Integer polytopes

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LP Relaxation of IP Problems

Definition

From the following integer programming (IP) problem

Minimize	$c^{\mathsf{T}}x$-t
subject to	$x \in \mathcal{P}$
	$x \in \mathbb{Z}^n$,

if we omit the condition $x \in \mathbb{Z}^n$, we obtain the associated linear programming (LP) problem

Minimize	$c^{\mathsf{T}}x$-t
subject to	$x \in \mathcal{P}$.

This is called the LP relaxation of the original IP problem.

Relationship Between IP Problems and Their LP Relaxations

• IP problems are very general. $\mathcal{NP}$-complete problems can easily be formulated as IP problems. It cannot generally be expected to be efficiently solvable.

• LP problems, however, are efficiently manageable.

• In general, this relaxation is a real simplification. Nevertheless, it also provides useful information about the original problem.

Observation

If the optimum of the original IP problem is p_I^* and that of the LP relaxation is p^*, then

$$p^* \leq p_I^*.$$

• Through the LP relaxation, we easily obtain a lower bound on the optimal value.

Integral Polyhedra

Definition

A polyhedron $\mathcal{P} = \langle g_1, g_2, \ldots, g_k \rangle_{\text{convex}} + \langle h_1, h_2, \ldots, h_\ell \rangle_{\text{cone}}$ is integral iff all generating vectors can be chosen from $\mathbb{Z}^n$.

Definition

$\mathcal{P} = \{x \colon Ax \preceq b\}$ is a regular polyhedron integral if $\mathrm{ext}(\mathcal{P}) \subseteq \mathbb{Z}^n$, meaning that every extremal point has integer coordinates, and $A \in \mathbb{Q}^{k \times n}$, $b \in \mathbb{Q}^k$.

• For polytopes, the previous definition is equivalent to $\mathcal{P}$ being integral if the convex hull of finitely many $\mathbb{Z}^n$ points.

• From the above, if the IP problem defined by the continuous constraints is integral, then the LP relaxation will have integral optimal points (since the vertices of $\mathcal{P}$ are integral). In this case, $p_I^* = p^*$.

Conditions Guaranteeing Integrality of Polyhedra I

Edmonds–Giles Theorem

Let $\mathcal{P} = \{x\colon Ax \preceq b\} \neq \emptyset$ be a polyhedron, $A \in \mathbb{Q}^{k\times n}$, $b \in \mathbb{Q}^k$. Then the following are equivalent:

(i) $\mathcal{P}$ is an integral polyhedron (i.e., $\mathrm{ext}(\mathcal{P}) \subseteq \mathbb{Z}^n$).

(ii) For every $c \in \mathbb{R}^n$ objective vector, the LP problem

Minimize	$c^{\mathsf{T}}x$-t
subject to	$x \in \mathcal{P}$

either has $p^* = -\infty$ or has an optimal point in $\mathbb{Z}^n$.

(iii) For every $c \in \mathbb{Z}^n$, the optimal value of the LP problem

Minimize	$c^{\mathsf{T}}x$-t
subject to	$x \in \mathcal{P}$

is either $-\infty$ or integral.

Initial notes

Note

The equivalence of (i) with (ii) follows from earlier results.

The (i)$\Rightarrow$(iii) is indeed true, as if a linear function attains its minimum on a polyhedron, it does so at a vertex.

If the coordinates of this optimal point and the objective function are integers, then the objective function value is also integral. (Of course, this also establishes the validity of (ii) in this case.)

The proof

(iii)$\Rightarrow$(i)

• Let $e \in \mathrm{ext}(\mathcal{P})$, which means that there exists $\nu \neq 0 \in \mathbb{R}^n$ and $\tau \in \mathbb{R}$ such that

$$(\star) \qquad \mathcal{P} \subset \{x \colon \nu^{\mathsf{T}} x \geq \tau\} \text{ and } \nu^{\mathsf{T}} e = \tau.$$

• The normal vector ν is not unique. Obviously, it can be multiplied by a positive scalar to obtain another possible ν (with a new τ). Geometrically, it is *sensible* and easy to see that for a suitable positive ε, any ν within a distance of at most ε from the original one is also suitable as a normal vector.

• Based on these two remarks, there exists a $\nu \in \mathbb{Z}^n$ vector such that both ν and the vectors $(\nu + e_i)$ are suitable for satisfying $(\star)$, where e_i are the standard unit vectors in n dimensions ($e_i = (0, \ldots, 0, 1, 0, \ldots, 0)^{\mathsf{T}}$).

