Lagrange dualization, examples

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$$\boxed{\begin{array}{lll} \text{Minimize} & c(x) & \\ \text{Subject to} & f_i(x) \le 0, & i = 1, \dots, k, \\ & g_j(x) = 0, & j = 1, \dots, \ell, \end{array}} \qquad \text{(P)}$$

where $x \in \mathbb{R}^n$, $c : \text{dom }(c)(\subset \mathbb{R}^n) \to \mathbb{R}$, $x = (x_1, \dots, x_n)^{\mathsf{T}}$, and f_i and g_j are real-valued functions of n variables.

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- Let's introduce a concise and simple notation. Let

$$f = \begin{pmatrix} f_1 \\ \vdots \\ f_k \end{pmatrix} : \cap_{i=1}^{k} \text{dom}\ f_i \subset \mathbb{R}^n \to \mathbb{R}^k, \text{ and } g = \begin{pmatrix} g_1 \\ \vdots \\ g_\ell \end{pmatrix} : \cap_{j=1}^{\ell} \text{dom}\ g_j \subset \mathbb{R}^n \to \mathbb{R}^\ell$$

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- It's always good to keep in mind what the concise notation represents. For example, in the above, the 0s represent zero vectors in $\mathbb{R}^k$ and $\mathbb{R}^\ell$.

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$$\lambda = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_k \end{pmatrix}, \quad \mu = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_\ell \end{pmatrix}.$$

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$$L(x;\lambda,\mu) = c(x)+\sum_{i=1}^{k}\lambda_i f_i(x)+\sum_{j=1}^{\ell}\mu_j g_j(x) = c(x)+\lambda^{\mathsf{T}} f(x)+\mu^{\mathsf{T}} g(x).$$

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- The domain of the Lagrange function coincides with the domain of the original optimization problem denoted by (P), which we labeled as $\mathcal{D}$.

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- Thus, for every non-negative coordinate λ and any μ, we obtain a lower bound for $c(x)$ by evaluating L.

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It immediately follows from the previous remark that for $x \in \mathcal{L}$ and $\lambda \succeq 0$,

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This is because $c(x) \geq L(\lambda, \mu, x) \geq \widetilde{c}(\lambda, \mu)$.

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- We denote the dual problem as (D), and its optimal value as d^*. (The original problem (P) is the primal problem; its optimal value is p^*).

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Theorem

$\widetilde{c}(\lambda, \mu)$ is concave, thus $-\widetilde{c}(\lambda, \mu)$ is convex.

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- However, this is not necessary.
- When $p^* - d^* > 0$, we say there is a (positive) duality gap.

Break Time

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$$= (c - \lambda + A^{\mathsf{T}}\mu)^{\mathsf{T}}x - b^{\mathsf{T}}\mu.$$

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- To determine the value of the dual objective function at (λ,μ), we need to find the global minimum of a linear function.

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Remark

$$\inf_{x\in\mathbb{R}^n} a^{\mathsf{T}}x + \alpha = \begin{cases} \alpha, & a = 0 \\ -\infty, & a \neq 0 \end{cases}$$

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$$\widetilde{c}(\lambda,\mu) = \inf_{x\in\mathbb{R}^n} (c-\lambda+A^\mathsf{T}\mu)^\mathsf{T} x - b^\mathsf{T}\mu = \begin{cases} -b^\mathsf{T}\mu, & \text{if } c-\lambda+A^\mathsf{T}\mu = 0, \\ -\infty, & \text{otherwise.} \end{cases}$$

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- Thus,

$$\widetilde{c}(\lambda) = \begin{cases} -b^{\mathsf{T}} \lambda, & \text{if } c + A^T \lambda = 0, \\ -\infty, & \text{otherwise.} \end{cases}.$$

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Strong Duality Theorem for LP

Consider any LP problem. Exactly one of the following two possibilities holds:

(1)
$$p^* = \infty > -\infty = d^*,$$

(2)
$$p^* = d^*.$$

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Example

$\mathcal{H} = (\overrightarrow{G}, s, t, c)$ is a network, where $\overrightarrow{G}$ is a directed graph, s and t are two distinguished vertices (source and sink), and c is the capacity function. $c : E(\overrightarrow{G}) \to \mathbb{R} \equiv c \in \mathbb{R}^{E(\overrightarrow{G})}$.

