Testing $\mathcal{MP}(G)$

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Reminder

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Edmonds Polytope Theorem)

$\mathcal{MP}(G)$ consists exactly of the vectors $(x_e)_{e\in E} \in \mathbb{R}^{E(G)}$ that satisfy the following three types of inequalities:

$$(E_e): \qquad x_e \geq 0 \qquad \forall e \in E(G)$$

$$(E_v): \qquad \sum_{e:v I e} x_e \leq 1 \qquad \forall v \in V(G)$$

$$(E_S): \qquad \sum_{e \subseteq S} x_e \leq \frac{|S|-1}{2} \qquad \forall S \in \mathcal{O}$$

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Some of these may be redundant (we saw that for bipartite G this can lead to significant simplification), but the polytope is generally complicated (the number of required inequalities is exponential in the number of vertices and edges of the graph).

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Find an efficient algorithm that, given a $x \in \mathbb{Q}^{E(G)}$, outputs either the true information that x is an element of $\mathcal{MP}(\mathcal{G})$, or a defining inequality violated by the x vector.

By efficient we mean polynomial time in the size of G. So the naive algorithm (substitute x coordinates into every inequality and after checks announce the result) won't do.

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In the first case, we're done, as we now know that $x \notin \mathcal{MP}(G)$ and have found a violated inequality.

So for the rest, we can assume that the (E_v) and (E_e) inequalities hold for the vector x to be tested.

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So $(x_e)_{e \in E} \in \mathbb{R}^{E(G)}$ is an edge-weighted graph.

We assumed the weights to be non-negative. At every vertex, the sum of weights of incident edges is at most 1.

Vector vs Function Perspective

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Edge weighting can also be naturally thought of as a function $x : E(G) \to \mathbb{R}_+$ (we know that x satisfies the (E_e) inequalities). So $x(e) = x_e$ will denote the weight of edge e.

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If $R \subset V(G)$, then

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At first glance, these conventions may be confusing. The meaning of $x(\cdot)$ depends on whether the parentheses contain an edge, an edge set, or a vertex set. Let's take the time and effort to get used to it.

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$2^{|V|}$ inequalities

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Summarizing, the (E_S) conditions *only* mean that the above estimate (which holds for every vertex set) can be sharpened by $1/2$ for subsets with odd cardinality.

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Since the components of x are non-negative, for any vertex set R

$$2 \cdot x(R) \leq x(\partial R) + 2 \cdot x(R) \leq |R|, \quad \text{thus} \quad x(R) \leq \frac{|R|}{2}.$$

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Given a non-negative edge weighting, assuming that for every vertex the sum of weights of incident edges is at most 1.

From a (G, x) construct a $\widetilde{G}, \widetilde{x}$ pair, which already satisfies the (E_v) inequalities with equality (the sum of weights of incident edges at each vertex is exactly 1).

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- Consider a complete matching between the *twin vertices*. The $\widetilde{x}$ weighting in G and G' is the same as x, and the crossing edges' weight in $\widetilde{x}$ is

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Claim supporting the 1st Goal

For (G, x), then all (E_S) conditions hold if and only if they hold for $(\widetilde{G}, \widetilde{x})$.

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Let $S \subset V(\widetilde{G})$ be arbitrary. $R = S \cap V(G)$ and $T' = S \cap V(G')$ are the two parts of set S. Think of T' as a twin of a vertex set $T \subset V(G)$.

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If S has an odd cardinality, then one of S and T is odd, and the other has an even cardinality.

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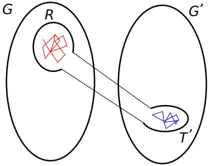

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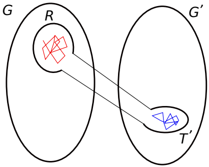

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This is a straightforward case. Then the set of edges $E(R)$ inside R and the set of edges $E(T')$ inside T' together give the set of edges $E(S)$ inside S. Specifically, $x(S) = x(R) + x(T') = x(R) + x(T)$.