Proof (continued)

- The ν and $(\nu + e_i)$ are potential c values in condition (iii) of the theorem.
- Furthermore, their corresponding optimal values are $\nu^{\mathsf{T}} e$ and $(\nu + e_i)^{\mathsf{T}} e$.
- Indeed. If we minimize $\nu^{\mathsf{T}} x$ over $\mathcal{P}$, then we obtain at least the same value as if we optimize over the half-space containing $\mathcal{P}$ defined by $\{x \colon \nu^{\mathsf{T}} x \geq \tau\}$. That is, the minimum value is at least τ, which is achieved at the vertex e.
- Therefore, according to (iii), the optimal values of $\nu^{\mathsf{T}} e$ and $(\nu + e_i)^{\mathsf{T}} e$ are integers.
- Specifically, the i-th coordinate of $e \in \operatorname{ext}(\mathcal{P})$: $e_i^{\mathsf{T}} e = (\nu + e_i)^{\mathsf{T}} e - \nu^{\mathsf{T}} e$ is also an integer.
- Thus, each component of e is an integer, i.e., e is an integer vector.

Break

Conditions Guaranteeing Integrality II: Totally Unimodular Matrices

Definition

A matrix $M \in \mathbb{R}^{k\times n}$ is called totally unimodular (TU) if for every square submatrix N, we have $\det N \in \{-1, 0, 1\}$.

- Specifically, a TU matrix has a determinant of 0 or ± 1 for every 1×1 sized submatrix. That is, its elements can only be -1, 0, or 1.

Theorem

If $A \in \mathbb{R}^{k\times n}$ is a totally unimodular matrix and $b \in \mathbb{Z}^k$, then $\mathcal{P} = \{x \in \mathbb{R}^n \colon Ax \preceq b\}$ is an integral polyhedron.

Proof

- Let $e \in \mathrm{ext}(\mathcal{P})$. Specifically, $e \in \mathcal{P}$ and the inequalities that e sharpens are such that the vectors on the left sides span $\mathbb{R}^n$.
- That is, A has rows such as $a_{i_1}^{\mathsf{T}}, \ldots, a_{i_n}^{\mathsf{T}}$, which are linearly independent and satisfy

$$\begin{gathered} a_{i_1}^{\mathsf{T}} e = b_{i_1} \\ \vdots \\ a_{i_n}^{\mathsf{T}} e = b_{i_n}. \end{gathered}$$

- From this (given A and b), e can be expressed using Cramer's rule.

When calculating each coordinate, we work with integers, and there is only one division involved. The divisor is the determinant of a square submatrix of A. The submatrix does not degenerate, so its determinant cannot be 0. Thus, its value is -1 or 1. Dividing by this does not lead to non-integer results.

Totally Unimodular Matrices: Example I

• The vertex-edge incidence matrix of a loopless graph G is denoted by $\mathcal{B}_G$, where the rows correspond to the vertices and the columns correspond to the edges, and at the intersection of a vertex $v \in V$ row and an edge $e \in E$ column, we have

$$(\mathcal{B}_G)_{v,e} = \begin{cases} 1, & \text{if } v \text{ is incident to } e, \\ 0, & \text{otherwise.} \end{cases}$$

• Note that each column of $\mathcal{B}_G$ contains exactly two non-zero elements, two 1s.

• Let G be a complete graph on three vertices: Then

$$\mathcal{B}_{K_3} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

• The complete matrix is a square submatrix of itself. Since $\det \mathcal{B}_{K_3} = 2$, $\mathcal{B}_{K_3}$ is not TU.

Totally Unimodular Matrices: Examples II

- Thus, if G contains a clique of size three, then $\mathcal{B}_G$ contains the above submatrix, specifically, it is not TU.

- Similarly, it can be shown that the vertex-edge incidence matrix of a cycle with an odd length (which is square) also has a determinant of 2.

- Specifically, if a graph contains an odd-length cycle (which is equivalent to being non-bipartite), then its vertex-edge incidence matrix is not TU.

- We will see that if G is a bipartite graph, then $\mathcal{B}_G$ is a TU matrix.

Totally Unimodular Matrices: Examples III

Example

Example $\overrightarrow{G}$ is a loopless directed graph. Then, the vertex-edge incidence matrix $\mathcal{D}$ of $\overrightarrow{G}$ has an element $\mathcal{D}_{v,e}$ (the element at the intersection of the row corresponding to vertex v and the column corresponding to edge e) given by:

$$\mathcal{D}_{v,e} = \begin{cases} +1, & \text{if the edge "enters" the vertex} \\ -1, & \text{if the edge "leaves" the vertex} \\ 0, & \text{otherwise.} \end{cases}$$

It can be seen that each column of $\mathcal{D}_G$ contains one 1 and one (-1), with the remaining elements being 0.

- We will show that for any directed graph G, $\mathcal{D}_G$ matrix is totally unimodular.

Totally Unimodular Matrices: Operations

Lemma

Let A be a totally unimodular matrix. Form $\widetilde{A}$ from A by the following rules/operations:

(i) Multiplying rows/columns by -1.

(ii) Deleting rows/columns.

(iii) Repeating existing rows/columns.

(iv) Adding rows/columns with e_i where e_i contains exactly one non-zero element which is 1.

(v) Transposing.

Then the resulting $\widetilde{A}$ matrix is also totally unimodular.

