note

At the end of the previous proof, we saw that we have great freedom in choosing the set satisfying $(\star)$, as any color class of an optimal coloring suffices. Based on this, we can replace $(\star)$ with a stricter condition, the fulfillment of which remains equivalent to the perfection of G for every spanning subgraph.

Lemma$^+$

G is perfect if and only if every spanning subgraph satisfies

$(\star)^+$: for any $x \in V(G)$, there exists an independent set containing x, intersecting every optimal clique.

For the final lemma, we will need a graph-theoretical operation.

Definition

Let G be a graph with an associated vector $n = (n_v)_{v \in V} \in \mathbb{N}^V$. Each component assigns a natural number to every vertex.

The *blow up* of G (with the vector n) is the graph obtained by replacing each vertex with a clique of size n_v, and connecting the vertices of these resulting cliques (each with all others) if and only if the corresponding vertices in the original graph were connected.

Graph Theory Preliminaries, 2nd Lemma

2nd Lemma

Let G be a perfect graph and $n \in \mathbb{N}^V$, $n = (n_v)_{v \in V}$ an arbitrary vector. Then G remains perfect when blown up with the vector n.

Remarks

1st Remark

Let H be the graph obtained from G by blow up with n. According to the previous lemma, it suffices to show that $(\star)$ holds for all spanning subgraphs of H. However, H's subgraphs are blown up graphs of G with different vectors, so once we prove that H satisfies $(\star)$ for an arbitrary vector n, we need not concern ourselves with its spanning subgraphs.

2nd Remark

A blow up can be done step by step, only one vertex blown up at a time. Thus, we can assume that $n = (1, \ldots, 1, n_v, 1, \ldots, 1)$, as the ones in the vector represent cliques of a single vertex. The blown up vertex is denoted by v.

Proof of 2nd Lemma

H is the graph obtained from the perfect graph G by blowing up the vertex v.

According to the previous lemma, we know $(\star)^+$ for G, so there exists an independent set F in G such that $v \in F$ and it intersects every optimal clique.

Given F, we can construct an independent set F' in H, defined as $\{v$ or one of the vertices blown up from $v\} \cup (F \setminus v)$. We need to show that F' satisfies $(\star)$ for H, i.e., intersects every optimal clique.

Notice that the optimal cliques in H can be of two types:

a) Contains the entire blown up vertex set of v,

b) Corresponds to an optimal clique in G which does not contain v.

Type a) cliques are intersected by F' because it includes one of the vertices blown up from v. Type b) cliques are intersected because of $F \setminus \{v\}$. This completes the proof of the lemma.

G is Perfect $\Rightarrow_{(A)}$ Vertices of $\widehat{\mathcal{PP}}(G)$ are Integers

Let G be perfect, $e \in \widehat{\mathcal{PP}(G)}$ a vertex of the polytope.

Our goal is to show that $e \in \mathbb{Z}^V$.

We know that $e \in \mathbb{Q}^V$. Let $e = (\frac{n_v}{N})_{v \in V}$ be the common denominator form of the coordinates.

We use the previous lemma on the graph G and the vector $n = (n_v)_{v \in V}$, thus constructing the blown up graph H.

The lemma assures us that H is perfect.

Let's find the largest clique in H, or in other words, the optimal clique K of G for which $\sum_{v \in K} n_v$ is maximal.

Proof of (A) (continued)

Since $(\frac{n_v}{N}) \in \widehat{\mathcal{PP}(G)}$, we have

$$\sum_{v \in K} \frac{n_v}{N} \leq 1 \quad \text{and}$$
$$\sum_{v \in K} n_v \leq N,$$

implying that H has no clique larger than N.

From the perfection of H, we deduce that it can be colored with N colors.

Thus, there exist independent sets $F_1, \ldots, F_N$ whose union covers $V(H)$.

Proof of (A) (continued)

The sets F_i can be projected back to G: let $F_i \subset V(H)$ correspond to $\Phi_i \subset V(G)$.

The union of Φ_i covers each vertex v n_v times. Thus,

$$\sum_{i=1}^{N} \chi_{\Phi_i} = (n_1, n_2, \ldots).$$

Dividing by N, we obtain:

$$\sum_{i=1}^{N} \frac{\chi_{\Phi_i}}{N} = e.$$

Conclusion of (A)

- The left-hand side represents an *average* of the independent set characteristic vectors (specifically, $\widehat{\mathcal{PP}}$ elements). On the right-hand side, we see a vertex of $\widehat{\mathcal{PP}(G)}$.
- This implies that e is one of the χ_{Φ_i} (indeed, all of them are equal to e). Therefore, e is an integer.
- Indeed: If e is a vertex of $\widehat{\mathcal{PP}}$, then a suitable half-space $\{x \in \mathbb{R}^V : \nu^T x \leq b\}$ contains $\widehat{\mathcal{PP}}$ such that only e satisfies the inequality as an equality.
- Assume that there is a vector in the average that is not e. (Indirect proof)
- This vector strictly satisfies the defining inequality of the half-space mentioned above. The other terms also satisfy the inequality. However, these inequalities can also be averaged. The result will be a strict inequality, meaning that the average vector cannot be e.
- This contradiction proves the claimed relationship.

Proving $\overline{G}$ Perfect if Vertices of $\widehat{\mathcal{PP}}(G)$ are Integers

The vertices of $\widehat{\mathcal{PP}}(G)$ are integers, meaning that they represent characteristic vectors of independent sets.

The half-space $\{x \in \mathbb{R}^V : 1^{\mathsf{T}}x \leq \alpha(G)\}$ contains the $\widehat{\mathcal{PP}}(G)$ polytope.

The boundaries of this half-space correspond to the characteristic vectors of maximal independent sets.

More precisely,

$$\widehat{\mathcal{PP}}(G) \cap \{x \in \mathbb{R}^V : 1^{\mathsf{T}}x \leq \alpha(G)\} = \\ \text{conv}\,\{\chi_F : \ F \text{ is a maximal independent set of size } \alpha(G)\}.$$

This forms a facet of the polytope.

Proof of (B) (continued)

One of the supporting hyperplanes derived from the inequalities defining the polytope must contain it.

Moreover, such a hyperplane must also exist among the clique constraints.

If the hyperplane $\sum_{v:v\in K} x_v = 1$ associated with a clique K contains all characteristic vectors of maximal independent sets, then $\alpha(G-K) < \alpha(G)$.

Conclusion of Proof of (B)

The conclusion of proving statement (B) is that if the vertices of $\widehat{\mathcal{PP}}(G)$ are integers, then the vertices of $\widehat{\mathcal{PP}}(R)$ are integers for every spanning subgraph $R \sqsubseteq G$.

Seeing that we found the above clique K in G, we can find a similar clique K' in $G - K$.

Continuing this process, we can cover $V(G)$ with $\alpha(G)$ cliques. This precisely means that $\overline{G}$ is perfect.

The same line of reasoning holds for every spanning subgraph of $\overline{G}$, proving that $\overline{G}$ is perfect.

This is the end!

Thank you for your attention!