Geometry of linear programming

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Basics of Linear Programming

• There exist various normal forms. The one we most commonly use is the following:

Minimize	$c^{\mathsf{T}}x$-t
subject to	$Ax \preceq b$

where $c \in \mathbb{R}^n$, $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{k\times n}$, $b \in \mathbb{R}^k$.

• In this normal form, only linear inequalities are allowed among the constraints.

• Another common normal form is:

Minimize	$c^{\mathsf{T}}x$-t
subject to	$Ax = b,$
	$x \succeq 0.$

LP Duality

LP Duality

For any LP problem, exactly one of the following two conditions holds:

(i) $p^* = d^*$, i.e., strong duality holds,

(ii) $d^* = -\infty < \infty = p^*$.

- For example, if $\mathcal{L} \neq \emptyset$ (where $\mathcal{L}$ is the feasible solutions set), and c is bounded below (which is often the case in practical applications), then $p^* = d^* \in \mathbb{R}$.
- If $p^* = -\infty$, weak duality guarantees strong duality.
- The only loophole for an LP problem to evade strong duality is to have $p^* = \infty$ and $d^* = -\infty$. That is, both primal and dual problems are infeasible. This possibility is not theoretical; it can occur in concrete examples.

Solution Set of a Linear Equation

LINEAR ALGEBRA	GEOMETRY
$v \in \mathbb{R}^n$ is a vector.	V is a point in $\mathbb{R}^n$, its position vector is v.
$\nu \in \mathbb{R}^n - \{0\}$. $\nu^\mathsf{T} x = 0$ is a nontrivial, homogeneous linear equation solution set.	$\nu \in \mathbb{R}^n - \{0\}$ is a normal vector. $\nu^\mathsf{T} x = 0$ is the equation of vectors perpendicular to ν. It describes a hyperplane passing through the origin O.
$\nu \in \mathbb{R}^n - \{0\}$, $b \in \mathbb{R}$. $\nu^\mathsf{T} x = b$ is a nontrivial linear equation solution set.	$\nu \in \mathbb{R}^n - \{0\}$ is a normal vector. $\nu^\mathsf{T} x = b = \nu^\mathsf{T} v_0$ is the equation of vectors perpendicular to ν and passing through v_0.

Solution Set of Linear Inequalities

LINEAR ALGEBRA	GEOMETRY
$\nu \in \mathbb{R}^n - \{0\}$. The solution set of the non-trivial linear homogeneous inequality $\nu^\mathsf{T} x \leq 0/\nu^\mathsf{T} x \geq 0$ is not trivial.	$\nu \in \mathbb{R}^n - \{0\}$ is a normal vector. The inequality $\nu^\mathsf{T} x \leq 0/\nu^\mathsf{T} x \geq 0$ defines a CLOSED half-space bounded by a hyperplane passing through the origin and perpendicular to ν.
$\nu \in \mathbb{R}^n - \{0\}$, $b \in \mathbb{R}$. The solution set of the non-trivial linear inequality $\nu^\mathsf{T} x \leq b/\nu^\mathsf{T} x \geq b$ is not trivial.	$\nu \in \mathbb{R}^n - \{0\}$ is a normal vector. The inequality $\nu^\mathsf{T} x \leq b = \nu^\mathsf{T} v_0/\nu^\mathsf{T} x \geq b$ defines a CLOSED half-space bounded by a hyperplane passing through v_0 and perpendicular to ν.

Formal Definitions

Definition

Let $\nu \in \mathbb{R}^n$ be a nonzero vector, τ any real number. Then the set $\{x \in \mathbb{R}^n : \nu^\mathsf{T} x = \tau\}$ is called a hyperplane in $\mathbb{R}^n$. The sets of the form $\{x \in \mathbb{R}^n : \nu^\mathsf{T} x \leq \tau\}$ are called (closed) half-spaces.

Remark

Every hyperplane defines two closed half-spaces, which share the same boundary.

Lemma

Half-spaces and hyperplanes are convex.

Solution Sets of Inequality Systems

LINEAR ALGEBRA	GEOMETRY
$A \in \mathbb{R}^{k\times n}$. Solution set of the homogeneous linear equation system $Ax = 0$.	Intersection of finitely many hyperplanes passing through the origin $\equiv$ *linear subspace*.
$A \in \mathbb{R}^{k\times n}, b \in \mathbb{R}^{k}$. Solution set of the linear equation system $Ax = b$.	Intersection of finitely many hyperplanes $\equiv$ *affine subspace*.
$A \in \mathbb{R}^{k\times n}$. Solution set of the homogeneous linear inequality system $Ax \preceq 0$.	Intersection of finitely many closed half-spaces passing through the origin $\equiv$ *polyhedral (closed, convex) cone*.
$A \in \mathbb{R}^{k\times n}, b \in \mathbb{R}^{k}$. Solution set of the linear inequality system $Ax \preceq b$.	Intersection of finitely many closed half-spaces $\equiv$ *(convex, closed) polyhedron*.

