Integer polytopes

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- The half-space $c^{\mathsf{T}}x = \lambda$ lies on one side of $\mathcal{P}$.
- The minimal objective value is attained when λ is increased (pushing the hyperplane towards $\mathcal{P}$) until the moving hyperplane touches $\mathcal{P}$.
- Then $\mathcal{P}$ supports the hyperplane. The supporting points are the optimal points.

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(i) For every $c \in \mathbb{R}^n$, either $p^* = -\infty$ or there exists $x \in \mathrm{ext}(\mathcal{P})$ as an optimal point.

(ii) For every $x \in \mathrm{ext}(\mathcal{P})$, there exists c such that x is the unique optimal point.

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• Indeed. For $\alpha \geq 0$, $\alpha k \in \mathcal{C}$, so $t + \alpha k \in \mathcal{P}$. If $c^\mathsf{T} k < 0$, then the objective function can take arbitrarily small values.

• If $c^\mathsf{T} k \geq 0$, we can assume $k = 0$, i.e., o falls into the *polytope part* of our polyhedron.

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- Obviously, $c = \nu$ is a good choice.

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If $p^* \in \mathbb{R}$, then there exists $x \in \mathbb{Q}^n$ as an optimal point.

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- By Cramer's rule, the components of e are the ratio of the determinants of two matrices containing rational numbers, specifically rational.

Break Time

LP Relaxation of IP Problems

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Definition

From the following integer programming (IP) problem

Minimize	$c^T x$-t
subject to	$x \in \mathcal{P}$
	$x \in \mathbb{Z}^n$,

if we omit the condition $x \in \mathbb{Z}^n$, we obtain the associated linear programming (LP) problem

Minimize	$c^T x$-t
subject to	$x \in \mathcal{P}$.

This is called the LP relaxation of the original IP problem.

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If the optimum of the original IP problem is p_I^* and that of the LP relaxation is p^*, then

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- Through the LP relaxation, we easily obtain a lower bound on the optimal value.

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Definition

A polyhedron $\mathcal{P} = \langle g_1, g_2, \ldots, g_k \rangle_{\text{convex}} + \langle h_1, h_2, \ldots, h_\ell \rangle_{\text{cone}}$ is integral iff all generating vectors can be chosen from $\mathbb{Z}^n$.

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Definition

$\mathcal{P} = \{x \colon Ax \preceq b\}$ is a regular polyhedron integral if $\mathrm{ext}(\mathcal{P}) \subseteq \mathbb{Z}^n$, meaning that every extremal point has integer coordinates, and $A \in \mathbb{Q}^{k \times n}$, $b \in \mathbb{Q}^k$.

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- For polytopes, the previous definition is equivalent to $\mathcal{P}$ being integral if the convex hull of finitely many $\mathbb{Z}^n$ points.

- From the above, if the IP problem defined by the continuous constraints is integral, then the LP relaxation will have integral optimal points (since the vertices of $\mathcal{P}$ are integral). In this case, $p_I^* = p^*$.

Conditions Guaranteeing Integrality of Polyhedra I

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Edmonds–Giles Theorem

Let $\mathcal{P} = \{x\colon Ax \preceq b\} \neq \emptyset$ be a nice polyhedron, $A \in \mathbb{Q}^{k\times n}$, $b \in \mathbb{Q}^k$. Then the following are equivalent:

(i) $\mathcal{P}$ is an integral polyhedron (i.e., $\mathrm{ext}(\mathcal{P}) \subseteq \mathbb{Z}^n$).

(ii) For every $c \in \mathbb{R}^n$ objective vector, the LP problem

Minimize	$c^\mathsf{T}x$-t
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either has $p^* = -\infty$ or has an optimal point in $\mathbb{Z}^n$.

(iii) For every $c \in \mathbb{Z}^n$, the optimal value of the LP problem

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is either $-\infty$ or integral.

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The equivalence of (i) with (ii) follows from earlier results.

The (i)$\Rightarrow$(iii) is indeed true, as if a linear function attains its minimum on a polyhedron, it does so at a vertex.