Totally Unimodular Matrices: Examples and Proofs

Theorem

(i) Let G be any bipartite graph. Then $\mathcal{B}_G$ is a TU matrix.

(ii) Let $\overrightarrow{G}$ be any directed graph. Then $\mathcal{D}_{\overrightarrow{G}}$ is a TU matrix.

• We prove the two statements in parallel for a while. We use complete induction on k.

• The statement holds for $k = 1$ since all elements of both matrices are from the set $\{-1, 0, 1\}$.

• Induction step. Suppose that for square submatrices of size k or less, we know that their determinants are ± 1 or 0. Let N be a $k \times k$ sized submatrix. We need to prove that its determinant is ± 1 or 0.

Proof (continued): 3 cases

Case 1: One of the columns of N contains only 0s. In this case, $\det N = 0$ and we are done.

Case 2: One of the columns of N contains exactly one non-zero value and 0s. We know that the non-zero element is -1 or 1 (in the even case, it can only be 1). Then there exists an expansion for this column, and the induction hypothesis gives the result.

Case 3: The complement of the above two cases. For the two types of matrices, this means that each column contains exactly two non-zero elements.

The proof now splits into two branches. For $\mathcal{D}_{\vec{G}}$, we know that the sum of rows will be the zero vector. For $\mathcal{B}_G$, we know that the sum of rows corresponding to bottom vertices and the sum of rows corresponding to top vertices will both be the all-ones vector. In both cases, there exists a non-trivial linear dependency among the rows. Hence, the determinant is 0, and we are done.

Consequences: Weighted Matching Problem

The Weighted Matching Problem

Given a graph G with an edge weighting $c : E(G) \to \mathbb{R}_+$.

Find a maximum-weight matching, where the weight of a matching/edge set is $\sum_{e:e\in M} c(e)$.

Consequence

The weighted matching problem on bipartite graphs can be solved using an LP algorithm.

Consequences: Weighted Matching Problem as an IP

- Identifying the weight function c with a vector $c \in \mathbb{R}^{E(G)}$, the problem becomes the following

$$
\boxed{
\begin{array}{lll}
\text{Minimize} & c^T x\text{-t} & \\
\text{subject to} & \sum\limits_{e:v I e} x_e \le 1, & \forall\, v \in V \\
 & 0 \le x_e, & \forall\, e \in E \\
 & x_e \in \mathbb{Z}, & \forall\, e \in E
\end{array}
}
$$

integer programming formulation.

- We obtain its LP relaxation by removing the $x_e \in \mathbb{Z}$ constraints:

$$
\boxed{
\begin{array}{ll}
\text{Minimize} & c^T x\text{-t} \\
\text{subject to} & \sum\limits_{e:v I e} x_e \le 1, \\
 & x_e \ge 0.
\end{array}
}
$$

Consequences: Weighted Matching Problem: LP Relaxation

• This is now an LP problem and the matrix is totally unimodular IF G IS BIPARTITE.

• Indeed. The essential part of the matrix is the vertex-edge incidence matrix of the bipartite graph, which we have shown to have the TU property. The TU property of the complete matrix easily follows from this.

• Thus, the LP relaxation's vertices are integer-coordinate, i.e., they correspond to matchings.

• So the LP relaxation is equivalent to the original formulation. An LP problem can be efficiently handled in many ways.

Consequences: Networks

Theorem

If all edge capacities in a network are integers, then there exists an optimal flow in which every edge carries an integer amount of material.

• This theorem was seen and proven in discrete mathematics lectures.

• It follows from the above. The algebraic description of the flow problem is an LP problem. The matrix is TU. Hence, the vertices of the polytope are integer-coordinate vectors. Among the optimal points, there is an integer one.

• The same applies to the dual problem. When searching for the optimal dual solution, we can confine ourselves to integral dual feasible solutions. We utilized this in one of our previous examples of duality.

Break

Conditions Guaranteeing Integrality III: TDI Inequality Systems

- Let $\mathcal{E} : Ax \preceq b$ be an inequality system. Suppose $A \in \mathbb{Q}^{k\times n}$, $b \in \mathbb{Q}^k$. Let $\mathcal{P} : \{x \in \mathbb{R}^n : Ax \preceq b\}$ be the corresponding non-empty polyhedron (solution set).
- Consider the following four optimization problems related to $\mathcal{E}$.

$(\mathrm{P})_{\mathbb{Z}}$:	(P) :	(D) :	$(\mathrm{D})_{\mathbb{Z}}$:
Min $c^T x$ st. $Ax \preceq b$ $x \in \mathbb{Z}^n$	Min $c^T x$ st. $Ax \preceq b$	Max $-b^T\lambda$ st. $c + A^T\lambda = 0$ $\lambda \succeq 0$	Max $-b^T\lambda$ st. $c + A^T\lambda = 0$ $\lambda \in \mathbb{N}^k$

$$p^*_{\mathbb{Z}} \geq p^* = d^* \geq d^*_{\mathbb{Z}},$$

where $p^*_{\mathbb{Z}}, p^*, d^*, d^*_{\mathbb{Z}}$ are the optimal values of the respective optimization problems (in the specified order).