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- The flow function $f : E(\overrightarrow{G}) \to \mathbb{R}$ can be described as $f \in \mathbb{R}^{E(\overrightarrow{G})}$, i.e., $x = (f(e_1), \ldots, f(e_m))^{\mathsf{T}} \in \mathbb{R}^{E}$ as a vector.

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- Capacities can also be handled as vectors.

Dualization Example III: Flow Problem

Example

$\mathcal{H} = (\overrightarrow{G}, s, t, c)$ is a network, where $\overrightarrow{G}$ is a directed graph, s and t are two distinguished vertices (source and sink), and c is the capacity function. $c : E(\overrightarrow{G}) \to \mathbb{R} \equiv c \in \mathbb{R}^{E(\overrightarrow{G})}$.

Flow assigns a quantity to each edge such that it lies between 0 and the capacity of the corresponding edge (capacity constraints). Furthermore, it ensures the conservation of flow at every vertex except the source and sink.

We seek the flow f with maximum value.

- The flow function $f : E(\overrightarrow{G}) \to \mathbb{R}$ can be described as $f \in \mathbb{R}^{E(\overrightarrow{G})}$, i.e., $x = (f(e_1), \ldots, f(e_m))^{\mathsf{T}} \in \mathbb{R}^E$ as a vector.
- Capacities can also be handled as vectors. Algebraically, the capacity constraints are: $0 \preceq x \preceq c$.

Dualization Example III: Flow Problem (continued)

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- The conservation law can also be written in algebraic form:

$$\sum_{e:vKe} x_e - \sum_{e:vBe} x_e = 0 \quad \text{for all } v \in V\setminus\{s,t\}.$$

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- The objective function/the value of the flow $(x = x(\text{flow}))$

$$c(x) = \text{val}(f) = \sum_{e:sKe} x_e - \sum_{e:sBe} x_e.$$

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- Before us is the LP form of the flow problem. We introduce a little *twist* into the obvious formalization.

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- Consider the following modification of the network: We add an edge of infinite capacity (an edge without capacity constraint), leading from t to s in $\overrightarrow{G}$.

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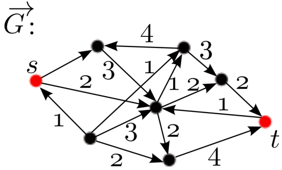

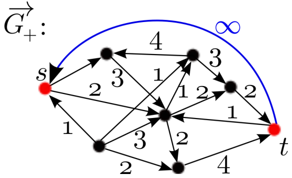

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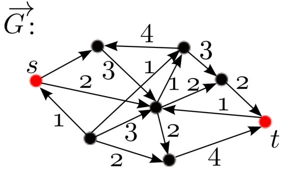

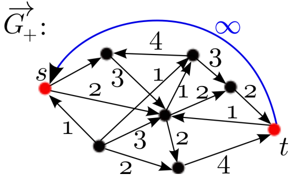

- In this $\overrightarrow{G}_+$ graph, a flow should remain on the old edges, and the e_+ edge should have the value of the flow. Thus, conservation is satisfied at every vertex (our network becomes a so-called circulation). Let $x_+ = \binom{x}{v}$ be the extended variable vector with the variable corresponding to the new edge, i.e., the new coordinate $v = \mathrm{val}(f)$, the value of the flow.