If the coordinates of this optimal point and the objective function are integers, then the objective function value is also integral. (Of course, this also establishes the validity of (ii) in this case.)

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(iii)⇒(i)

- Let $e \in \mathrm{ext}(\mathcal{P})$, which means that there exists $\nu \neq 0 \in \mathbb{R}^n$ and $\tau \in \mathbb{R}$ such that

$$(\star) \qquad \mathcal{P} \subset \{x \colon \nu^{\mathsf{T}} x \geq \tau\} \text{ and } \nu^{\mathsf{T}} e = \tau.$$

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- Based on these two remarks, there exists a $\nu \in \mathbb{Z}^n$ vector such that both ν and the vectors $(\nu + e_i)$ are suitable for satisfying $(\star)$,

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• Based on these two remarks, there exists a $\nu \in \mathbb{Z}^n$ vector such that both ν and the vectors $(\nu + e_i)$ are suitable for satisfying $(\star)$, where e_i are the standard unit vectors in n dimensions ($e_i = (0, \dots, 0, 1, 0, \dots, 0)^{\mathsf{T}}$).

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- Furthermore, their corresponding optimal values are $\nu^\mathsf{T} e$ and $(\nu + e_i)^\mathsf{T} e$.

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- Specifically, the i-th coordinate of $e \in \mathrm{ext}(\mathcal{P})$: $e_i^\mathsf{T} e = (\nu + e_i)^\mathsf{T} e - \nu^\mathsf{T} e$ is also an integer.

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- Therefore, according to (iii), the optimal values of $\nu^\mathsf{T} e$ and $(\nu + e_i)^\mathsf{T} e$ are integers.
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- Thus, each component of e is an integer, i.e., e is an integer vector.

Break

Conditions Guaranteeing Integrality II: Totally Unimodular Matrices

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Theorem

If $A \in \mathbb{R}^{k\times n}$ is a totally unimodular matrix and $b \in \mathbb{Z}^k$, then $\mathcal{P} = \{x \in \mathbb{R}^n \colon Ax \preceq b\}$ is an integral polyhedron.

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- That is, A has rows such as $a_{i_1}^{\mathsf{T}}, \ldots, a_{i_n}^{\mathsf{T}}$, which are linearly independent and satisfy

$$a_{i_1}^{\mathsf{T}} e = b_{i_1}$$
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When calculating each coordinate, we work with integers, and there is only one division involved. The divisor is the determinant of a square submatrix of A. The submatrix does not degenerate, so its determinant cannot be 0. Thus, its value is -1 or 1. Dividing by this does not lead to non-integer results.

Totally Unimodular Matrices: Example I

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• The vertex-edge incidence matrix of a loopless graph G is denoted by $\mathcal{B}_G$, where the rows correspond to the vertices and the columns correspond to the edges, and at the intersection of a vertex $v \in V$ row and an edge $e \in E$ column, we have

$$(\mathcal{B}_G)_{v,e} = \begin{cases} 1, & \text{if } v \text{ is incident to } e, \\ 0, & \text{otherwise.} \end{cases}$$

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• Let G be a complete graph on three vertices: Then

$$\mathcal{B}_{K_3} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

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• The complete matrix is a square submatrix of itself. Since $\det \mathcal{B}_{K_3} = 2$, $\mathcal{B}_{K_3}$ is not TU.

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- Specifically, if a graph contains an odd-length cycle (which is equivalent to being non-bipartite), then its vertex-edge incidence matrix is not TU.

- We will see that if G is a bipartite graph, then $\mathcal{B}_G$ is a TU matrix.

Totally Unimodular Matrices: Examples III

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Example

Example $\overrightarrow{G}$ is a loopless directed graph. Then, the vertex-edge incidence matrix $\mathcal{D}$ of $\overrightarrow{G}$ has an element $\mathcal{D}_{v,e}$ (the element at the intersection of the row corresponding to vertex v and the column corresponding to edge e) given by:

$$\mathcal{D}_{v,e} = \begin{cases} +1, & \text{if the edge "enters" the vertex} \\ -1, & \text{if the edge "leaves" the vertex} \\ 0, & \text{otherwise.} \end{cases}$$

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- We will show that for any directed graph G, $\mathcal{D}_G$ matrix is totally unimodular.