Comments

- Examples can be provided for an inequality system $\mathcal{E}$ and a vector c so that the first and last inequalities can be arbitrarily sharp.
  - For suitable inequality system $\mathcal{E}$ and vector c, equality can be maintained throughout for both the first and last inequalities.
  - For suitable inequality system $\mathcal{E}$ and vector c, the first inequality can be strict, while the last one can be an equality.
  - For suitable inequality system $\mathcal{E}$ and vector c, the first inequality can be an equality, while the last one can be strict.
  - For suitable inequality system $\mathcal{E}$ and vector c, both the first and last inequalities can be strict.

TPI Systems

• The situation is different if c is not fixed but an arbitrary vector in $\mathbb{Z}^n$.

• There are inequality systems for which for every $c \in \mathbb{Z}^n$, $p^*_{\mathbb{Z}} = p^*$ These are called *totally primal integral* (TPI) systems.

• Thus, specifically for a TPI system (since $p^*_{\mathbb{Z}}$ is obviously integral if finite), p^* is also integral (if finite).

• We know that this is equivalent to $\mathcal{P}$ being an integral polyhedron.

TDI Systems

Definition

Definition Let $A \in \mathbb{Q}^{k \times n}$, $b \in \mathbb{Q}^k$. The inequality system $\mathcal{E} : Ax \preceq b$ is dual integral (TDI) if for every $c \in \mathbb{Z}^n$, $d^* = d^*_{\mathbb{Z}}$ (assuming d^* is finite).

- The TDI property fundamental theorem states that if for every $c \in \mathbb{Z}^n$, the last inequality in our inequality chain is an equality, then necessarily the first inequality is also an equality for every $c \in \mathbb{Z}^n$.

Edmonds—Giles Theorem

If $\mathcal{E} : Ax \preceq b$ is TDI and $b \in \mathbb{Z}^k$, then it is also TPI. Thus, $\mathcal{P}$ is an integral polyhedron.

Important Note

- The statement is NOT about the polyhedron $\mathcal{P} = \{x : Ax \preceq b\}$ itself.

Example

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \preceq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

is not TDI.

Example

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \preceq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

is TDI.

Important Note (continued)

• It is known that for every integral polyhedron, there exists a description with matrix A, vector b such that $Ax \preceq b$ is TDI.

• So if we want to prove that a polyhedron is integral, our plan could be as follows:

(1) We „cleverly" express the polyhedron as $\{x : Ax \preceq b\}$.

(2) We show that $Ax \preceq b$ is a TDI system. That is, we show that the

Minimize	$b^\mathsf{T}x$-t
subject to	$A^\mathsf{T}\lambda = -c$
	$\lambda \succeq 0$

problem has an integral optimal solution for every $c \in \mathbb{Z}^n$.

(3) We conclude the integrality of $\mathcal{P}$.

Edmonds—Giles Theorem: The Proof

• Our assumption is that $b \in \mathbb{Z}^k$.

• Based on the TDI property, we know that for every $c \in \mathbb{Z}^n$, $p^* = d^* = d^*_{\mathbb{Z}}$. Since $b \in \mathbb{Z}^k$, $d^*_{\mathbb{Z}}$ is integral. Thus, p^* is integral for every $c \in \mathbb{Z}^n$.

• We have seen („earlier" Edmonds—Giles Theorem) that from this, we can deduce the integrality of $\mathcal{P}$.

$\mathcal{MP}(G)$ Matching Polytope (G loopless)

- Consider the convex hull of characteristic vectors of matchings:

$$\mathcal{MP}(G) = \operatorname{conv}\{\chi_M : M \text{ matching}\}.$$

- Any linear inequality that holds for all χ_M vectors (M matching) holds for all elements of the convex hull. If a half-space contains all χ_M, then its convex hull also does.

- Thus, it is easy to provide an „upper bound" for the convex hull:

$$\operatorname{conv}\{\chi_M : M \text{ matching}\} \subseteq \left\{x \in \mathbb{R}^{E(G)} : x_e \geq 0, \sum_{e:v|e} x_e \leq 1, v \in V(G)\right\} \subseteq \mathbb{R}^{E(G)}$$

- If G is a bipartite graph, equality holds. In the general case, more inequalities are needed to describe the convex hull.

Edmonds Polyhedron Theorem

Edmonds' Polyhedron Theorem

Let G be any simple graph. Then

$$\operatorname{conv}\{\chi_M : M \text{ matching}\} = \{x \in \mathbb{R}^{E(G)} : \begin{array}{l} x_e \geq 0 \quad \forall e \in E(G) \\ \displaystyle\sum_{e:vIe} x_e \leq 1 \quad \forall v \in V(G) \\ \displaystyle\sum_{\substack{e=xy\in E(G):\\ x\in S, y\notin S}} x_e \leq \frac{|S|-1}{2} \quad \forall S \in \mathcal{O}\}, \end{array}$$

where $\mathcal{O}$ is the set of subsets of V with odd number of elements.