Formal Definitions

Definition: Linear Combination of Vectors

Let $v_1, v_2, \ldots, v_N \in \mathbb{R}^n$ be vectors in a finite system and $\lambda_1, \lambda_2, \ldots, \lambda_N \in \mathbb{R}$ be a system of real numbers. Then

$$\lambda_1 v_1 + \lambda_2 v_2 + \ldots + \lambda_N v_N$$

is called the linear combination of the v_i vectors.

Definition: Linear Subspace of $\mathbb{R}^n$

$\mathcal{L} \subset \mathbb{R}^n$ is a linear subspace if $0 \in \mathcal{L}$ and closed under linear combination.

Example

Example: Finitely Generated Linear Subspace

$$\langle v_1, v_2, \ldots, v_N \rangle_{\text{lin}} = \{\lambda_1 v_1 + \lambda_2 v_2 + \ldots + \lambda_N v_N : \lambda_i \in \mathbb{R}\}.$$

Formal Definitions (continued)

Definition: Affine Combination of Vectors

Let $v_1, v_2, \ldots, v_N \in \mathbb{R}^n$ be vectors in a finite system and $\lambda_1, \lambda_2, \ldots, \lambda_N \in \mathbb{R}$ be a system of real numbers such that $\lambda_1 + \lambda_2 + \ldots + \lambda_N = 1$. Then

$$\lambda_1 v_1 + \lambda_2 v_2 + \ldots + \lambda_N v_N$$

is called the affine combination of the v_i vectors.

Definition: Affine Subspace of $\mathbb{R}^n$

$\mathcal{A} \subset \mathbb{R}^n$ is an affine subspace if closed under affine combination.

Example

Example: Finitely Generated Affine Subspace

$\langle v_1, v_2, \ldots, v_N \rangle_{\text{affine}} = \{\lambda_1 v_1 + \lambda_2 v_2 + \ldots + \lambda_N v_N : \lambda_i \in \mathbb{R}, \sum_i \lambda_i = 1\}.$

Formal Definitions (continued)

Definition: Cone Combination of Vectors

Let $v_1, v_2, \ldots, v_N \in \mathbb{R}^n$ be vectors in a finite system and $\lambda_1, \lambda_2, \ldots, \lambda_N \in \mathbb{R}_+$ be nonnegative real numbers. Then

$$\lambda_1 v_1 + \lambda_2 v_2 + \ldots + \lambda_N v_N$$

is called the cone combination of the v_i vectors.

Definition: Cone in $\mathbb{R}^n$

$\mathcal{C} \subset \mathbb{R}^n$ is a (convex) cone if closed under cone combination.

Example

Example: Finitely Generated Cone

$$\langle v_1, v_2, \ldots, v_N \rangle_{\text{cone}} = \{\lambda_1 v_1 + \lambda_2 v_2 + \ldots + \lambda_N v_N : \lambda_i \in \mathbb{R}_+\}.$$

Formal Definitions (continued)

Definition: Convex Combination of Vectors

Let $v_1, v_2, \ldots, v_N \in \mathbb{R}^n$ be vectors in a finite system and $\lambda_1, \lambda_2, \ldots, \lambda_N \in \mathbb{R}_+$ be nonnegative real numbers such that $\lambda_1 + \lambda_2 + \ldots + \lambda_N = 1$. Then

$$\lambda_1 v_1 + \lambda_2 v_2 + \ldots + \lambda_N v_N$$

is called the convex combination of the v_i vectors.

Definition: Convex Set in $\mathbb{R}^n$

$\mathcal{K} \subset \mathbb{R}^n$ is a convex point set if closed under convex combination.

Example

Example: Finitely Generated Convex Set

$$\langle v_1, v_2, \ldots, v_N \rangle_{\text{convex}} = \{\lambda_1 v_1 + \lambda_2 v_2 + \ldots + \lambda_N v_N : \lambda_i \in \mathbb{R}_+, \sum_i \lambda_i = 1\}.$$

Theorems

Theorem

Let $0 \in \mathcal{L} \subset \mathbb{R}^n$. Then the following are equivalent:

(i) Closed under line joining.