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subject to	$0 \preceq x \preceq c$,
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Minimize	$-v$-t
subject to	$-x \preceq 0$,
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Dualization Example III: Flow Problem (continued)

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- The Lagrange function is given by:

$$L(x_+;\lambda_1,\lambda_2,\mu) = -v + \lambda_1^{\mathsf{T}}(-x) + \lambda_2^{\mathsf{T}}(x-c) + \underbrace{\underbrace{\mu^{\mathsf{T}} A_+ x_+}_{(A_+^{\mathsf{T}}\mu)^{\mathsf{T}} x_+}}_{(A^{\mathsf{T}}\mu)^{\mathsf{T}} x + (\mu_s - \mu_t)v} =$$

$$=(-1+\mu_s-\mu_t)v + (\lambda_2 - \lambda_1 + A^{\mathsf{T}}\mu)^{\mathsf{T}} x - \lambda_2^{\mathsf{T}} c$$

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- From here, the dual objective function is:

$$\widetilde{c} = \begin{cases} -\lambda_2^{\mathsf{T}} c, & \text{if } (-1+\mu_s-\mu_t) = 0 \text{ and } \lambda_2 - \lambda_1 + A^{\mathsf{T}}\mu = 0 \\ -\infty, & \text{otherwise} \end{cases}$$

Dualization Example III: Flow Problem (continued)

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- The (D) dual problem is:

Maximize	$-\lambda_2^{\mathsf{T}} c$-t
subject to	$\mu_s - \mu_t = 1$
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- Below, we reconsider the dual problem using elementary methods.

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1st Observation

The goal is to minimize the components of λ_2, i.e., to make the coordinates of the (non-negative) variable vector as close to zero as possible. To achieve this, it suffices to choose μ and λ_1 components wisely.

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There exists an integral optimal solution.

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2nd Observation

There exists an integral optimal solution.

- This is not trivial. We will prove it later in the semester.

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In the constraints of the dual problem, the μ vector appears only as differences of two μ coordinates: $e = \overrightarrow{uv}$ edge has $(A^{\mathsf{T}}\mu)_e = \mu_v - \mu_u$. Moreover, it is advantageous if this difference — when negative — is as close to 0 as possible.

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- If $\mu \in \mathbb{R}^V$ is a feasible solution, then for any constant c,

$\mu + \begin{pmatrix} c \\ c \\ \vdots \\ c \end{pmatrix} = \mu + c \cdot 1^{\mathsf{T}}$ is also a feasible solution and equivalent to the original μ.

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$\mu + \begin{pmatrix} c \\ c \\ \vdots \\ c \end{pmatrix} = \mu + c \cdot 1^{\mathsf{T}}$ is also a feasible solution and equivalent to the original μ.
- Due to such shifts, we can assume that the μ vector satisfies $\mu_s = 1$ and $\mu_t = 0$ (normalization).

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subject to	$\mathcal{V}$ is an s-t cut.

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Break

Dualization Example IV: Least Squares Problem

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Example

Minimize	$x^{\mathsf{T}}x$-t
subject to	$Ax = b$,

where $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{\ell\times n}$, $b \in \mathbb{R}^{\ell}$.

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- Then
$$L(x;\mu) = x^{\mathsf{T}}x + \mu^{\mathsf{T}}(Ax - b).$$
- Expressing $\widetilde{c}$:
$$\widetilde{c}(\mu) = \inf_{x\in\mathbb{R}^n} L(\mu, x) = \inf_{x\in\mathbb{R}^n}(x^{\mathsf{T}}x + \underbrace{\mu^{\mathsf{T}}(Ax)}_{(A^{\mathsf{T}}\mu)^{\mathsf{T}}x}) - \underbrace{\mu^{\mathsf{T}}b}_{\text{independent of } x}$$

Dualization Example IV: Least Squares Problem (continued)

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- The infimum over x is taken for a function depending on x as

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- We know that at an extremum, the gradient is zero. $\nabla_x L = 0$ if and only if $x = -\frac{1}{2}A^{\mathsf{T}}\mu$.

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- The zero gradient does not necessarily mean a minimum, but in our case, we have a convex function, so there will definitely be a minimum here. Therefore, substituting x with $-\frac{1}{2}A^{\mathsf{T}}\mu$, we get that

$$\widetilde{c}(\mu) = \left(-\frac{1}{2}A^{\mathsf{T}}\mu\right)^{\mathsf{T}} \cdot \left(-\frac{1}{2}A^{\mathsf{T}}\mu\right) + (A^{\mathsf{T}}\mu)^{\mathsf{T}} \cdot \left(-\frac{1}{2}A^{\mathsf{T}}\mu\right) - b^{\mathsf{T}}\mu =$$
$$= -\frac{1}{4}\mu^{\mathsf{T}}AA^{\mathsf{T}}\mu - b^{\mathsf{T}}\mu.$$

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• So the dual (D) problem is an unconstrained optimization question.