Totally Unimodular Matrices: Operations

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Lemma

Let A be a totally unimodular matrix. Form $\widetilde{A}$ from A by the following rules/operations:

(i) Multiplying rows/columns by -1.

(ii) Deleting rows/columns.

(iii) Repeating existing rows/columns.

(iv) Adding rows/columns with e_i where e_i contains exactly one non-zero element which is 1.

(v) Transposing.

Then the resulting $\widetilde{A}$ matrix is also totally unimodular.

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Consequence

The weighted matching problem on bipartite graphs can be solved using an LP algorithm.

Consequences: Weighted Matching Problem as an IP

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- Identifying the weight function c with a vector $c \in \mathbb{R}^{E(G)}$, the problem becomes the following

Minimize	$c^T x$-t	
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integer programming formulation.

- We obtain its LP relaxation by removing the $x_e \in \mathbb{Z}$ constraints:

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- Thus, the LP relaxation's vertices are integer-coordinate, i.e., they correspond to matchings.

- So the LP relaxation is equivalent to the original formulation. An LP problem can be efficiently handled in many ways.

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• The same applies to the dual problem. When searching for the optimal dual solution, we can confine ourselves to integral dual feasible solutions. We utilized this in one of our previous examples of duality.

Break

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Min $c^T x$ st. $Ax \preceq b$, $x \in \mathbb{Z}^n$	Min $c^T x$ st. $Ax \preceq b$	Max $-b^T\lambda$ st. $c + A^T\lambda = 0$, $\lambda \succeq 0$	Max $-b^T\lambda$ st. $c + A^T\lambda = 0$, $\lambda \in \mathbb{N}^k$

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$(\mathrm{P})_{\mathbb{Z}}$:	(P) :	(D) :	$(\mathrm{D})_{\mathbb{Z}}$:
Min $c^T x$ st. $Ax \preceq b$ $x \in \mathbb{Z}^n$	Min $c^T x$ st. $Ax \preceq b$	Max $-b^T \lambda$ st. $c + A^T \lambda = 0$ $\lambda \succeq 0$	Max $-b^T \lambda$ st. $c + A^T \lambda = 0$ $\lambda \in \mathbb{N}^k$

$$p^*_{\mathbb{Z}} \geq p^* = d^* \geq d^*_{\mathbb{Z}},$$

where $p^*_{\mathbb{Z}}, p^*, d^*, d^*_{\mathbb{Z}}$ are the optimal values of the respective optimization problems (in the specified order).

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- Thus, specifically for a TPI system (since $p^*_{\mathbb{Z}}$ is obviously integral if finite), p^* is also integral (if finite).

- We know that this is equivalent to $\mathcal{P}$ being an integral polyhedron.

TDI Systems

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Definition Let $A \in \mathbb{Q}^{k \times n}$, $b \in \mathbb{Q}^k$. The inequality system $\mathcal{E} : Ax \preceq b$ is dual integral (TDI) if for every $c \in \mathbb{Z}^n$, $d^* = d^*_{\mathbb{Z}}$ (assuming d^* is finite).

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- The TDI property fundamental theorem states that if for every $c \in \mathbb{Z}^n$, the last inequality in our inequality chain is an equality, then necessarily the first inequality is also an equality for every $c \in \mathbb{Z}^n$.

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If $\mathcal{E} : Ax \preceq b$ is TDI and $b \in \mathbb{Z}^k$, then it is also TPI. Thus, $\mathcal{P}$ is an integral polyhedron.

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• So if we want to prove that a polyhedron is integral, our plan could be as follows:

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• So if we want to prove that a polyhedron is integral, our plan could be as follows:

(1) We „cleverly" express the polyhedron as $\{x : Ax \preceq b\}$.

(2) We show that $Ax \preceq b$ is a TDI system. That is, we show that the

Minimize	$b^{\mathsf{T}}x$-t
subject to	$A^{\mathsf{T}}\lambda = -c$
	$\lambda \succeq 0$

problem has an integral optimal solution for every $c \in \mathbb{Z}^n$.