(ii) Closed under linear combination.

(iii) Solution set of $Ax = 0$ for some $A \in \mathbb{R}^{k \times n}$.

(iv) Finitely generated linear subspace.

Theorem

Let $\mathcal{A} \subset \mathbb{R}^n$. Then the following are equivalent:

(i) Closed under line joining.

(ii) Closed under affine combination.

(iii) Solution set of $Ax = b$ for some $A \in \mathbb{R}^{k \times n}$, $b \in \mathbb{R}^k$.

(iv) Finitely generated affine subspace.

Theorems (continued)

Minkowski-Weyl Theorem

Let $\mathcal{C} \subset \mathbb{R}^n$. Then the following are equivalent:

(i) Solution set of $Ax \preceq 0$ for some $A \in \mathbb{R}^{k \times n}$.

(ii) Finitely generated cone.

Fundamental Theorem of Polytopes

Let $\mathcal{T} \subset \mathbb{R}^n$. Then the following are equivalent:

(i) Bounded polyhedron ($\equiv$ polytope).

(ii) Finitely generated convex set.

Minkowski-Weyl Theorem

Let $\mathcal{P} \subset \mathbb{R}^n$. Then the following are equivalent:

(i) Polyhedron, i.e., solution set of $Ax \preceq b$ for some $A \in \mathbb{R}^{k \times n}$, $b \in \mathbb{R}^k$.

(ii) $\mathcal{T} + \mathcal{C}$, where $\mathcal{T}$ is a polytope/finitely generated convex set and $\mathcal{C}$ is a polyhedral/finitely generated cone.

Nice Polyhedrons

Definition

Let $\mathcal{P}$ be a polyhedron. $\mathcal{P}$ is called nice if it does not contain a line.

Lemma

Let $\mathcal{P}$ be a polyhedron in $\mathbb{R}^n$: $\mathcal{P} = \{x\colon Ax \preceq b\}$. Then the following are equivalent:

(i) Not nice. That is, there exists a nonzero vector v such that for some $p \in \mathcal{P}$, the line in the direction of v through p is a subset of $\mathcal{P}$.

(ii) There exists a nonzero vector v such that for every $p \in \mathcal{P}$, the line in the direction of v through p is a subset of $\mathcal{P}$.

(iii) The row rank of A is less than n (number of columns/dimension/number of variables).

(iv) $\operatorname{ext} \mathcal{P} = \emptyset$.

Further Decomposition Theorems

• When decomposing a non-nice polyhedron according to the above fundamental theorem, a component will be a line segment.

Definition: Pointed Cone

Among cones, those that do not contain a line are called *pointed cones*.

• These are exactly those cones for which there exists a hyperplane passing through the origin, such that all nonzero vectors of the cone lie strictly on one side of it. (This needs to be proved!)

• Every cone is a sum of a linear subspace and a pointed cone.

Theorem

Let $\mathcal{P}$ be an arbitrary polyhedron. Then

$$\mathcal{P} = \mathcal{T} + \mathcal{C}_{\text{pointed}} + \mathcal{L},$$

where $\mathcal{T}$ is polytope, $\mathcal{C}_{\text{pointed}}$ is a pointed cone, and $\mathcal{L}$ is a linear subspace.

Break

Vertices of Polyhedra

LINEAR ALGEBRA	GEOMETRY
	If the polyhedron $\mathcal{P} : Ax \preceq b$ is contained in the half-space $\mathcal{F} : \nu^{\mathsf{T}} x \leq \beta$ and $\mathcal{P} \cap \mathcal{H} \neq \emptyset$, where $\mathcal{H} : \nu^{\mathsf{T}} x = \beta$ (that is, $\mathcal{F}$ is a closed half-space border), then $\mathcal{F}$ is a half-space and the hyperplane $\mathcal{H}$ is the supporting face, or supporting hyperplane, of the polyhedron $\mathcal{P}$.
A solution m of a linear inequality system $Ax \preceq b$ (assuming A has no zero rows) is exactly an interior point of m (and any neighborhood of m contains only solutions) if every condition is satisfied with strict inequalities. That is, every condition is tight.	The boundary points of a polyhedron $\mathcal{P}$ are those points that have both $\mathcal{P}$-interior and $\mathcal{P}$-exterior points in every neighborhood. The set of boundary points, or the boundary itself, is denoted by $\partial\mathcal{P}$. The polyhedron $\mathcal{P}$ is closed, thus $\partial\mathcal{P} \subseteq \mathcal{P}$.