Dualization Example V: Maximum Cut

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Consider a simple graph G. The task is to find such a cut $\mathcal{V}$ (a partition of the vertices into two classes) where the $|E(\mathcal{V})|$ is maximized (maximize the number of edges crossing).

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- We can describe a cut by encoding with an additional plus or minus 1 component for each vertex to indicate which side of the cut it falls on:

$$\mathcal{V} \equiv x \in \{-1, 1\}^V \subset \mathbb{R}^V.$$

Dualization Example V: Maximum Cut (continued)

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- This formalization of the problem is $\mathcal{NP}$-hard.

Dualization Example V: Maximum Cut (continued)

- Let $A = A_G$ be the adjacency matrix of G.
- For the quadratic form $x^{\mathsf{T}} Ax$, each edge $e = uv$ contributes $2x_u x_v$. The value of $x_u x_v$ is $+1$ if edge e belongs to one side of the cut, and -1 if edge e belongs to the cut set (crosses).
- It's easy to calculate that

$$x^{\mathsf{T}} Ax = 2\,|E(G)| - 4\,|E(\mathcal{V})|\,.$$

- So the original problem's formalization is

Minimize	$x^{\mathsf{T}} Ax,$	$x \in \mathbb{R}^V$-t
subject to	$x_v^2 = 1,$	for all $v \in V$.

- This formalization of the problem is $\mathcal{NP}$-hard. Of course, we can determine its dual.

Dualization Example V: Maximum Cut (continued)

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- The Lagrange function of the dual problem is

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L(\mu, x) &= x^T A x + \sum_{v \in V} \mu_v \left(x_v^2 - 1\right) \\
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where for a vector $a \in \mathbb{R}^n$ diag $(a) = \begin{pmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & a_n \end{pmatrix}$ is an $n \times n$ diagonal matrix.

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- From this, the dual objective function is

$$\widetilde{c}(\mu) = -\underline{1}^T \mu + \inf_{x \in \mathbb{R}^n} x^T \left(A + \text{diag } \mu\right) x.$$

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- After the digression, we can now determine the dual objective function:

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Dualization Example V: Maximum Cut (continued)

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Consequence

Let G be any arbitrary simple graph, and λ_{min} be the smallest eigenvalue of graph G (the adjacency matrix A). Then

$$\frac{1}{2}|E(G)| \overset{(1)}{\leq} \max_{\mathcal{V}\text{ cut}} |E(\mathcal{V})| \overset{(2)}{\leq} \frac{1}{2}|E(G)| - \frac{\lambda_{min}}{4}|V(G)|.$$

Dualization Example V: Maximum Cut (continued)

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- The inequality (1) follows from

$$\max_{\mathcal{V}\text{ cut}} |E(\mathcal{V})| \geq \mathbb{E}(|E(\underline{\mathcal{V}})|),$$

where $\underline{\mathcal{V}}$ is a random cut in graph G (vector chosen uniformly at random from $\{-1, 1\}^n$).

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$$\xi_e = \begin{cases} 1, & \text{if } e \in E(\underline{\mathcal{V}}), \\ 0, & \text{if } e \notin E(\underline{\mathcal{V}}), \end{cases}$$

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$$\mathbb{E}(E(\underline{\mathcal{V}})) = \mathbb{E}\left(\sum_{e \in E(G)} \xi_e\right) = \sum_{e \in E(G)} \mathbb{E}(\xi_e) = \sum_{e \in E(G)} \frac{1}{2} = \frac{1}{2}|E(G)|.$$

Dualization Example V: Maximum Cut (continued)

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$$\mu = -\lambda_{min}1 = \begin{pmatrix} -\lambda_{min} \\ -\lambda_{min} \\ \vdots \\ -\lambda_{min} \end{pmatrix}.$$

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Rearranging this gives us equality (2).