Proof: Cunningham—Marsh Theorem

- We need to show that the vertices of the right-hand side polytope are integral.
- This directly follows from the following theorem:

Cunningham—Marsh Theorem

The inequality system appearing in the Edmonds' description of $\mathcal{MP}(G)$ is dual integral (TDI).

- That is, for any $c \in \mathbb{Z}^n$,

Minimize	$\sum_{v\in V(G)} \lambda_v + \sum_{S\in\mathcal{O}} \frac{\lvert S\rvert-1}{2} \cdot \lambda_S$-t
subject to	$-c_e + \lambda_u + \lambda_v + \sum_{\substack{S\in\mathcal{O}\\ u,v\in S}} \lambda_S - \lambda_e = 0$
	$\forall e = uv \in E(G)$, and $\lambda \succeq 0$.

Then there exists an integral optimal solution.

Break

Edmonds' Polyhedron Theorem II

Edmonds' Theorem, II

$$\begin{aligned}\mathcal{PMP}(G) = \mathrm{conv}\{\chi_M \colon M \text{ perfect matching}\} = \\ = \{x \in \mathbb{R}^{E(G)} \colon \quad & x_e \geq 0, \quad e \in E(G) \\ & \textstyle\sum_{e: v \mathrm{I} e} x_e \leq 1, \quad v \in V(G), \\ & \textstyle\sum_{\substack{e=xy \in E(G): \\ x \in S, y \notin S}} x_e \geq 1, \quad S \subseteq V(G), \\ & |S| \text{ odd}\}.\end{aligned}$$

- $\mathcal{PMP}(G)$ is the perfect matching polytope of graph G.

Consequence of Edmonds' Polyhedron Theorem

Theorem

G is a k-regular, k-edge-connected graph with an even number of vertices. Then there exists a positive integer t such that

$$\chi_e(t \times G) = t \cdot k,$$

where $t \times G$ is the graph obtained from G by multiplying its edges by t (alternatively, we add $t-1$ "twin copies" to each edge of G).

• The χ_e in the theorem is the edge chromatic number: Color the edges of the graph in such a way that converging edges have different colors, i.e., edges belonging to the same color class form a perfect matching.

• The minimum number of colors needed for this is the edge chromatic number of the graph.

Reminder

Reminder: Vizing's Theorem

If G is a simple graph, then

$$D(G) \leq \chi_e(G) \leq D(G) + 1,$$

where $D(G)$ denotes the maximum degree of the graph.

• For non-simple graphs, the corresponding upper bound does not hold.

Reminder: Shannon's Theorem

$$D(G) \leq \chi_e(G) \leq \frac{3}{2} \cdot D(G),$$

• The theorem is tight: $\chi_e(t \times K_3) = 3t$, while $D(t \times K_3) = 2t$. Thus, by multiplying edges, we can reach up to the upper bound given by Shannon's estimate.

Consequence: Proof

- Observe that $\frac{1}{k}\cdot \underline{1} \in \mathcal{MP}(\mathcal{G})$, where $\frac{1}{k}\cdot \underline{1} \in \mathbb{Q}^E$ is the vector containing all $1/k$ coordinates.

- To show this, it suffices to verify that $\mathcal{MP}(\mathcal{G})$ satisfies each condition in the Edmonds' description.

- Obviously, its components are nonnegative. The sum of components corresponding to converging edges at each vertex is a sum of k $1/k$ terms, totaling exactly 1.

- We check the third type condition for $S \in \mathcal{O}$ ($|V|$ even, so $S \neq \emptyset, V$): First, for an arbitrary (x_e) vector, sum the component sums corresponding to edges meeting at vertices in S:

$$\sum_{v\in S}\sum_{e:vIe} x_e = 2\sum_{e=xy:x,y\in S} x_e + \sum_{e\in\partial S} x_e.$$

Consequence: Proof (Continuation)

- Rearranging,

$$\sum_{e\subseteq S} x_e = \frac{\sum_{v\in S}\sum_{e:v I e} x_e - \sum_{e\in\partial S} x_e}{2}$$

- The k-edge connectivity implies that $|\partial S| \geq k$.

- Now if we apply this to $(x_e) = \frac{1}{k}\cdot \underline{1}$, the subtracted term in the numerator is at least 1 (at least k $1/k$ terms are summed). This gives us the third type inequality to check.

- Summing up: $\frac{1}{k}\underline{1} \in \mathcal{MP}(\mathcal{G})$

Consequence: Proof (Continuation)

• By Edmonds' theorem, we know that our vector can be represented as a convex combination of the vertex vectors of the polytope:

$$\frac{1}{k}\cdot \underline{1} = \sum_{M \text{ matching}} \alpha_M \chi_M = \sum_{M \text{ matching}} \frac{\ell_M}{L}\chi_M,$$

where $\sum_{M:\ \text{matching}} \alpha_M = 1$, $\alpha_M \geq 0$.