Boundary Points Revisited

Theorem

A polyhedron is a closed, convex set.

• If A has a zero row, then the resulting inequality can have either all $x \in \mathbb{R}^n$ as solutions or none at all. In a special case ($A = 0 \in \mathbb{R}^{k\times n}, b = 0 \in \mathbb{R}^k$), the entire space is a polyhedron. The empty set is also a polyhedron.

• Even in two dimensions, it is easy to give a closed set and a point on its boundary such that no supporting hyperplane can be placed on it. This is not the case in the convex setting.

Theorem

Let $K \subseteq \mathbb{R}^n$ be a closed convex set. The following are equivalent:

(i) $p \in \partial K$,

(ii) $p \in K$ and a supporting hyperplane can be placed on it.

Faces of Polyhedra

Definition

Let K be a closed convex set. A face of K is a subset of its boundary that can be intersected by an appropriate supporting hyperplane.

- Of course, faces are also closed, convex sets, subsets of ∂K.

Definition

Let K be a convex set and F be a face. Let $\text{aff}(F)$ be the affine hull of the set F, i.e., the smallest affine subspace containing F. The dimension of F is $\dim(\text{aff}(F))$.

Special Faces: Vertices

Theorem

Let $\mathcal{P} : \{x : Ax \preceq b\} \subset \mathbb{R}^n$ be a polyhedron, $e \in \mathcal{P}$. Then the following are equivalent:

(i) There exists a supporting hyperplane that intersects $\mathcal{P}$ only at e.

(ii) There is no line segment in $\mathcal{P}$ that contains e as an interior point.

(iii) Let $I = \{i : a_i^\top e = b_i\}$. Then I is such that $\{a_i : i \in I\}$ spans $\mathbb{R}^n$.

General Faces

- The surfaces of polyhedra are formed by the faces. We've only looked at the vertices in a bit more detail.

Definition

Let $\mathcal{P}$ be a polyhedron, $p \in \partial\mathcal{P}$

$$C_p := \{\nu \in \mathbb{R}^n \setminus \{0\} : \exists\, \alpha \in \mathbb{R} \text{ such that } \{x \colon \nu^{\mathsf{T}} x \le \alpha\} \supseteq \mathcal{P} \text{ and } \nu p = \alpha\} \cup \{\underline{0}\}.$$

Lemma

C_p is a convex cone.

Special Faces: Vertices (again)

• The cone associated with boundary points provides a new, alternative description of the vertices.

Theorem

Let $\mathcal{P}$ be a polyhedron, $\mathcal{P} = \{x\colon Ax \preceq b\}$, $p \in \partial\mathcal{P}$. The following are equivalent:

(i) $p \in \mathrm{ext}(\mathcal{P})$,

(ii) C_p has an interior point (in $\mathbb{R}^n$),

(iii) there exist row vectors $a_{i_1}^{\mathsf{T}}, a_{i_2}^{\mathsf{T}}, \ldots, a_{i_n}^{\mathsf{T}}$ in A such that

  (1) they are linearly independent,

  (2) $a_{i_j}^{\mathsf{T}} p = b_{i_j}$ for every $j = 1, 2, \ldots, n$.

• That is, C_p is full-dimensional if and only if p is a vertex. Generally, the dimension of C_p determines the dimension of the interior point of the boundary p point.

Refinement of Minkowski-Weyl Theorem

• Let $\mathcal{P}$ be a polyhedron, i.e., for some $A \in \mathbb{R}^{k \times n}$, $b \in \mathbb{R}^k$, $\mathcal{P} = \{x \in \mathbb{R}^n : Ax \preceq b\}$.

• If $\mathcal{P}$ is not nice, it's easy to recognize this based on linear algebraic knowledge. Moreover, we can decompose it into the sum of an affine space and a nice polyhedron. We can assume that our polyhedron is nice.

Theorem

Let $\mathcal{P} = \{x \in \mathbb{R}^n : Ax \preceq b\}$ be an arbitrary nice polyhedron.
Let $\mathcal{C} = \{x \in \mathbb{R}^n : Ax \preceq 0\}$ be a polyhedral/cone.
Let $\mathcal{T} = \langle ext(\mathcal{P}) \rangle_{\text{conv}}$ be a finitely generated convex set/polytope.
Then

$$\mathcal{P} = \mathcal{T} + \mathcal{C}.$$

Break Time

LP Geometrically

- The fundamental task of LP is to minimize a linear function, $c^{\mathsf{T}}x$, over a polyhedron.