Break Time

Dualization Example VI: Minimizing Norm under Linear Constraints

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Example

Minimize	$\|x\|$,	$x \in \mathbb{R}^n$,-t
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where $\|.\| : \mathbb{R}^n \to \mathbb{R}$ is an arbitrary norm.

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- Now we seek an infimum of this, and here too we need a little digression, as in the previous example.

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- Scaling leads to the following Lemma.

Digression (continued)

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Lemma

$$\left| v^T x \right| \leq \|v\|_* \, \|x\| \, .$$

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- Using the dual norm, the sought infimum can be expressed as follows:

$$\inf_{x \in \mathbb{R}^n} \|x\| + v^T x = \begin{cases} 0, & \text{if } \|v\|_* \leq 1, \\ -\infty, & \text{if } \|v\|_* > 1. \end{cases}$$

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- In the first case, it is equivalent to $\|v\|_* \leq 1$ implying $\|x\| + v^T x \geq 0$ for all x vectors.

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$$\left|v^T x\right| \leq \|v\|_* \|x\| .$$

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$$\inf_{x\in\mathbb{R}^n} \|x\| + v^T x = \begin{cases} 0, & \text{if } \|v\|_* \leq 1, \\ -\infty, & \text{if } \|v\|_* > 1. \end{cases}$$

- In the first case, it is equivalent to $\|v\|_* \leq 1$ implying $\|x\| + v^T x \geq 0$ for all x vectors. This can be easily deduced from the above statement.

Digression (continued)

Lemma

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• In the first case, it is equivalent to $\|v\|_* \leq 1$ implying $\|x\| + v^T x \geq 0$ for all x vectors. This can be easily deduced from the above statement.

• In the second case, if we reach a negative value, then through scaling we can achieve arbitrarily large negative values, leading to the $-\infty$ infimum.

Dualization Example VI: Minimizing Norm under Linear Constraints (continued)

Dualization Example VI: Minimizing Norm under Linear Constraints (continued)

The dual objective function is as follows:

$$\widetilde{c}(\mu) = \begin{cases} -b^T\mu, & \text{if } \|A^T\mu\|_* \leq 1, \\ -\infty, & \text{if } \|A^T\mu\|_* > 1. \end{cases}$$

Dualization Example VI: Minimizing Norm under Linear Constraints (continued)

The dual objective function is as follows:

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Thus, the dual problem is as follows:

Maximize	$-b^T\mu$-t
subject to	$\left\|A^T\mu\right\|_* \leq 1.$

Dualization Example VII: Optimization under Linear Constraints

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Example

Minimize	$c(x)$-t
subject to	$Ax \preceq b,$
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Dualization Example VII: Optimization under Linear Constraints

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Minimize	$c(x)$-t
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- The problem is much more general than an LP problem, where we also work with linear constraints. Here, the objective function can be any function.
- The Lagrange function of the problem is

$$\begin{aligned} L(\lambda, \mu, x) &= c(x) + \lambda^T(Ax - b) + \mu^T(Cx - d) \\ &= c(x) + \left(A^T\lambda + C^{\mathsf{T}}\mu\right)^T x - (b^T\lambda + d^T\mu). \end{aligned}$$

Digression

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- Now we are seeking the infimum of an expression in the form $f(x) + v^T x$ over the domain dom (f).

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Definition

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$$f^*(u) = \sup_{x \in \text{dom }(f)} u^T x - f(x).$$

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Definition

The convex conjugate, or alternatively the Fenchel conjugate, of the function f is defined as

$$f^*(u) = \sup_{x \in \text{dom }(f)} u^T x - f(x).$$

- In the expression to be minimized, the role of u in the objective function will be played by $-v$:

$$\inf_x f(x) + v^\mathsf{T} x = -\sup_x -f(x) - v^\mathsf{T} x = -\sup_x -v^\mathsf{T} x - f(x) = -f^*(-v).$$

Dualization Example VII: Optimization under Linear Constraints (continued)