(3) We conclude the integrality of $\mathcal{P}$.

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- Based on the TDI property, we know that for every $c \in \mathbb{Z}^n$, $p^* = d^* = d^*_{\mathbb{Z}}$. Since $b \in \mathbb{Z}^k$, $d^*_{\mathbb{Z}}$ is integral. Thus, p^* is integral for every $c \in \mathbb{Z}^n$.

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• We have seen („earlier" Edmonds—Giles Theorem) that from this, we can deduce the integrality of $\mathcal{P}$.

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- If G is a bipartite graph, equality holds. In the general case, more inequalities are needed to describe the convex hull.

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Let G be any simple graph. Then

$$\operatorname{conv}\{\chi_M : M \text{ matching}\} = \{x \in \mathbb{R}^{E(G)} : \\ x_e \geq 0 \quad \forall e \in E(G) \\ \sum_{e:v I e} x_e \leq 1 \quad \forall v \in V(G) \\ \sum_{\substack{e=xy\in E(G):\\ x\in S, y\notin S}} x_e \leq \frac{|S|-1}{2} \quad \forall S \in \mathcal{O}\},$$

where $\mathcal{O}$ is the set of subsets of V with odd number of elements.

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Cunningham—Marsh Theorem

The inequality system appearing in the Edmonds' description of $\mathcal{MP}(G)$ is dual integral (TDI).

- That is, for any $c \in \mathbb{Z}^n$,

Minimize	$\sum_{v\in V(G)} \lambda_v + \sum_{S\in\mathcal{O}} \frac{\lvert S\rvert-1}{2} \cdot \lambda_S$-t
subject to	$-c_e + \lambda_u + \lambda_v + \sum_{\substack{S\in\mathcal{O}\\ u,v\in S}} \lambda_S - \lambda_e = 0$
	$\forall e = uv \in E(G)$, and $\lambda \succeq 0$.

Then there exists an integral optimal solution.

Break

Edmonds' Polyhedron Theorem II

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$$\mathcal{PMP}(G) = \mathrm{conv}\{\chi_M : M \text{ perfect matching}\} =$$

$$= \{x \in \mathbb{R}^{E(G)} : \begin{array}{rl} x_e \geq 0, & e \in E(G) \\ \sum_{e: v \mathrm{I} e} x_e \leq 1, & v \in V(G), \\ \sum_{\substack{e=xy \in E(G): \\ x \in S, y \notin S}} x_e \geq 1, & S \subseteq V(G), \\ & |S| \text{ odd}\}. \end{array}$$

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$$\sum_{\substack{e=xy \in E(G): \\ x \in S, y \notin S}} x_e \geq 1, \quad S \subseteq V(G),$$
$$|S| \text{ odd}\}.$$

- $\mathcal{PMP}(G)$ is the perfect matching polytope of graph G.

Consequence of Edmonds' Polyhedron Theorem

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Theorem

G is a k-regular, k-edge-connected graph with an even number of vertices. Then there exists a positive integer t such that

$$\chi_e(t \times G) = t \cdot k,$$

where $t \times G$ is the graph obtained from G by multiplying its edges by t (alternatively, we add $t-1$ "twin copies" to each edge of G).

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- The minimum number of colors needed for this is the edge chromatic number of the graph.

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Reminder: Vizing's Theorem

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• Since the vertices of the polytope are integral, our vector is rational, so the α_M's can be assumed to be rational, i.e., $(\alpha_M) \in \mathbb{Q}^E$, thus $L \in \mathbb{N}_+$, $\ell_M \in \mathbb{N}$.