- The level sets of $c^{\mathsf{T}}x$ are hyperplanes.

- A lower bound, λ, on the objective function over a non-empty polytope $\mathcal{P}$ means that the half-space $\{x : c^{\mathsf{T}}x \geq \lambda\}$ contains the polyhedron $\mathcal{P}$.

- The half-space $c^{\mathsf{T}}x = \lambda$ lies on one side of $\mathcal{P}$.

- The minimal objective value is attained when λ is increased (pushing the hyperplane towards $\mathcal{P}$) until the moving hyperplane touches $\mathcal{P}$.

- Then $\mathcal{P}$ supports the hyperplane. The supporting points are the optimal points.

Optimal Points and Vertices

Theorem

Let $\mathcal{P} = \{x : Ax \preceq b\}$ be a non-empty nice polyhedron. Consider the

Minimize	$c^{\mathsf{T}}x$-t
subject to	$Ax \preceq b$,

LP problems (where c varies).

Then

(i) For every $c \in \mathbb{R}^n$, either $p^* = -\infty$ or there exists $x \in \mathrm{ext}(\mathcal{P})$ as an optimal point.

(ii) For every $x \in \mathrm{ext}(\mathcal{P})$, there exists c such that x is the unique optimal point.

Proof

(i) We know that $P = \mathcal{T} + \mathcal{C}$, where $\mathcal{T}$ is a polytope and $\mathcal{C}$ is a cone.

- Assume $p^* \neq -\infty$.

- Let o be an optimal point: $o \in \mathcal{P} = \mathcal{T} + \mathcal{C}$, i.e., $o = t + k$, where $t \in \mathcal{T}$ and $k \in \mathcal{C}$.

- Firstly, $c^\mathsf{T} k \geq 0$.

- Indeed. For $\alpha \geq 0$, $\alpha k \in \mathcal{C}$, so $t + \alpha k \in \mathcal{P}$. If $c^\mathsf{T} k < 0$, then the objective function can take arbitrarily small values.

- If $c^\mathsf{T} k \geq 0$, we can assume $k = 0$, i.e., o falls into the *polytope part* of our polyhedron.

Proof (continued)

- Then o is a convex combination of $ext(\mathcal{T})$ points.

- Thus $c^{\mathsf{T}}o$ is a convex combination of $c^{\mathsf{T}}e$ values ($e \in ext(\mathcal{C})$).

In particular,

$$c^{\mathsf{T}}o \geq \min\{c^{\mathsf{T}}e : e \in \mathrm{ext}(\mathcal{T})\}.$$

This proves the statement.

(ii) Consider a supporting hyperplane ($\{x : \nu^{\mathsf{T}}x \geq b\}$), where $\{x : \nu^{\mathsf{T}}x = b\} \cap \mathcal{P} = \{x\}$.

- Obviously, $c = \nu$ is a good choice.

Rational Optimal Points

Theorem

For the

Minimize	$c^{\mathsf{T}}x$-t
subject to	$Ax \preceq b$

LP problem, assume that $A \in \mathbb{Q}^{k\times n}$, $b \in \mathbb{Q}^k$. Moreover, assume that $\{x : Ax \preceq b\}$ is a nice polyhedron.

If $p^* \in \mathbb{R}$, then there exists $x \in \mathbb{Q}^n$ as an optimal point.

Proof

- If $p^* \in \mathbb{R}$, then we can choose $e \in \text{ext}(\mathcal{P})$ as an optimal point.

- Then the inequalities $a_i^\mathsf{T} x \leq b_i$ satisfied by e are such that the corresponding a_i vectors span $\mathbb{R}^n$.

- Specifically, we can write a system of n equations, whose matrix is a submatrix of A, constants are the components of b, and e is the unique solution.

- By Cramer's rule, the components of e are the ratio of the determinants of two matrices containing rational numbers, specifically rational.

Break Time

Farkas' Lemma: First Alternative Form

Farkas' Lemma, First Alternative Form

Let $Ax \preceq b$ be a system of equations, where $A \in \mathbb{R}^{k \times n}$,

$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$, and $b \in \mathbb{R}^k$. Then exactly one of the following two statements holds:

(i) The system of equations is solvable, i.e., there exists $x_0 \in \mathbb{R}^n$ such that $Ax_0 \preceq b$.