Dualization Example VII: Optimization under Linear Constraints (continued)

- Following this, the dual problem becomes

Maximize	$-c^*(-A^\mathsf{T}\lambda - C^\mathsf{T}\mu) - (b^\mathsf{T}\lambda + d^\mathsf{T}\mu)$-t
subject to	$\lambda \succeq 0.$

Dualization Example VIII: Maximizing Entropy

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Example

Minimize	$\sum_{i=1}^{n} x_i \log x_i$-t
subject to	$Ax \preceq b,$
	$1^{\mathsf{T}} x = 1.$

Dualization Example VIII: Maximizing Entropy

Example

Minimize	$\sum_{i=1}^{n} x_i \log x_i$-t
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- The objective function is negative entropy, resolving the apparent contradiction between the maximization sign and the minimization optimization task.

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• Due to the appearance of the logarithm function, the components of the feasible solutions are positive. $1^{\mathsf{T}} x = 1$ indicates that x encodes a probability distribution. The linear inequalities $Ax \preceq b$ can be statistical observations about the distribution. For example, its expected value, variance, moments, estimation of the tail of the distribution, etc.

Dualization Example VIII: Maximizing Entropy (continued)

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$$(x \log x)^* = e^{y-1}.$$

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$$c^*(x) = \sum_{i=1}^{n} e^{y_i - 1}.$$

- Now we write down $\widetilde{c}\,(\lambda, \mu)$ function:

$$\widetilde{c}\,(\lambda, \mu) = -b^T\lambda - \mu - \sum_{i=1}^{n} e^{-a_i^T\lambda - \mu - 1} = -b^T\lambda - \mu - e^{-\mu-1}\sum_{i=1}^{n} e^{-a_i^T\lambda},$$

where a_i^{T} is the i-th row of the A^{T} matrix, i.e., the i-th column of A.

Dualization Example VIII: Maximizing Entropy (continued)

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Maximize	$\widetilde{c}\,(\lambda, \mu)$-t
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Maximize	$\widetilde{c}(\lambda, \mu)$-t
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- This can be easily simplified: If λ is fixed, then $\widetilde{c}$ is a single-variable real function, and thus a good μ value can be determined for each λ:

$$\mu = \log \sum_{i=1}^{n} e^{-a_i^\mathsf{T} \lambda} - 1.$$

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$$\mu = \log \sum_{i=1}^{n} e^{-a_i^{\mathsf{T}}\lambda} - 1.$$

- Substituting, the dual becomes equivalent to the following:

Maximize	$-b^{\mathsf{T}}\lambda - \log\left(\sum_{i=1}^{n} e^{-a_i^{\mathsf{T}}\lambda}\right)$-t
subject to	$\lambda \succcurlyeq 0.$

Break

Dualization Example IX

Dualization Example IX

Example

Minimize	$x_1 x_2$-t
subject to	$x_1 \geq 0$
	$x_2 \geq 0$
	$x_1^2 + x_2^2 \leq 1$

Dualization Example IX

Example

Minimize	$x_1 x_2$-t
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- The optimization problem is trivial:

Dualization Example IX

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- The optimization problem is trivial: The product of non-negative numbers is non-negative,

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- Thus, $p^* = 0$.

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- The optimization problem is trivial: The product of non-negative numbers is non-negative, in our case $(0, 0)$ is a feasible solution.
- Thus, $p^* = 0$.
- Nevertheless, let's practice the learned dualization formalism.

Dualization Example IX (continued)

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- The dual variables are λ_1, λ_2, and λ_3.

