Integer polytopes

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2024. Fall

LP Relaxation of IP Problems

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Definition

From the following integer programming (IP) problem

Minimize	$c^{\mathsf{T}}x$-t
subject to	$x \in \mathcal{P}$
	$x \in \mathbb{Z}^n$,

if we omit the condition $x \in \mathbb{Z}^n$, we obtain the associated linear programming (LP) problem

Minimize	$c^{\mathsf{T}}x$-t
subject to	$x \in \mathcal{P}$.

This is called the LP relaxation of the original IP problem.

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If the optimum of the original IP problem is p_I^* and that of the LP relaxation is p^*, then

$$p^* \leq p_I^*.$$

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If the optimum of the original IP problem is p_I^* and that of the LP relaxation is p^*, then

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• Through the LP relaxation, we easily obtain a lower bound on the optimal value.

Integral Polyhedra

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A polyhedron $\mathcal{P} = \langle g_1, g_2, \ldots, g_k \rangle_{\text{convex}} + \langle h_1, h_2, \ldots, h_\ell \rangle_{\text{cone}}$ is integral iff all generating vectors can be chosen from $\mathbb{Z}^n$.

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$\mathcal{P} = \{x \colon Ax \preceq b\}$ is a regular polyhedron integral if $\mathrm{ext}(\mathcal{P}) \subseteq \mathbb{Z}^n$, meaning that every extremal point has integer coordinates, and $A \in \mathbb{Q}^{k \times n}$, $b \in \mathbb{Q}^k$.

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• For polytopes, the previous definition is equivalent to $\mathcal{P}$ being integral if the convex hull of finitely many $\mathbb{Z}^n$ points.

• From the above, if the IP problem defined by the continuous constraints is integral, then the LP relaxation will have integral optimal points (since the vertices of $\mathcal{P}$ are integral). In this case, $p_I^* = p^*$.

Conditions Guaranteeing Integrality of Polyhedra I

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Edmonds–Giles Theorem

Let $\mathcal{P} = \{x\colon Ax \preceq b\} \neq \emptyset$ be a polyhedron, $A \in \mathbb{Q}^{k\times n}$, $b \in \mathbb{Q}^k$. Then the following are equivalent:

(i) $\mathcal{P}$ is an integral polyhedron (i.e., $\mathrm{ext}(\mathcal{P}) \subseteq \mathbb{Z}^n$).

(ii) For every $c \in \mathbb{R}^n$ objective vector, the LP problem

Minimize	$c^{\mathsf{T}}x$-t
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either has $p^* = -\infty$ or has an optimal point in $\mathbb{Z}^n$.

(iii) For every $c \in \mathbb{Z}^n$, the optimal value of the LP problem

Minimize	$c^{\mathsf{T}}x$-t
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is either $-\infty$ or integral.

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The (i)$\Rightarrow$(iii) is indeed true, as if a linear function attains its minimum on a polyhedron, it does so at a vertex.

If the coordinates of this optimal point and the objective function are integers, then the objective function value is also integral. (Of course, this also establishes the validity of (ii) in this case.)

The proof

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- Let $e \in \mathrm{ext}(\mathcal{P})$, which means that there exists $\nu \neq 0 \in \mathbb{R}^n$ and $\tau \in \mathbb{R}$ such that

$$(\star) \qquad \mathcal{P} \subset \{x\colon \nu^\mathsf{T} x \geq \tau\} \text{ and } \nu^\mathsf{T} e = \tau.$$

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- The normal vector ν is not unique.

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• Based on these two remarks, there exists a $\nu \in \mathbb{Z}^n$ vector such that both ν and the vectors $(\nu + e_i)$ are suitable for satisfying $(\star)$, (iii)$\Rightarrow$(i)

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• Based on these two remarks, there exists a $\nu \in \mathbb{Z}^n$ vector such that both ν and the vectors $(\nu + e_i)$ are suitable for satisfying $(\star)$, where e_i are the standard unit vectors in n dimensions ($e_i = (0, \ldots, 0, 1, 0, \ldots, 0)^{\mathsf{T}}$).