(ii) There exists $0 \preceq \lambda \in \mathbb{R}^k$ such that $\lambda^\mathsf{T} A = 0^\mathsf{T}$ and $\lambda^\mathsf{T} b = -1$.

Second Alternative Form

Farkas' Lemma, Second Alternative Form

Consider the system of equations $\begin{cases} Ax = b \\ x \succeq 0 \end{cases}$, where $A \in \mathbb{R}^{\ell \times n}$,

$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$, and $b \in \mathbb{R}^{\ell}$. Then exactly one of the following two statements holds:

(i) The system of equations is solvable, i.e., there exists $0 \preceq x_0 \in \mathbb{R}^n$ such that $Ax_0 = b$.

(ii) There exists $\lambda \in \mathbb{R}^{\ell}$ such that $\lambda^{\mathsf{T}} A \succeq 0^{\mathsf{T}}$ and $\lambda^{\mathsf{T}} b = -1$.

Farkas' Lemma: Geometric Form

Let $\mathcal{C} \subset \mathbb{R}^n$ be a finitely generated cone. That is, there exists a matrix $G \in \mathbb{R}^{n \times k}$ such that

$$\mathcal{C} = \{G\lambda : 0 \preceq \lambda \in \mathbb{R}^k\}.$$

The columns of G are the generators of the cone.

- Alternatively, $b \in \mathcal{C}_G$ if and only if $\begin{cases} Gx = b, \\ 0 \preceq x \end{cases}$ is solvable.

- The infeasibility of such a system of inequalities is precisely one alternative of Farkas' Lemma. What is the other alternative?

Farkas' Lemma: Geometric Form (continued)

• According to Farkas' Lemma, the infeasibility of $\begin{cases} Gx = b, \\ 0 \preceq x \end{cases}$ is equivalent to the existence of a vector $\lambda \in \mathbb{R}^n$ such that

$$\lambda^{\mathsf{T}} G \succeq 0 \text{ and } \lambda^{\mathsf{T}} b = -1.$$

• In other words, the hyperplane $\mathcal{H} : \lambda^{\mathsf{T}} x = 0$ passing through the origin separates the cone and the point b, where one side $\mathcal{F}^{\geq} : \lambda^{\mathsf{T}} x \geq 0$ contains the cone $\mathcal{C}$, while the other side $\mathcal{F}^{\leq} : \lambda^{\mathsf{T}} x \leq 0$ contains b.

Farkas' Lemma: Geometric Form

Let $\mathcal{C} \subset \mathbb{R}^n$ be a finitely generated cone, $b \notin \mathcal{C}$. Then there exists a hyperplane $\mathcal{H} : \lambda^{\mathsf{T}} x = 0$ that separates the cone and b.

Proof of Weyl's Theorem: If a cone is finitely generated, then it's polyhedral

Let $\mathcal{G} = \{G\lambda : 0 \preceq \lambda\}$ be a finitely generated cone.

Let
$$\widehat{\mathcal{G}} = \left\{ \begin{pmatrix} \lambda \\ y \end{pmatrix} : y = G\lambda, 0 \preceq \lambda \right\}.$$

Clearly, $\widehat{\mathcal{G}}$ is a polyhedron.

Obviously, $\mathcal{G}$ can be obtained from the projections of $\widehat{\mathcal{G}}$.

Theorem

The projection of a polyhedron is also a polyhedron.

We know that $\mathcal{G}$ is both a polyhedron and a cone.

Lemma

We know that $\mathcal{C}$ is both a polyhedron and a cone. Then $\mathcal{C}$ is a polyhedral cone.

Minkowski's Lemma

Lemma

Suppose that
$$\{x : Ax \preceq 0\} = \{G\lambda : 0 \preceq \lambda\}.$$
Then
$$\{x : G^{\mathsf{T}}x \preceq 0\} = \{A^{\mathsf{T}}\lambda : 0 \preceq \lambda\}.$$

• We can interpret the condition of the lemma as two containment relations:
$$\{x : Ax \preceq 0\} \supset \{G\lambda : 0 \preceq \lambda\}.$$

$$\{x : Ax \preceq 0\} \subset \{G\lambda : 0 \preceq \lambda\}.$$

Minkowski's Lemma: The First Condition

$$\{x : Ax \preceq 0\} \supset \{G\lambda : 0 \preceq \lambda\}.$$

- The elements on the left side are cone combinations of the columns of G. By containment, each of these vectors is contained in the left-hand set.

- This is equivalent to saying that the columns of G are contained in the left-hand set.

- This is equivalent to saying that

the elements of AG are all non-positive.