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$$x_1x_2 - \lambda_1 x_1 - \lambda_2 x_2 + \lambda_3(x_1^2 + x_2^2 - 1).$$

- The dual objective function is:

$$\widetilde{c}(\lambda) = \inf_{x\in\mathbb{R}^2} \left(x_1x_2 - \lambda_1 x_1 - \lambda_2 x_2 + \lambda_3(x_1^2 + x_2^2 - 1)\right) =$$

$$= \inf_{x\in\mathbb{R}^2} \left((x_1, x_2) \begin{pmatrix} \lambda_3 & 1/2 \\ 1/2 & \lambda_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - (\lambda_1, \lambda_2) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \lambda_3 \right).$$

Dualization Example IX (continued)

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- The matrix of the quadratic part is positive definite if $\lambda_3 > 1/2$, positive semidefinite if $\lambda_3 = 1/2$, and indefinite if $\lambda_3 < 1/2$.
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- It can be easily seen that in the positive semidefinite case ($\lambda_3 = 1/2$), if $\lambda_1 - \lambda_2 \neq 0$, the $\widetilde{c}(\lambda_1, \lambda_2, \lambda_3)$ objective function can be arbitrarily small.

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- For positive semidefinite matrices and $\lambda_1 = \lambda_2$, the value of $\widetilde{c}(\lambda_1, \lambda_2, \lambda_3)$ is $-\lambda_3$.
- In the case of positive definite matrix, it can be easily seen that a finite minimum exists. With our analytical knowledge, the minimum value can be easily determined as:

$$-\frac{1}{4}(\lambda_1, \lambda_2)\begin{pmatrix}\lambda_3 & 1/2\\ 1/2 & \lambda_3\end{pmatrix}^{-1}\begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix} - \lambda_3.$$

Dualization Example IX (continued)

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- The dual problem is:

Maximize	$\widetilde{c}(\lambda_1, \lambda_2, \lambda_3)$-t
subject to	$\lambda_1 \geq 0$
	$\lambda_2 \geq 0$
	$\lambda_3 \geq \frac{1}{2}$

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- Elementary consideration suggests that the optimal points are: $(\lambda, \lambda, 1/2)$ and $d^* = -1/2$.

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- Weak duality inequality is of course satisfied, but as a strict inequality.

Dualization Example $\widetilde{\text{IX}}$

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Example

Minimize	$x_1 x_2$-t
subject to	$x_1 \geq 0$
	$x_2 \geq 0$
	$x_1^2 + x_2^2 \leq 1$
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Dualization Example $\widetilde{IX}$

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- We repeated the previous example

Dualization Example $\widetilde{IX}$

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- Of course, p^* remains 0.

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- The dual objective function is:

$$\widetilde{c}(\lambda) = \inf_{x\in\mathbb{R}^2} \left(x_1x_2 - \lambda_1 x_1 - \lambda_2 x_2 + \lambda_3(x_1^2 + x_2^2 - 1) - \lambda_4 x_1 x_2\right) =$$

$$= \inf_{x\in\mathbb{R}^2} \left((x_1, x_2) \begin{pmatrix} \lambda_3 & \frac{1-\lambda_4}{2} \\ \frac{1-\lambda_4}{2} & \lambda_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - (\lambda_1, \lambda_2) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \lambda_3 \right).$$

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- The optimal point and the dual optimal value were controlled by the fourth (corresponding to the redundant) dual variable.

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• The calculation results in: the optimal point is $(0, 0, 0, 1)$ and $d^* = 0$, strong duality holds.

• The optimal point and the dual optimal value were controlled by the fourth (corresponding to the redundant) dual variable. Adding the seemingly redundant constraint can be justified afterwards.

Dualization Example X

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Example

Minimize	$c(u)$-t
subject to	$u \leq 0$

where $c(u) = -\left(\frac{u+1}{2}\right)^2, \ dom\,(c) = [-1, 1]$, meaning the graph of c is a parabolic arc.

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- The domain of the objective function is unnatural. With this strange, unnatural domain, we built constraints into the objective function. This is not fair. This is cheating.

- Our goal is not to present an application, but to demonstrate the geometric view of duality gap.

Dualization Example X (continued)

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- First, let's dualize the original problem. We use the notation $v = c(u)$.