• Since the vertices of the polytope are integral, our vector is rational, so the α_M's can be assumed to be rational, i.e., $(\alpha_M) \in \mathbb{Q}^E$, thus $L \in \mathbb{N}_+$, $\ell_M \in \mathbb{N}$.

• The relationship sorted becomes

$$L\cdot \underline{1} = \sum (k\cdot \ell_M)\chi_M.$$

Consequence: Proof (Continuation)

- We demonstrate that this equality precisely means that our claim is true for $t = L$.
- Indeed, consider each M matching $k \cdot \ell_M$ times. The matchings form possible color classes.
- Based on the above equality, each edge in G is covered L times by these matchings. That is, they form a partition of $L \times G$, a good edge coloring.
- The color demand:

$$\begin{aligned}\sum_{M \text{ matching}} k\ell_M =& k \sum_{M \text{ matching}} \ell_M = kL \sum_{M \text{ matching}} \frac{\ell_M}{L} = \\ =& kL \sum_{M \text{ matching}} \alpha_M = kL, \text{ since } \sum_{M \text{ matching}} \alpha_M = 1.\end{aligned}$$

Break

Reminder: Cunningham—Marsh Theorem

- For any $c \in \mathbb{Z}^n$,

Minimize	$\sum_{v \in V(G)} \lambda_v + \sum_{S \in \mathcal{O}} \frac{\lvert S\rvert - 1}{2} \cdot \lambda_S$-t
subject to	$-c_e + \lambda_u + \lambda_v + \sum_{\substack{S \in \mathcal{O} \\ u,v \in S}} \lambda_S - \lambda_e = 0$
	$\forall e = uv \in E(G)$, and $\lambda \succeq 0$.

Then there exists an integer feasible solution.

- Equivalently:

Minimize	$\sum_{v \in V(G)} \lambda_v + \sum_{S \in \mathcal{O}} \frac{\lvert S\rvert - 1}{2} \cdot \lambda_S$-t
subject to	$\lambda_u + \lambda_v + \sum_{\substack{S \in \mathcal{O} \\ u,v \in S}} \lambda_S \geq c_e$
	$\forall e = uv \in E(G)$, and $\lambda \succeq 0$.

Then there exists an integer feasible solution.

New Form of Cunningham—Marsh Theorem

Cunningham—Marsh Theorem

Let $(c_e)_{c\in E(G)} \in \mathbb{Z}^{E(G)}$ be an arbitrary integral edge weighting of G. Then there exists $(\lambda_v) \in \mathbb{R}_+^V, (\lambda_S) \in \mathbb{R}_+^{\mathcal{O}}$ satisfying

$$\lambda_u + \lambda_v + \sum_{\substack{S\in\mathcal{O} \\ u,v\in S}} \lambda_S \geq c_e \quad \forall e = uv \in E(G)$$

and

$$\sum_{v\in V(G)} \lambda_v + \sum_{S\in\mathcal{O}} \frac{|S|-1}{2}\lambda_S \leq \nu_c(G),$$

furthermore, these are integral solutions.

New Form of Cunningham—Marsh Theorem: Justification

- The system of conditions is the dualized conditions of the primal problem with the natural elimination of the λ_e (with sign constraints) variables (these did not appear in the objective function).

- The disappearance of the optimization is due to the additional condition.

- Satisfying the additional condition, we have

$$\sum_{v\in V(G)} \lambda_v + \sum_{S\in\mathcal{O}} \frac{|S|-1}{2}\lambda_S \leq \nu_c(G) \leq p^* \leq d^*$$

(The last inequality holds due to the weak duality for maximization problems), thus guaranteeing that our possible dual solution is optimal.

Proof of Cunningham—Marsh Theorem: Initial Steps

• If we know the theorem for connected graphs, then from the dual solutions found for the components we can construct a solution for the entire G. To those sets of odd cardinality containing multiple components, we assign 0 values.

• Parallel edges can be handled easily. From now on, we assume that our graph is simple.

• If any component of the $(c_e)_{e\in E(G)}$ vector is not positive, then in the dual problem, the edge imposes no constraint. Thus, these edges can be removed from our graph. Hence, we may assume that $(c_e)_{e\in E(G)} \in \mathbb{N}_+^{E(G)}$.

• We carry out a complete induction on $|V|+|E|+\sum_{e\in E(G)} c(e)$. Verification of the cases of small graphs (with small weights) is straightforward, left as an exercise for the interested reader.

Proof of Cunningham—Marsh Theorem: Case 1 and Scheme

Case 1: Let G and c be such that there exists a vertex $v \in V(G)$ such that every c-optimal matching covers v. By c-optimal matching, we mean a matching M such that $c(M) = \nu_c(G)$.