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- Specifically, the i-th coordinate of $e \in \mathrm{ext}(\mathcal{P})$: $e_i^{\mathsf{T}} e = (\nu + e_i)^{\mathsf{T}} e - \nu^{\mathsf{T}} e$ is also an integer.

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- Thus, each component of e is an integer, i.e., e is an integer vector.

Break

Conditions Guaranteeing Integrality II: Totally Unimodular Matrices

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Theorem

If $A \in \mathbb{R}^{k \times n}$ is a totally unimodular matrix and $b \in \mathbb{Z}^k$, then $\mathcal{P} = \{x \in \mathbb{R}^n : Ax \preceq b\}$ is an integral polyhedron.

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- That is, A has rows such as $a_{i_1}^{\mathsf{T}}, \ldots, a_{i_n}^{\mathsf{T}}$, which are linearly independent and satisfy

$$a_{i_1}^{\mathsf{T}} e = b_{i_1}$$
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When calculating each coordinate, we work with integers, and there is only one division involved. The divisor is the determinant of a square submatrix of A. The submatrix does not degenerate, so its determinant cannot be 0. Thus, its value is -1 or 1. Dividing by this does not lead to non-integer results.

Totally Unimodular Matrices: Example I

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• The vertex-edge incidence matrix of a loopless graph G is denoted by $\mathcal{B}_G$, where the rows correspond to the vertices and the columns correspond to the edges, and at the intersection of a vertex $v \in V$ row and an edge $e \in E$ column, we have

$$(\mathcal{B}_G)_{v,e} = \begin{cases} 1, & \text{if } v \text{ is incident to } e, \\ 0, & \text{otherwise.} \end{cases}$$

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• Let G be a complete graph on three vertices: Then

$$\mathcal{B}_{K_3} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

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$$\mathcal{B}_{K_3} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

• The complete matrix is a square submatrix of itself. Since $\det \mathcal{B}_{K_3} = 2$, $\mathcal{B}_{K_3}$ is not TU.

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- Specifically, if a graph contains an odd-length cycle (which is equivalent to being non-bipartite), then its vertex-edge incidence matrix is not TU.

- We will see that if G is a bipartite graph, then $\mathcal{B}_G$ is a TU matrix.

Totally Unimodular Matrices: Examples III

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Example

Example $\overrightarrow{G}$ is a loopless directed graph. Then, the vertex-edge incidence matrix $\mathcal{D}$ of $\overrightarrow{G}$ has an element $\mathcal{D}_{v,e}$ (the element at the intersection of the row corresponding to vertex v and the column corresponding to edge e) given by:

$$\mathcal{D}_{v,e} = \begin{cases} +1, & \text{if the edge "enters" the vertex} \\ -1, & \text{if the edge "leaves" the vertex} \\ 0, & \text{otherwise.} \end{cases}$$

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- We will show that for any directed graph G, $\mathcal{D}_G$ matrix is totally unimodular.

Totally Unimodular Matrices: Operations

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Lemma

Let A be a totally unimodular matrix. Form $\widetilde{A}$ from A by the following rules/operations:

(i) Multiplying rows/columns by -1.

(ii) Deleting rows/columns.

(iii) Repeating existing rows/columns.

(iv) Adding rows/columns with e_i where e_i contains exactly one non-zero element which is 1.

(v) Transposing.

Then the resulting $\widetilde{A}$ matrix is also totally unimodular.

Theorem

(i) Let G be any bipartite graph. Then $\mathcal{B}_G$ is a TU matrix.

(ii) Let $\overrightarrow{G}$ be any directed graph. Then $\mathcal{D}_{\overrightarrow{G}}$ is a TU matrix.

Totally Unimodular Matrices: Examples and Proofs

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- The statement holds for $k = 1$ since all elements of both matrices are from the set $\{-1, 0, 1\}$.