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$$L(u, \lambda) = -\left(\frac{u+1}{2}\right)^2 + \lambda u = \lambda u + v.$$

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- The dual objective function is:

$$\widetilde{c}(\lambda) = \inf_{u \in [-1,1], v = \frac{1}{4}(u+1)^2} \lambda u + v.$$

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- The dual is:

Maximize	$\widetilde{c}(\lambda) = \inf_{u \in [-1,1], v = \frac{1}{4}(u+1)^2} \lambda u + v$-t
subject to	$\lambda \geq 0$

Dualization Example X: The Figure

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- With the following figure, we can visualize the solutions to the primal and dual problems:

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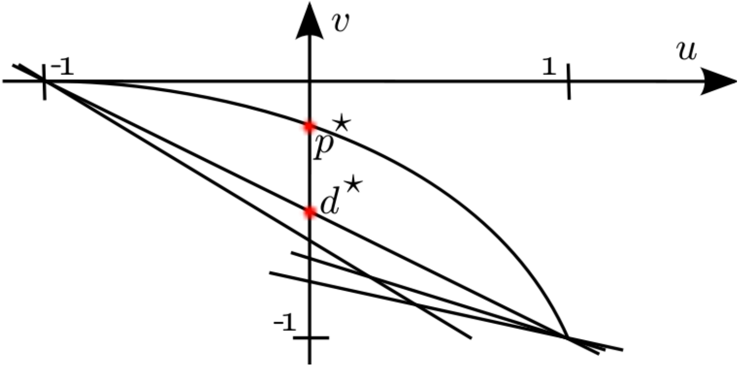

The function $v = -\frac{1}{4}(u+1)^2$ is plotted in the u-v coordinate plane. Additionally, there are $v + \lambda u = \alpha$-type functions visible, which bound the objective function from below.

Dualization Example X: The Figure (continued)

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- The solution to the primal is evident: On the feasible values ($[-1, 0]$), the function is monotonically decreasing, thus attaining a minimum at 0.

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- The Lagrange function is of the form $v + \lambda u$.

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- The solution to the primal is evident: On the feasible values ($[-1,0]$), the function is monotonically decreasing, thus attaining a minimum at 0. So, $x^* = 0$ and $p^* = -1/4$.

- The Lagrange function is of the form $v + \lambda u$. If it takes the value α_0 for some λ_0, it bounds the objective function from below for $u \in [-1,1]$.

Dualization Example X: The Figure (continued)

- The solution to the primal is evident: On the feasible values ($[-1,0]$), the function is monotonically decreasing, thus attaining a minimum at 0. So, $x^* = 0$ and $p^* = -1/4$.

- The Lagrange function is of the form $v + \lambda u$. If it takes the value α_0 for some λ_0, it bounds the objective function from below for $u \in [-1,1]$. $v + \lambda_0 u \geq \alpha$ intersects the parabolic arc in the half-space.

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- In the figure, the optimal value d^* is clearly visible, as well as the strict inequality $d^* < p^*$. There is no strong duality.

Dualization Example XI

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Example

Let $n = 2$, $c(x, y) : \mathbb{R} \times \mathbb{R}_{\geq 0} \to \mathbb{R}$, $(x, y) \mapsto e^{-x}$. Our optimization problem is the following:

Minimize	$c(x, y)$-t
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- To satisfy the constraint, due to the non-negativity of x^2 and $y > 0$, x must be 0.
- The feasible solution set for the problem is

$$\{(x, y) : x = 0, y > 0\}.$$

Dualization Example XI (continued)

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$$L(x, y; \lambda) = c(x, y) + \lambda \frac{x^2}{y} = e^{-x} + \lambda \frac{x^2}{y}.$$

- Then, the dual objective function is as follows:

$$\widetilde{c}(\lambda) = \inf_{(x,y)\in\mathcal{D}} L(x, y; \lambda) = \inf_{(x,y)\in\mathcal{D}} \left(e^{-x} + \lambda \frac{x^2}{y} \right) = \begin{cases} 0 & \text{if } \lambda \geq 0 \\ -\infty & \text{otherwise} \end{cases}$$

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- So we can write down the dual optimization problem:

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- Note that the weak duality theorem holds in this case, since $p^* \geq d^*$.
- In our case, the inequality is strict, creating a so-called *duality gap*, because $p^* - d^* = 1 > 0$.

This is the End!

Thank you for your attention!