- The scheme of our proof will be as follows:

$$G, c \xrightarrow[\text{step}]{\text{back-}} G' = G \text{ (the graph remains the same)}$$

$$c'_e = \begin{cases} c_e - 1, & \text{if } vIe \\ c_e, & \text{otherwise.} \end{cases}$$

$$\Big\downarrow \text{induction assumption}$$

possible, integral dual λ'

$$\sum_{v \in V(G)} \lambda'_v + \sum_{S \in \mathcal{O}} \frac{|S|-1}{2} \lambda'_S \le \nu_{c'}(G)$$

$$\longleftarrow$$

$$\lambda_u = \begin{cases} \lambda'_v + 1, & \text{if } u = v \\ \lambda'_u, & \text{otherwise,} \end{cases}$$

$$\lambda_S = \lambda'_S \text{ for all } S \in \mathcal{O}$$

Proof of Case 1

Claim

The λ defined in the above scheme satisfies the assertion. That is, they are possible integral dual solutions and fulfill the inequality proving the theorem.

- Non-negativity and integrality are obvious.
- From the induction assumption, we know that

$$\sum_x \lambda'_x + \sum_S \frac{|S|-1}{2}\lambda'_S \leq \nu_{c'}(G).$$

- The question is:

$$\sum_x \lambda_x + \sum_S \frac{|S|-1}{2}\lambda_S \leq \nu_c(G).$$

Proof of Case 1 (continued)

- How do the two sides of the first inequality change when we drop the primes?
- The condition of Case 1 and the definition of c' guarantee that the right side increases by one. On the left side, the same obviously happens.
- For each edge, we need to verify the prescribed condition for feasible solutions. Let $e = xy$ be an arbitrary edge. We know the following:

$$\lambda'_x + \lambda'_y + \sum_{\substack{S \in \mathcal{O} \\ xy \in S}} \lambda_S \geq c'_e.$$

- We need to show that

$$\lambda_x + \lambda_y + \sum_{\substack{S \in \mathcal{O} \\ xy \in S}} \lambda_S \geq c_e.$$

Proof of Case 1 (conclusion)

• If v does not match with e, then $\lambda'_x = \lambda_x$, $\lambda'_y = \lambda_y$, $c'_e = c_e$, from which the claim is obvious.

• If v matches with e, then again we analyze the change between the known and the to-be-proven inequalities.

• It is easy to see that by dropping the primes, both sides increase by 1, from which the claim follows.

Proof of Cunningham—Marsh Theorem: Case 2 and Scheme

Case 2: For every vertex v, there exists an M c-optimal matching that does not cover (skips) v.

- The scheme of our proof will be as follows:

$$G, c \xrightarrow[\text{step}]{\text{back-}} G' = G \text{ (the graph remains the same)}$$

$$c' = c - 1$$

$$\Big\downarrow \text{induction step}$$

possible, integral dual λ'

$$\sum_{v \in V(G)} \lambda'_v + \sum_{S \in \mathcal{O}} \frac{|S| - 1}{2} \lambda'_S \leq \nu_{c'}(G)$$

$$\xleftarrow[\text{CLAIM}]{\star\ \text{CONDITION}}$$

$$\lambda_v = \lambda'_v$$

$$\lambda_S = \begin{cases} \lambda'_S + 1, & \text{if } S = V(G) \\ \lambda'_S, & \text{otherwise.} \end{cases}$$

- When discussing Case 2, we assume

$\star$ CONDITION

The c'-optimal matching leaves only one vertex unmatched.

Specifically, the cardinality of V is odd, thus $V \in \mathcal{O}$.

Proof of Case 2 with $\star$ CONDITION

Claim

The λ defined in the above scheme satisfies the assertion.

- Non-negativity and integrality are obvious.
- To prove the inequality ensuring optimality, we know that

$$\sum_x \lambda'_x + \sum_S \frac{|S|-1}{2}\lambda'_S \leq \nu_{c'}(G).$$

- We need to show:

$$\sum_x \lambda_x + \sum_S \frac{|S|-1}{2}\lambda_S \leq \nu_c(G).$$

Proof of Case 2 with $\star$ CONDITION (continued)

• How do the two sides of the first inequality change when we drop the primes?

• The definition of c' guarantees that the right side increases by one, where M is a c'-optimal matching.

• The CONDITION ensures that the increase is $|M|$, where M is a c'-optimal matching.

• On the left side, only one term changes: the dual variable indexed by V. Its coefficient is $\frac{|V|-1}{2}$, and its value increases by 1. The claim is obvious.

Justification of $\star$ CONDITION: 1st Lemma

Claim

After Case 1/In Case 2, the $\star$ CONDITION can be assumed.

- This follows from the following two lemmas. In the justification, we assume that the conditions of Case 2 are satisfied.

Lemma

A c'-optimal matching cannot be a perfect matching.

- Let M be a c-optimal matching. Since we are in Case 2, we may assume that M is not perfect.

- Let M' be a c'-optimal matching. Indirectly, assume that M' is perfect.