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Consequence

The weighted matching problem on bipartite graphs can be solved using an LP algorithm.

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- Identifying the weight function c with a vector $c \in \mathbb{R}^{E(G)}$, the problem becomes the following

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integer programming formulation.

- We obtain its LP relaxation by removing the $x_e \in \mathbb{Z}$ constraints:

$$\begin{array}{|ll|}\hline \text{Minimize} & c^T x\text{-t} \\ \text{subject to} & \sum_{e:v|e} x_e \le 1, \\ & x_e \ge 0. \\ \hline \end{array}$$

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• Thus, the LP relaxation's vertices are integer-coordinate, i.e., they correspond to matchings.

• So the LP relaxation is equivalent to the original formulation. An LP problem can be efficiently handled in many ways.

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Break

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$$p^*_{\mathbb{Z}} \geq p^* = d^* \geq d^*_{\mathbb{Z}},$$

where $p^*_{\mathbb{Z}}, p^*, d^*, d^*_{\mathbb{Z}}$ are the optimal values of the respective optimization problems (in the specified order).

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- Thus, specifically for a TPI system (since $p^*_{\mathbb{Z}}$ is obviously integral if finite), p^* is also integral (if finite).
- We know that this is equivalent to $\mathcal{P}$ being an integral polyhedron.

TDI Systems

Definition

Definition Let $A \in \mathbb{Q}^{k \times n}$, $b \in \mathbb{Q}^k$. The inequality system $\mathcal{E} : Ax \preceq b$ is dual integral (TDI) if for every $c \in \mathbb{Z}^n$, $d^* = d^*_{\mathbb{Z}}$ (assuming d^* is finite).

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- The TDI property fundamental theorem states that if for every $c \in \mathbb{Z}^n$, the last inequality in our inequality chain is an equality, then necessarily the first inequality is also an equality for every $c \in \mathbb{Z}^n$.

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- So if we want to prove that a polyhedron is integral, our plan could be as follows:

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Minimize	$b^{\mathsf{T}}x$-t
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	$\lambda \succeq 0$

problem has an integral optimal solution for every $c \in \mathbb{Z}^n$.

(3) We conclude the integrality of $\mathcal{P}$.

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- Based on the TDI property, we know that for every $c \in \mathbb{Z}^n$, $p^* = d^* = d^*_{\mathbb{Z}}$. Since $b \in \mathbb{Z}^k$, $d^*_{\mathbb{Z}}$ is integral. Thus, p^* is integral for every $c \in \mathbb{Z}^n$.

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- We have seen („earlier" Edmonds—Giles Theorem) that from this, we can deduce the integrality of $\mathcal{P}$.

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Let G be any simple graph. Then

$$\operatorname{conv}\{\chi_M : M \text{ matching}\} = \{x \in \mathbb{R}^{E(G)} : \quad x_e \geq 0 \quad \forall e \in E(G) \quad \sum_{e:v I e} x_e \leq 1 \quad \forall v \in V(G) \quad \sum_{\substack{e=xy \in E(G): \\ x \in S, y \notin S}} x_e \leq \frac{|S|-1}{2} \quad \forall S \in \mathcal{O}\},$$

where $\mathcal{O}$ is the set of subsets of V with odd number of elements.

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The inequality system appearing in the Edmonds' description of $\mathcal{MP}(G)$ is dual integral (TDI).

- That is, for any $c \in \mathbb{Z}^n$,

Minimize	$\sum_{v\in V(G)} \lambda_v + \sum_{S\in\mathcal{O}} \frac{\lvert S\rvert-1}{2} \cdot \lambda_S$-t
subject to	$-c_e + \lambda_u + \lambda_v + \sum_{\substack{S\in\mathcal{O}\\ u,v\in S}} \lambda_S - \lambda_e = 0$
	$\forall e = uv \in E(G)$, and $\lambda \succeq 0$.