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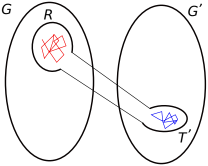

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We can estimate $x(R)$ and $x(T)$ by $|R|/2$ and $|T|/2$, respectively, and even the upper bound sharpened by $1/2$ for the set with an odd cardinality.

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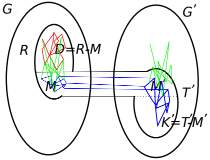

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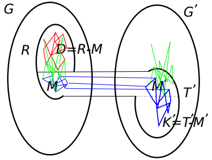

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We can assume that $D \subset V(G)$ is a set with an odd number of elements:

$$x(D) \leq \frac{|D| - 1}{2}.$$

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We will be more cautious when estimating $x(M \cup M' \cup K')$.

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Regarding the previously neglected, *halved* term, it is obvious that

$$x(E(D, M)) \leq \frac{1}{2}(x(\partial_G(M) + \partial_{G'}(M' \cup K')),$$

where $E(D, M)$ is the number of edges crossing between D and M.

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Combining these two inequalities yields the desired result straightforwardly.

We have thus obtained that testing the (E_S) inequalities is equivalent for (G, x) and $(\widetilde{G}, \widetilde{x})$.

Break

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- In what follows, we only deal with edge-weighted (G, x) graphs where the (E_v) inequalities hold with equality, that is, for every $v \in V(G)$ vertex

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• We change the language slightly: ∂S is the boundary of set S. However, this can be regarded as the edge set $E(\mathcal{V})$ of the cut $\mathcal{V} = (S, \overline{S})$. Let $x(\mathcal{V}) = x(E(\mathcal{V}))$. The $\mathcal{V}$ cut is odd if both its sides are sets with an odd cardinality (we already assume G has an even number of vertices).

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Given an edge-weighted, non-negative, and even-sized graph (G, x). Determine efficiently the minimum weight odd cut.

- If Goal 2 is achievable, then it implies solving the Edmonds' polytope testing problem.

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- However, if we seek $\min_{\mathcal{V}=(S,T),|S|=|T|} x(\mathcal{V})$, this is an $\mathcal{NP}$-complete problem.
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- Thus, if we impose oddness on $\mathcal{V}$, the complexity of our question is not clear.
- If our vertex set has an odd cardinality, then one side of every cut would be odd. Thus, determining the minimum weight among odd sets would be equivalent to searching among all subsets.

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- Solving this efficiently requires introducing a new concept.

Gomory–Hu Tree

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Consider a tree F on $V(G)$. Then F has $n-1$ edges, and note that deleting any edge of F separates F into two components. If e was the deleted edge, let the vertex sets be S_e and T_e. Then $\mathcal{V}_e = (S_e, T_e)$ is a cut.

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The tree F is a *Gomory–Hu tree* if for every $e = xy \in E(F)$ the cut (S_e, T_e) is w-optimal as an xy cut in G, meaning

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The T tree is a graph on the vertex set of G. However, its edges have *nothing* to do with G. It is not necessarily a subgraph.

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Beyond explicit optimality in the definition, additional information can be extracted from a Gomory–Hu tree.

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$\mathcal{V}$ is an optimal xy cut.

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This establishes the claim and hence the lemma.

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Theorem

Among the $n-1$ cuts implied by F, the smallest weight odd-odd cut exists.

Proof of the Theorem

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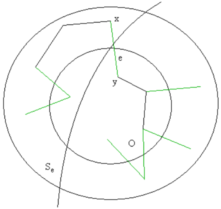
Let O be an optimal odd set.



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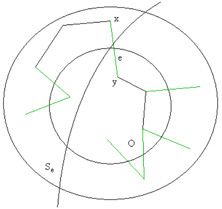

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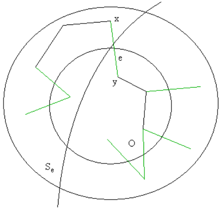

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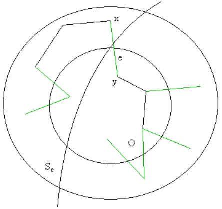

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Proof of the Stronger Theorem

$$\sum_{e\in\partial_F O} |S_{\overrightarrow{e}}| \underset{\pmod 2}{\equiv} \sum_{\substack{x\in O,\ e\in E(F),\\ \overrightarrow{e} \text{ points out of } x}} |S_{\overrightarrow{e}}| =$$

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The first congruence holds because the extra terms in the sum come in pairs (one for each e edge adjacent to O), and each pair contributes $|V|$, which is even.