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- The relationship sorted becomes

$$L \cdot \underline{1} = \sum (k \cdot \ell_M) \chi_M.$$

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- The color demand:

$$\sum_{M \text{ matching}} k\ell_M = k \sum_{M \text{ matching}} \ell_M = kL \sum_{M \text{ matching}} \frac{\ell_M}{L} =$$

$$= kL \sum_{M \text{ matching}} \alpha_M = kL, \text{ since } \sum_{M \text{ matching}} \alpha_M = 1.$$

Break

Reminder: Cunningham—Marsh Theorem

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Minimize	$\sum_{v\in V(G)} \lambda_v + \sum_{S\in\mathcal{O}} \frac{\lvert S\rvert-1}{2} \cdot \lambda_S$-t
subject to	$-c_e + \lambda_u + \lambda_v + \sum_{\substack{S\in\mathcal{O}\\ u,v\in S}} \lambda_S - \lambda_e = 0$
	$\forall e = uv \in E(G)$, and $\lambda \succeq 0$.

Then there exists an integer feasible solution.

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- Equivalently:

Minimize	$\sum_{v\in V(G)} \lambda_v + \sum_{S\in\mathcal{O}} \frac{\lvert S\rvert-1}{2} \cdot \lambda_S$-t
subject to	$\lambda_u + \lambda_v + \sum_{\substack{S\in\mathcal{O}\\ u,v\in S}} \lambda_S \geq c_e$
	$\forall e = uv \in E(G)$, and $\lambda \succeq 0$.

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New Form of Cunningham—Marsh Theorem

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Cunningham—Marsh Theorem

Let $(c_e)_{c\in E(G)} \in \mathbb{Z}^{E(G)}$ be an arbitrary integral edge weighting of G. Then there exists $(\lambda_v) \in \mathbb{R}_+^V, (\lambda_S) \in \mathbb{R}_+^{\mathcal{O}}$ satisfying

$$\lambda_u + \lambda_v + \sum_{\substack{S\in\mathcal{O}\\ u,v\in S}} \lambda_S \geq c_e \quad \forall e = uv \in E(G)$$

and

$$\sum_{v\in V(G)} \lambda_v + \sum_{S\in\mathcal{O}} \frac{|S|-1}{2}\lambda_S \leq \nu_c(G),$$

furthermore, these are integral solutions.

New Form of Cunningham—Marsh Theorem: Justification

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- Satisfying the additional condition, we have

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(The last inequality holds due to the weak duality for maximization problems), thus guaranteeing that our possible dual solution is optimal.

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• We carry out a complete induction on $|V| + |E| + \sum_{e\in E(G)} c(e)$. Verification of the cases of small graphs (with small weights) is straightforward, left as an exercise for the interested reader.

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$$c'_e = \begin{cases} c_e - 1, & \text{if } v I e \\ c_e, & \text{otherwise.} \end{cases}$$

$$\Big\downarrow \text{ induction assumption}$$

possible, integral dual λ'

$$\sum_{v \in V(G)} \lambda'_v + \sum_{S \in \mathcal{O}} \frac{|S|-1}{2} \lambda'_S \leq \nu_{c'}(G)$$

$$\longleftarrow$$

$$\lambda_u = \begin{cases} \lambda'_v + 1, & \text{if } u = v \\ \lambda'_u, & \text{otherwise,} \end{cases}$$

$$\lambda_S = \lambda'_S \text{ for all } S \in \mathcal{O}$$

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Claim

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- It is easy to see that by dropping the primes, both sides increase by 1, from which the claim follows.

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The c'-optimal matching leaves only one vertex unmatched.

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Specifically, the cardinality of V is odd, thus $V \in \mathcal{O}$.

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- The definition of c' guarantees that the right side increases by one, where M is a c'-optimal matching.
- The CONDITION ensures that the increase is $|M|$, where M is a c'-optimal matching.
- On the left side, only one term changes: the dual variable indexed by V. Its coefficient is $\frac{|V|-1}{2}$, and its value increases by 1. The claim is obvious.

Justification of ⋆ CONDITION: 1st Lemma

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A c'-optimal matching cannot be a perfect matching.

• Let M be a c-optimal matching. Since we are in Case 2, we may assume that M is not perfect.

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• This follows from the following two lemmas. In the justification, we assume that the conditions of Case 2 are satisfied.

Lemma

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• Let M be a c-optimal matching. Since we are in Case 2, we may assume that M is not perfect.

• Let M' be a c'-optimal matching. Indirectly, assume that M' is perfect.