Then there exists an integral optimal solution.

Break

Edmonds' Polyhedron Theorem II

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$$\mathcal{PMP}(G) = \text{conv}\{\chi_M : M \text{ perfect matching}\} =$$

$$= \{x \in \mathbb{R}^{E(G)} : \begin{array}{rl} x_e \geq 0, & e \in E(G) \\ \sum_{e: v \text{I} e} x_e \leq 1, & v \in V(G), \\ \sum_{\substack{e=xy \in E(G): \\ x \in S, y \notin S}} x_e \geq 1, & S \subseteq V(G), \\ & |S| \text{ odd}\}. \end{array}$$

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- $\mathcal{PMP}(G)$ is the perfect matching polytope of graph G.

Consequence of Edmonds' Polyhedron Theorem

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Theorem

G is a k-regular, k-edge-connected graph with an even number of vertices. Then there exists a positive integer t such that

$$\chi_e(t \times G) = t \cdot k,$$

where $t \times G$ is the graph obtained from G by multiplying its edges by t (alternatively, we add $t-1$ "twin copies" to each edge of G).

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- The minimum number of colors needed for this is the edge chromatic number of the graph.

Reminder

Reminder: Vizing's Theorem

If G is a simple graph, then

$$D(G) \leq \chi_e(G) \leq D(G) + 1,$$

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- Summing up: $\frac{1}{k}\underline{1} \in \mathcal{MP}(\mathcal{G})$

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• Since the vertices of the polytope are integral, our vector is rational, so the α_M's can be assumed to be rational, i.e., $(\alpha_M) \in \mathbb{Q}^E$, thus $L \in \mathbb{N}_+$, $\ell_M \in \mathbb{N}$.

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• The relationship sorted becomes

$$L \cdot \underline{1} = \sum (k \cdot \ell_M) \chi_M.$$

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- The color demand:

$$\sum_{M \text{ matching}} k\ell_M = k \sum_{M \text{ matching}} \ell_M = kL \sum_{M \text{ matching}} \frac{\ell_M}{L} =$$
$$= kL \sum_{M \text{ matching}} \alpha_M = kL, \text{ since } \sum_{M \text{ matching}} \alpha_M = 1.$$

Break

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Minimize	$\sum_{v\in V(G)} \lambda_v + \sum_{S\in\mathcal{O}} \frac{\lvert S\rvert-1}{2}\cdot\lambda_S$-t
subject to	$-c_e + \lambda_u + \lambda_v + \sum_{\substack{S\in\mathcal{O}\\ u,v\in S}} \lambda_S - \lambda_e = 0$
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- Equivalently:

Minimize	$\sum_{v\in V(G)} \lambda_v + \sum_{S\in\mathcal{O}} \frac{\lvert S\rvert-1}{2} \cdot \lambda_S$-t
subject to	$\lambda_u + \lambda_v + \sum_{\substack{S\in\mathcal{O}\\ u,v\in S}} \lambda_S \geq c_e$
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New Form of Cunningham—Marsh Theorem

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Cunningham—Marsh Theorem

Let $(c_e)_{c\in E(G)} \in \mathbb{Z}^{E(G)}$ be an arbitrary integral edge weighting of G. Then there exists $(\lambda_v) \in \mathbb{R}_+^V, (\lambda_S) \in \mathbb{R}_+^{\mathcal{O}}$ satisfying

$$\lambda_u + \lambda_v + \sum_{\substack{S\in\mathcal{O}\\ u,v\in S}} \lambda_S \geq c_e \quad \forall e = uv \in E(G)$$

and

$$\sum_{v\in V(G)} \lambda_v + \sum_{S\in\mathcal{O}} \frac{|S|-1}{2}\lambda_S \leq \nu_c(G),$$

furthermore, these are integral solutions.

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- Satisfying the additional condition, we have

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(The last inequality holds due to the weak duality for maximization problems), thus guaranteeing that our possible dual solution is optimal.