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The second congruence holds because an odd number of odd numbers is being summed up.

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Consequence

Given a graph G and $w \in \mathbb{Q}^{\mathrm{E}(G)}$, there exists a polynomial-time algorithm to decide whether w is an element of $\mathcal{MP}(G)$; if not, it provides an Edmonds condition violated by w.

Break Time

Introductory Lemma

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Lemma

The mapping $f = w \circ \partial : \mathcal{P}(V) \to \mathbb{R}_+$

(i) is symmetric, i.e., $f(S) = f(\overline{S}) \quad \forall S \subseteq V$,

(ii) is submodular, i.e.,
$f(S) + f(T) \geq f(S \cap T) + f(S \cup T) \quad \forall S, T \subseteq V$,

(iii) is posimodular, i.e.,
$f(S) + f(T) \geq f(S \setminus T) + f(T \setminus S) \quad \forall S, T \subseteq V$.

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(ii): Submodularity holds: Summing weights on both sides. In the left expression, each edge is counted at least as many times as in the right expression (by case analysis). Since weights are nonnegative, the inequality holds.

(iii): Posimodularity follows from the previous two properties:

$$f(S)+f(T)=f(S)+f(\overline{T})\geq f(S\cap\overline{T})+f(S\cup\overline{T})=f(S\setminus T)+f(\overline{T\setminus S})=$$

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This is equivalent to saying that cuts (S, T) and (S', T') are non-crossing cuts if $S \subseteq S'$ or $S' \subseteq S$ or $S \cap S' = \emptyset$.

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Case 2: x', y' are on different sides of $\mathcal{V}$ (suppose x' falls on the side of x).

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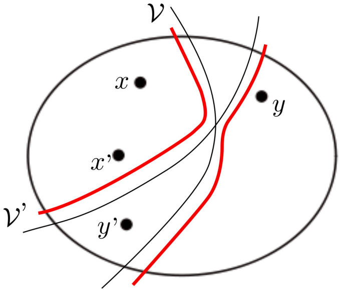

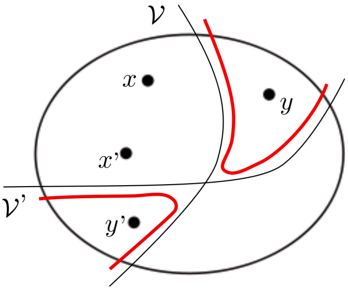

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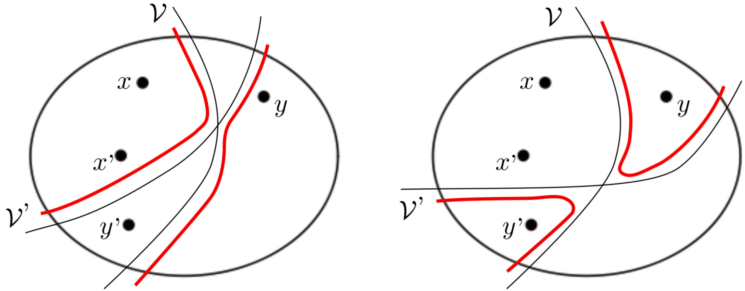

Among the cuts marked in red on the figure, one is an xy cut (let this be $\mathcal{V}^*$), and the other is an $x'y'$ cut (let this be $\mathcal{V}^{**}$).