Validity of $\star$ CONDITION: 1st Lemma (continued)

- Since M is c-optimal, we have $c(M') \leq c(M)$. Knowing that M is not perfect, we can say something about the weight under c' as well:

$$c'(M) = c(M) - |M| > c(M) - \frac{|V|}{2} \geq c(M') - \frac{|V|}{2}.$$

- Moreover, M' being a perfect matching implies

$$c'(M') = c(M') - |M'| = c(M') - \frac{|V|}{2} (< c'(M)).$$

- This contradicts the fact that M' is a c'-optimal matching.

Validity of $\star$ CONDITION: 2nd Lemma

Lemma

It cannot be the case that every c'-optimal matching leaves at least two vertices unmatched.

• Indirectly assume that M' is c'-optimal and $x, y \in V$ such that M' does not cover x and y. Let (M', x, y) be such that $d(x, y)$ is minimized.

• $d(x, y) > 1$, because the connectivity between x and y would ensure that $M' \cup \{xy \text{ edge}\}$ is also a matching, contradicting the c'-optimality ($c' > 0$). (Generally, an optimal matching cannot leave two connected vertices unmatched.)

Validity of $\star$ CONDITION: 2nd Lemma (continued)

• Let x^+ be the first vertex following x on a shortest xy path (towards y). (Due to the above, $x^+ \neq y$.) Consider the following two matchings:

(1) M_{x^+}: a c-optimal matching that does not cover x^+ (such exists in Case 2).

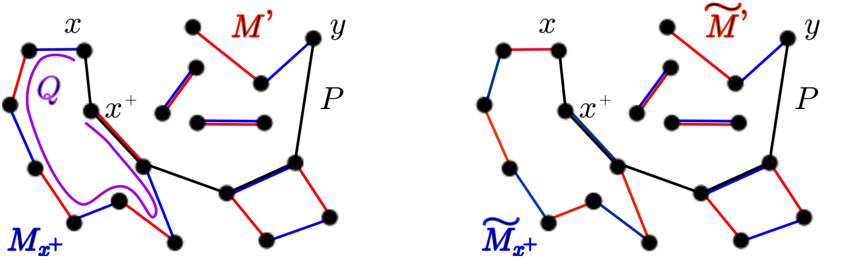
(2) M'. The c'-optimality ensures that M' covers x^+ (x and x^+ are connected).

• The components of the graph $\mathcal{M}$ formed by the edges of $M_{x^+}\Delta M'$ are cycles and paths (BSc Combinatorics course).

• Due to the properties of our matchings, x^+ is a degree 1 vertex in $\mathcal{M}$. Thus x^+ is an endpoint of a path Q in $\mathcal{M}$. Let

$$\widetilde{M}_{x^+} = M_x \Delta E(Q) \quad \text{and} \quad \widetilde{M}' = M' \Delta E(Q).$$

Validity of $\star$ CONDITION: 2nd Lemma: Figure



On the left, black edges denote the P path edges, red edges denote M_{x^+} edges, blue edges denote M' edges, purple indicates the Q path. On the right, the modified matchings ($\widetilde{M'}$ and $\widetilde{M_{x^+}}$): we exchange the red and blue edges along the Q/purple path. The total weight of red and blue edges remains the same on both sides.

Validity of $\star$ CONDITION: 2nd Lemma (continued)

- Then due to the c-optimality of M_{x^+}

$$c(\widetilde{M}_{x^+}) \leq c(M_{x^+}).$$

- The same reasoning applies to M' being c'-optimal: $c'(\widetilde{M'}) \leq c'(M')$.

- But with a little insight, we can say more: $\widetilde{M}'$ does not cover the vertex x^+. Moreover, either x or y remains uncovered (the exchange only changes the matching status at the endpoints of the Q path, and one endpoint must be x or y (not x^+)).

- Since $d(x^+, x) = 1 < d(x, y)$ and $d(x^+, y) = d(x, y) - 1 < d(x, y)$ also hold, then (M', x, y) being the choice implies $\widetilde{M}'$ cannot be c'-optimal:

$$c'(\widetilde{M'}) < c'(M').$$

Validity of $\star$ CONDITION: 2nd Lemma (continued)

- There is an edge of M' incident to x^+ (one of Q's endpoints) along the Q path. From this, it is obvious that Q contains at least as many edges of M' as of M_{x^+}. Thus, the number of edges in M' cannot increase with the exchange: $|\widetilde{M'}| \leq |M'|$.

- Thus,

$$c(\widetilde{M'}) = c'(\widetilde{M'}) + |\widetilde{M'}| < c'(M') + |\widetilde{M'}| \leq c'(M') + |M'| = c(M').$$

- The modification exchanged the roles of the two matchings along a path. The total weight of edges in the two matchings did not change, thus

$$c(\widetilde{M'}) + c(\widetilde{M_{x^+}}) = c(M') + c(M_{x^+}).$$

- This contradicts the sum of (1) and (2). From this, the assertion follows.

Conclusion of the Proof

- The two lemmas establish the validity of the ASSUMPTION.
- Thus, the consideration of Case 2 is justified.
- The proof is complete.

Thank you for your attention!