Minkowski's Lemma: The Second Condition

$$\{x : Ax \preceq 0\} \subset \{G\lambda : 0 \preceq \lambda\}.$$

- An element b from the left side is also in the right side. That is, if $Ab \preceq 0$, then the system $\begin{cases} G\lambda = b \\ 0 \preceq \lambda \end{cases}$ is solvable.
- By Farkas' Lemma, this can be reformulated as: The system $\begin{cases} Ab \preceq 0 \\ \mu^{\mathsf{T}} G \preceq 0 \\ \mu^{\mathsf{T}} b = 1 \end{cases}$ has no solution.

Minkowski's Lemma: The Conditions

- Based on the above, the conditions are

the elements of AG are all non-positive and $\begin{cases} Ab \preceq 0 \\ \mu^{\mathsf{T}} G \preceq 0 \\ \mu^{\mathsf{T}} b = 1 \end{cases}$ has no solu

- Alternatively,

the elements of $G^{\mathsf{T}} A^{\mathsf{T}}$ are all non-positive and $\begin{cases} G^{\mathsf{T}} \mu \preceq 0 \\ b^{\mathsf{T}} A^{\mathsf{T}} \preceq 0 \\ b^{\mathsf{T}} \mu = 1 \end{cases}$ has no

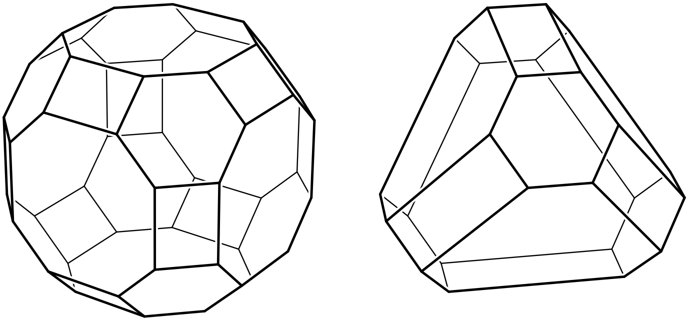
- These are equivalent to the proposition to be proven.

Polytopes

Definition

A polyhedron $\mathcal{P} \subset \mathbb{R}^n$ is called a polytope if it is bounded.

- Bounded polyhedra/polytopes play an important role in understanding polyhedra.



Fundamental Theorem of Convex Polytopes

Theorem

Let $\mathcal{P} \subset \mathbb{R}^d$. Then the following are equivalent:

(i) $\mathcal{P}$ is a bounded polyhedron.

(ii) $\mathcal{P}$ is the convex hull of finitely many points in $\mathbb{R}^d$.

Polyhedra: Coning, Homogenization

Let $\mathcal{P}$ be a polyhedron, i.e.,

$$\mathcal{P} = \{x : Ax \preceq b\} \subset \mathbb{R}^d.$$

Define

$$\widehat{\mathcal{P}} = \left\{ \begin{pmatrix} x \\ \lambda \end{pmatrix} : x \in \mathbb{R}^d, \lambda \in \mathbb{R}, Ax \preceq \lambda b, 0 \leq \lambda \right\} \subset \mathbb{R}^d \times \mathbb{R}_+ \subset \mathbb{R}^{d+1}.$$

Example

$$\mathcal{P} = \{(x, y)^\mathsf{T} : x \leq 0, y \leq 0\} \subset \mathbb{R}^2.$$

$$\widehat{\mathcal{P}} = \{(x, y, \lambda)^\mathsf{T} : x \leq 0, y \leq 0, \lambda \geq 0\} \subset \mathbb{R}^2 \times \mathbb{R}_+ \subset \mathbb{R}^3.$$

Coning of Polyhedra: The Observation

Observation

(i) $x \in \mathcal{P}$ if and only if $\begin{pmatrix} x \\ 1 \end{pmatrix} \in \widehat{\mathcal{P}}$.

(ii) $\widehat{\mathcal{P}}$ is a polyhedral cone.