Proof of the Main Lemma: Case 1 (Continued)

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Then by submodularity, or posimodularity (depending on which case we are in and how we label the S sides), we have

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However, equality must hold, since $\mathcal{V}$ and $\mathcal{V}'$ were optimal/minimal cuts and "performing their task" $\mathcal{V}^*$ and $\mathcal{V}^{**}$ also do. So,

$$f(\mathcal{V}') = f(\mathcal{V}^{**})$$

and $\mathcal{V}^{**}$ does not cross $\mathcal{V}$, thus $\mathcal{V}^{**}$ fulfills the desired property.

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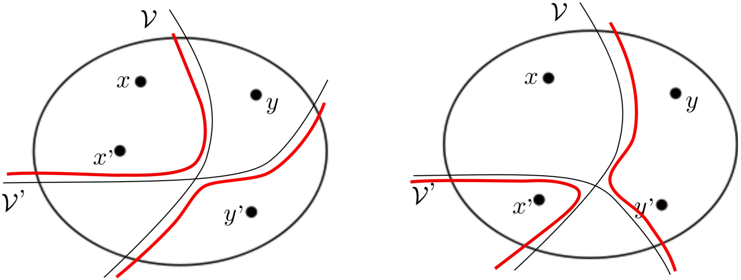
If $\mathcal{V}'$ does not separate x and y, then similarly as in the Case 1, another $x'y'$ optimal cut can be found that does not cross $\mathcal{V}$:



Figure

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The definition of G/S is similar, except here m_S is the new meta-vertex.

Bisection on a picture

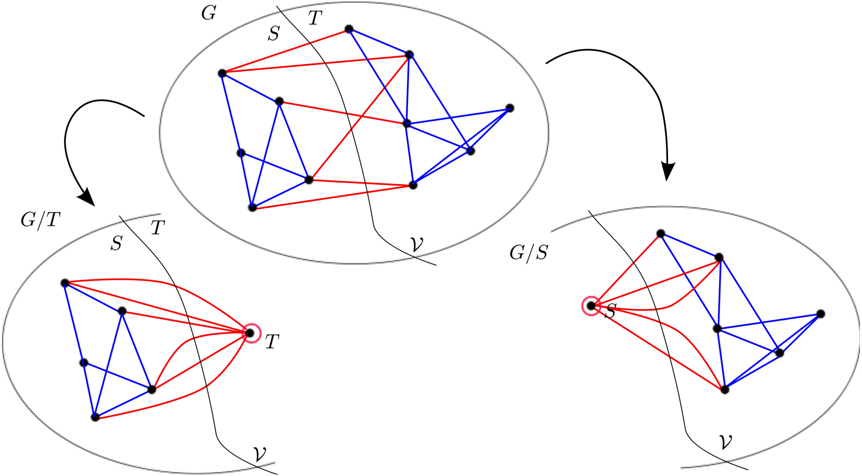

Figure: The initial bisection

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Gomory–Hu Algorithm: Recursive Bisection Algorithm

Perform the initial bisection.

While each part contains at least 2 original vertices, **repeat** the bisection (the vertices x and y defining the bisection are always original vertices).

The Gomory–Hu Algorithm in a Figure

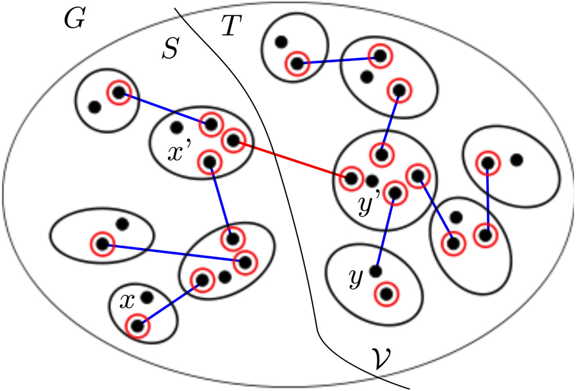

Figure: The meta-vertices are circled in red, and the edges passing through them are the meta-edges. x and y define the original $\mathcal{V}$ cut. Only one edge from the computed tree passes through this cut. This edge is not necessarily the xy edge.