Validity of $\star$ CONDITION: 1st Lemma (continued)

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• Since M is c-optimal, we have $c(M') \leq c(M)$. Knowing that M is not perfect, we can say something about the weight under c' as well:

$$c'(M) = c(M) - |M| > c(M) - \frac{|V|}{2} \geq c(M') - \frac{|V|}{2}.$$

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- This contradicts the fact that M' is a c'-optimal matching.

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- Indirectly assume that M' is c'-optimal and $x, y \in V$ such that M' does not cover x and y. Let (M', x, y) be such that $d(x, y)$ is minimized.

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• Indirectly assume that M' is c'-optimal and $x, y \in V$ such that M' does not cover x and y. Let (M', x, y) be such that $d(x, y)$ is minimized.

• $d(x, y) > 1$, because the connectivity between x and y would ensure that $M' \cup \{xy \text{ edge}\}$ is also a matching, contradicting the c'-optimality ($c' > 0$). (Generally, an optimal matching cannot leave two connected vertices unmatched.)

Validity of $\star$ CONDITION: 2nd Lemma (continued)

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- Let x^+ be the first vertex following x on a shortest xy path (towards y). (Due to the above, $x^+ \neq y$.) Consider the following two matchings:

(1) M_{x^+}: a c-optimal matching that does not cover x^+ (such exists in Case 2).

(2) M'. The c'-optimality ensures that M' covers x^+ (x and x^+ are connected).

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- The components of the graph $\mathcal{M}$ formed by the edges of $M_{x^+} \Delta M'$ are cycles and paths (BSc Combinatorics course).

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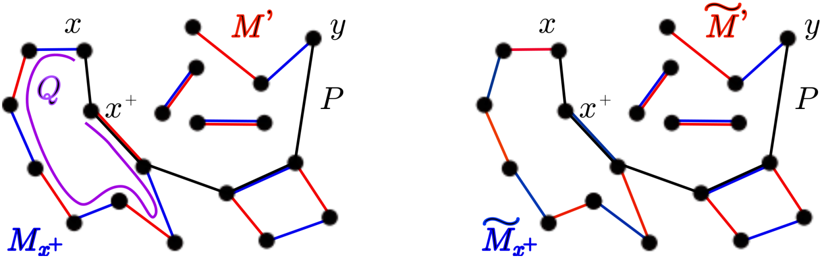
- Due to the properties of our matchings, x^+ is a degree 1 vertex in $\mathcal{M}$. Thus x^+ is an endpoint of a path Q in $\mathcal{M}$. Let

$$\widetilde{M}_{x^+} = M_x \Delta E(Q) \quad \text{and} \quad \widetilde{M}' = M' \Delta E(Q).$$

Validity of $\star$ CONDITION: 2nd Lemma: Figure

Validity of ⋆ CONDITION: 2nd Lemma: Figure



On the left, black edges denote the P path edges, red edges denote M_{x^+} edges, blue edges denote M' edges, purple indicates the Q path. On the right, the modified matchings ($\widetilde{M'}$ and $\widetilde{M_{x^+}}$): we exchange the red and blue edges along the Q/purple path. The total weight of red and blue edges remains the same on both sides.

Validity of $\star$ CONDITION: 2nd Lemma (continued)

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- Then due to the c-optimality of M_{x^+}

$$c(\widetilde{M}_{x^+}) \leq c(M_{x^+}).$$

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$$c(\widetilde{M}_{x^+}) \leq c(M_{x^+}).$$

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- But with a little insight, we can say more: $\widetilde{M}'$ does not cover the vertex x^+.

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- Since $d(x^+, x) = 1 < d(x, y)$ and $d(x^+, y) = d(x, y) - 1 < d(x, y)$ also hold, then (M', x, y) being the choice implies $\widetilde{M}'$ cannot be c'-optimal:

$$c'(\widetilde{M'}) < c'(M').$$

Validity of $\star$ CONDITION: 2nd Lemma (continued)

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- This contradicts the sum of (1) and (2). From this, the assertion follows.

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- Thus, the consideration of Case 2 is justified.
- The proof is complete.

This is the End!

Thank you for your attention!