Fundamental Theorem of Convex Polytopes: Proof (i)⇒(ii)

- Since $\mathcal{P}$ is bounded, the polyhedral cone $\widehat{\mathcal{P}}$ contains only 0 from the hyperplane $\lambda = 0$.
- By Weyl's theorem,

$$\widehat{\mathcal{P}} = \langle \widehat{g}_1, \widehat{g}_2, \ldots, \widehat{g}_k \rangle_{\text{cone}} = \left\langle \begin{pmatrix} g_1 \\ 1 \end{pmatrix}, \begin{pmatrix} g_2 \\ 1 \end{pmatrix}, \ldots, \begin{pmatrix} g_k \\ 1 \end{pmatrix} \right\rangle_{\text{cone}}$$

- Thus,

$$\begin{pmatrix} g \\ 1 \end{pmatrix} \in \widehat{\mathcal{P}}$$

if and only if

$$g \in \langle g_1, g_2, \ldots, g_k \rangle_{\text{convex}}$$

Fundamental Theorem of Convex Polytopes: Proof (ii)⇒(i)

Assume $\mathcal{P} = \langle g_1, g_2, \ldots, g_k \rangle_{\text{convex}}$. Clearly, $\mathcal{P}$ is bounded. Let

$$\widehat{\mathcal{P}} = \left\langle \begin{pmatrix} g_1 \\ 1 \end{pmatrix}, \begin{pmatrix} g_2 \\ 1 \end{pmatrix}, \ldots, \begin{pmatrix} g_k \\ 1 \end{pmatrix} \right\rangle_{\text{cone}},$$

a finitely generated polyhedral cone.

By Weyl's theorem, there exists a matrix $(A| - b)$ such that

$$\widehat{\mathcal{P}} = \left\{ \begin{pmatrix} x \\ \lambda \end{pmatrix} : (A| - b) \begin{pmatrix} x \\ \lambda \end{pmatrix} \preceq 0 \right\}.$$

Then

$$\mathcal{P} = \{x : Ax \preceq b\},$$

i.e., $\mathcal{P}$ is a polyhedron.

Combining Geometric Sets

Definition

Let $A, B \subset \mathbb{R}^d$. Then

$$A + B = \{a + b : a \in A, b \in B\}$$

is called the direct or Minkowski sum of sets A and B.

Minkowski–Weyl Theorem

Minkowski–Weyl Theorem

(i) Let $\mathcal{P}$ be any polyhedron. Then there exist finitely generated convex sets/polytopes $\mathcal{T}$ and $\mathcal{C}$

$$\mathcal{P} = \mathcal{T} + \mathcal{C}.$$

(ii) Let $\mathcal{T}$ be a finitely generated convex set/polytope and $\mathcal{C}$ be a finitely generated cone. Then $\mathcal{T} + \mathcal{C}$ is a polyhedron.

Minkowski–Weyl Theorem: Proof: (i)

- For $\mathcal{P}$, we defined a $\widehat{\mathcal{P}}$ polyhedral cone.
- By Weyl's theorem,

$$\widehat{\mathcal{P}} = \left\langle \begin{pmatrix} g_1 \\ 1 \end{pmatrix}, \begin{pmatrix} g_2 \\ 1 \end{pmatrix}, \ldots, \begin{pmatrix} g_k \\ 1 \end{pmatrix}, \begin{pmatrix} h_1 \\ 0 \end{pmatrix}, \begin{pmatrix} h_2 \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} h_\ell \\ 0 \end{pmatrix} \right\rangle_{\text{cone}},$$

- Then

$$\mathcal{P} = \langle g_1, g_2, \ldots, g_k \rangle_{\text{convex}} + \langle h_1, h_2, \ldots, h_\ell \rangle_{\text{cone}},$$

Minkowski–Weyl Theorem: Proof: (ii)

Assume $\mathcal{P} = \langle g_1, g_2, \ldots, g_k \rangle_{\text{convex}} + \langle h_1, h_2, \ldots, h_\ell \rangle_{\text{cone}}$.
Let

$$\widehat{\mathcal{P}} = \left\langle \begin{pmatrix} g_1 \\ 1 \end{pmatrix}, \begin{pmatrix} g_2 \\ 1 \end{pmatrix}, \ldots, \begin{pmatrix} g_k \\ 1 \end{pmatrix}, \begin{pmatrix} h_1 \\ 0 \end{pmatrix}, \begin{pmatrix} h_2 \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} h_\ell \\ 0 \end{pmatrix} \right\rangle_{\text{cone}},$$

a finitely generated cone.

By Weyl's theorem, there exists a matrix $(A| - b)$ such that

$$\widehat{\mathcal{P}} = \left\{ \begin{pmatrix} x \\ \lambda \end{pmatrix} : (A| - b) \begin{pmatrix} x \\ \lambda \end{pmatrix} \preceq 0 \right\}.$$

Then

$$\mathcal{P} = \{x : Ax \preceq b\},$$

i.e., $\mathcal{P}$ is a polyhedron.

Thank you for your attention!