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- Thus, the parts computed by the algorithm are identified with the vertices.
- The meta-edges connect different parts corresponding to different vertices.
- So, the meta-edges can be viewed as the edges between the original vertices.
- These meta-edges constitute the computed graph (on the vertex set of G).

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- We compute an $n-1$ edge graph on n vertices.
- It is clear from the recursion (and can be formally proved by induction) that the output is a connected graph.

Correctness of the Gomory–Hu Algorithm

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Theorem

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The assertion is to show that each cut defined by a meta-edge is an optimal separation of its endpoints. That is, we need to prove $n-1$ statements.

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Let $\mathcal{V}$ be the cut corresponding to the initial bisection. The cuts of G can be grouped as follows:

1. Crossing cuts with $\mathcal{V}$,
2. Non-crossing cuts with $\mathcal{V}'$:
   a) $\mathcal{V}' = \mathcal{V}$
   b) $\mathcal{V}' = (S', T')$, $S' \subsetneq S$
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Observation

The cuts of G/T can be paired with $\mathcal{V}$ and cuts of type 2.b), while cuts of G/S can be paired with $\mathcal{V}$ and cuts of type 2.c). All cuts in G/T and G/S are present in either $G \backslash T$ or $G \backslash S$ (and nowhere else).

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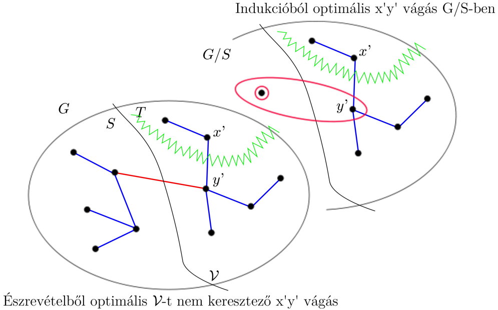

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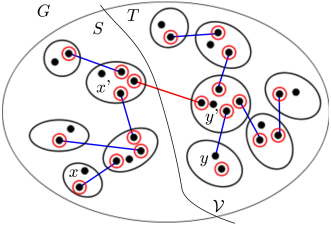

The meta-vertices are circled in red, and the edges passing through them are the meta-edges. x and y define the original $\mathcal{V}$ cut. Only one edge

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From the main lemma, we know that $\mathcal{V}' = (S', T')$ can be chosen such that $S' \subset S$ or $T' \subset T$. We may assume $S' \subsetneq S$.

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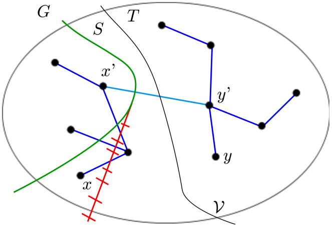

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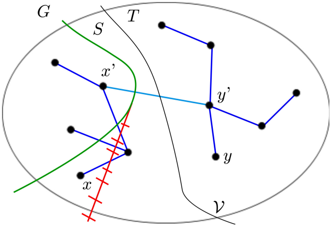

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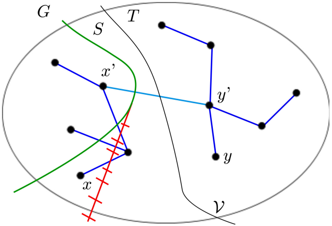

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That is, it cannot be that x and x' are on the same side while the entire T is on the other side, including y.

In this case, $\mathcal{V}'$ would be an xy cut, which is a contradiction. $\mathcal{V}'$ separates x and x'.

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By our initial observation, $\mathcal{V}''$ corresponds to a cut $\widetilde{\mathcal{V}''}$ in G. From the above, this is an xy cut, with weight smaller than $\mathcal{V}$'s weight, which contradicts.

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The contradiction proves the assertion, the only missing piece in proving the correctness of the Gomory–Hu algorithm.

Summary

Gomory–Hu Theorem

For every (G, w), there exists a Gomory–Hu tree F, which can be computed by determining $n-1$ minimal st cuts, achievable by applying the flow algorithm $n-1$ times. Specifically, the Gomory–Hu algorithm is polynomial.

This is the End!

Thnak you for your attention!