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• We carry out a complete induction on $|V| + |E| + \sum_{e\in E(G)} c(e)$. Verification of the cases of small graphs (with small weights) is straightforward, left as an exercise for the interested reader.

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- The scheme of our proof will be as follows:

$$G, c \quad \xrightarrow[\text{step}]{\text{back-}} \quad G' = G \text{ (the graph remains the same)}$$

$$c'_e = \begin{cases} c_e - 1, & \text{if } v I e \\ c_e, & \text{otherwise.} \end{cases}$$

$\downarrow$ induction assumption

possible, integral dual λ'

$$\sum_{v \in V(G)} \lambda'_v + \sum_{S \in \mathcal{O}} \frac{|S|-1}{2} \lambda'_S \leq \nu_{c'}(G)$$

$$\longleftarrow$$

$$\lambda_u = \begin{cases} \lambda'_v + 1, & \text{if } u = v \\ \lambda'_u, & \text{otherwise,} \end{cases}$$

$$\lambda_S = \lambda'_S \text{ for all } S \in \mathcal{O}$$

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The λ defined in the above scheme satisfies the assertion. That is, they are possible integral dual solutions and fulfill the inequality proving the theorem.

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Proof of Case 1 (conclusion)

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• It is easy to see that by dropping the primes, both sides increase by 1, from which the claim follows.

Proof of Cunningham—Marsh Theorem: Case 2 and Scheme

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$$G, c \xrightarrow[\text{step}]{\text{back-}} G' = G \text{ (the graph remains the same)}, \quad c' = c - 1$$

$$\Big\downarrow \text{ induction step}$$

$$\text{possible, integral dual } \lambda': \quad \sum_{v \in V(G)} \lambda'_v + \sum_{S \in \mathcal{O}} \frac{|S|-1}{2} \lambda'_S \leq \nu_{c'}(G)$$

$$\xleftarrow[\text{CLAIM}]{\star \text{ CONDITION}}$$

$$\lambda_v = \lambda'_v, \qquad \lambda_S = \begin{cases} \lambda'_S + 1, & \text{if } S = V(G) \\ \lambda'_S, & \text{otherwise.} \end{cases}$$

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- When discussing Case 2, we assume

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The c'-optimal matching leaves only one vertex unmatched.

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Specifically, the cardinality of V is odd, thus $V \in \mathcal{O}$.

Proof of Case 2 with $\star$ CONDITION

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- Non-negativity and integrality are obvious.
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$$\sum_x \lambda'_x + \sum_S \frac{|S|-1}{2}\lambda'_S \leq \nu_{c'}(G).$$

- We need to show:

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Proof of Case 2 with ⋆ CONDITION (continued)

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• How do the two sides of the first inequality change when we drop the primes?

• The definition of c' guarantees that the right side increases by one, where M is a c'-optimal matching.

• The CONDITION ensures that the increase is $|M|$, where M is a c'-optimal matching.

• On the left side, only one term changes: the dual variable indexed by V. Its coefficient is $\frac{|V|-1}{2}$, and its value increases by 1. The claim is obvious.

Justification of $\star$ CONDITION: 1st Lemma

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• Let M be a c-optimal matching. Since we are in Case 2, we may assume that M is not perfect.

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After Case 1/In Case 2, the $\star$ CONDITION can be assumed.

- This follows from the following two lemmas. In the justification, we assume that the conditions of Case 2 are satisfied.

Lemma

A c'-optimal matching cannot be a perfect matching.

- Let M be a c-optimal matching. Since we are in Case 2, we may assume that M is not perfect.

- Let M' be a c'-optimal matching. Indirectly, assume that M' is perfect.

Validity of $\star$ CONDITION: 1st Lemma (continued)

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- Since M is c-optimal, we have $c(M') \leq c(M)$. Knowing that M is not perfect, we can say something about the weight under c' as well:

$$c'(M) = c(M) - |M| > c(M) - \frac{|V|}{2} \geq c(M') - \frac{|V|}{2}.$$

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- This contradicts the fact that M' is a c'-optimal matching.

Validity of $\star$ CONDITION: 2nd Lemma

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- Indirectly assume that M' is c'-optimal and $x, y \in V$ such that M' does not cover x and y. Let (M', x, y) be such that $d(x, y)$ is minimized.

Lemma

It cannot be the case that every c'-optimal matching leaves at least two vertices unmatched.

- Indirectly assume that M' is c'-optimal and $x, y \in V$ such that M' does not cover x and y. Let (M', x, y) be such that $d(x, y)$ is minimized.

- $d(x, y) > 1$, because the connectivity between x and y would ensure that $M' \cup \{xy \text{ edge}\}$ is also a matching, contradicting the c'-optimality ($c' > 0$). (Generally, an optimal matching cannot leave two connected vertices unmatched.)

Validity of $\star$ CONDITION: 2nd Lemma (continued)

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• Let x^+ be the first vertex following x on a shortest xy path (towards y). (Due to the above, $x^+ \neq y$.) Consider the following two matchings:

(1) M_{x^+}: a c-optimal matching that does not cover x^+ (such exists in Case 2).

(2) M'. The c'-optimality ensures that M' covers x^+ (x and x^+ are connected).

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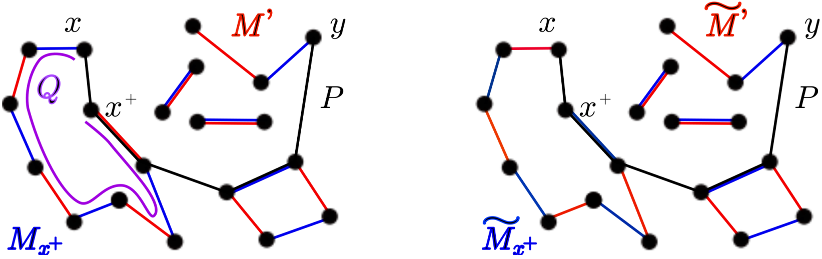
• Due to the properties of our matchings, x^+ is a degree 1 vertex in $\mathcal{M}$. Thus x^+ is an endpoint of a path Q in $\mathcal{M}$. Let

$$\widetilde{M}_{x^+} = M_x \Delta E(Q) \quad \text{and} \quad \widetilde{M}' = M' \Delta E(Q).$$

Validity of ⋆ CONDITION: 2nd Lemma: Figure



On the left, black edges denote the P path edges, red edges denote M_{x^+} edges, blue edges denote M' edges, purple indicates the Q path. On the right, the modified matchings ($\widetilde{M'}$ and $\widetilde{M_{x^+}}$): we exchange the red and blue edges along the Q/purple path. The total weight of red and blue edges remains the same on both sides.

Validity of $\star$ CONDITION: 2nd Lemma (continued)

- Then due to the c-optimality of M_{x^+}

$$c(\widetilde{M}_{x^+}) \leq c(M_{x^+}).$$

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- Since $d(x^+, x) = 1 < d(x, y)$ and $d(x^+, y) = d(x, y) - 1 < d(x, y)$ also hold, then (M', x, y) being the choice implies $\widetilde{M}'$ cannot be c'-optimal:

$$c'(\widetilde{M'}) < c'(M').$$

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- There is an edge of M' incident to x^+ (one of Q's endpoints) along the Q path. From this, it is obvious that Q contains at least as many edges of M' as of M_{x^+}.

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• There is an edge of M' incident to x^+ (one of Q's endpoints) along the Q path. From this, it is obvious that Q contains at least as many edges of M' as of M_{x^+}. Thus, the number of edges in M' cannot increase with the exchange: $|\widetilde{M'}| \leq |M'|$.

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- The proof is complete.

This is the End!

Thank you for your attention!