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Finite-Dimensional Cones[1]

1 Basic Definitions.

Definition 1. *A set $A \subseteq \mathbb{R}^N$ is a* cone *iff it is not empty and for any $a \in A$ and any $\gamma \geq 0$, $\gamma a \in A$.*

Definition 2. *A cone is* degenerate *iff it equals the origin.*

Definition 3. *A cone $A \subseteq \mathbb{R}^N$ is* pointed *iff it is non-degenerate and $a \in A$, $a \neq 0$, implies $-a \notin A$.*

Example 1. The set $\mathbb{R}^2_+$ is a closed, convex, pointed cone. The half plane $\{x \in \mathbb{R}^2 : x_2 \geq 0\}$ is a closed, convex cone that is not pointed. The union of the open half plane $\{x \in \mathbb{R}^2 : x_2 > 0\}$ and 0 is a somewhat pathological example of a convex cone that is pointed but not closed. □

Remark 1. There are several different definitions of "cone" in the mathematics. Some, for example, require the cone to be convex but allow the cone to omit the origin. The definition used here is sometimes referred to as the linear algebra definition. □

2 Finitely Generated Cones.

A cone $A \subseteq \mathbb{R}^N$ is *finitely generated* iff there is a finite set of vectors $Z = \{z^1, \ldots, z^K\}$ such that $a \in A$ iff there are numbers $\lambda_k \geq 0$, such that

$$a = \sum_{k=1}^{K} \lambda_k z^k.$$

I will also say that Z *positively spans* A. Obviously, all of the z^k are elements of A.

Theorem 1. *If A is finitely generated then it is convex and closed.*

Proof. Convex is trivial. As for closed, the claim holds vacuously if A is degenerate. Therefore, assume A is non-degenerate.

I first claim that if $a \in A$, $a \neq 0$, then it is possible to generate a from a linearly independent subset of Z, the finite subset of $\mathbb{R}^N$ that generates A.[2] To see this, suppose $\lambda_k > 0$ for all k but Z is not linearly independent. If Z is not linearly independent, there are γ_k, not all zero, such that

$$0 = \sum_{k=1}^{K} \gamma_k z^k.$$

Without loss of generality, one can assume that at least one $\gamma_k > 0$ (if all are non-positive, just multiply them all by -1). For ease of notation, relabel indices so that $\gamma_k > 0$ iff $k \geq J$, where J is some index ($J = 1$ is possible), and

$$\frac{\lambda_J}{\gamma_J} \geq \cdots \geq \frac{\lambda_K}{\gamma_K}.$$

In particular, $\gamma_K > 0$. Then

$$z^K = \sum_{k=1}^{K-1} -\frac{\gamma_k}{\gamma_K} z^k$$

and

$$\begin{aligned} a &= \lambda_K z^K + \sum_{k=1}^{K-1} \lambda_k b^k \\ &= \sum_{k=1}^{K-1} \left(\lambda_k - \frac{\gamma_k}{\gamma_K} \lambda_K \right) z^k. \end{aligned}$$

Because of the labeling,

$$\lambda_k - \frac{\gamma_k}{\gamma_K} \lambda_K \geq 0$$

for every $k \in \{1, \ldots, K-1\}$. Hence a can be generated by just $K-1$ of the vectors in Z. The same argument applies to any subset of Z that is not linearly independent: if $a \neq 0$ is generated by a subset of Z that is not linearly independent, then a is also generated by a strictly smaller subset of Z. This establishes the claim.

To complete the proof that A is closed, consider the cone generated by any linearly independent subset of Z. Possibly relabeling indices for ease of notation, if the vectors in the subset are $\{z^1, \ldots, z^L\}$, $L \leq N$, and if $V \subseteq \mathbb{R}^N$ is the L-dimensional vector space spanned by these vectors, then the linear function f :

[2] Note the order of quantifiers. I allow the linearly independent subset of Z to depend on a. I am not claiming that all of A is generated by a single linearly independent set of vectors. In general that won't be true. For example, consider a cone in $\mathbb{R}^3$ that is pyramid-shaped with a square cross section. Four vectors are needed to generate such a cone, but any independent set contains at most three vectors.

$V \to \mathbb{R}^L$ given by $f(x) = (\lambda_1, \ldots, \lambda_L)$ iff

$$x = \sum_{k=1}^{L} \lambda_k z^k,$$

is well defined since $\{z^1, \ldots, z^L\}$ is independent.

Note that the image under f of the cone in $\mathbb{R}^N$ generated by $\{z^1, \ldots, z^L\}$ is the weakly positive orthant in $\mathbb{R}^L$. Since the weakly positive orthant in $\mathbb{R}^L$ is closed and since f is continuous (it is a finite-dimensional linear function), the preimage under f of the weakly positive orthant in $\mathbb{R}^L$ is closed. But this preimage is the cone in $\mathbb{R}^N$ generated by $\{z^1, \ldots, z^L\}$. Since A is the union of such cones, for all possible combinations of linearly independent vectors in Z, and since there are only finitely many such combinations, A is closed. ■

Example 2. A closed convex cone that is *not* finitely generated is the cone generated by the origin and the unit disk: explicitly, take the cone to be the set of all a in $\mathbb{R}^3$ such that $a = \gamma x$ for $\gamma \geq 0$ and $x = (x_1, x_2, 1)$ with $x_1^2 + x_2^2 \leq 1$. Note that if I instead require that $x_1^2 + x_2^2 < 1$ then this cone, while still convex, is not closed: cones that are not finitely generated need not be closed. □

3 A Separating Hyperplane Theorem for Cones.

Theorem 2 (A Separating Hyperplane Theorem For Cones)**.** *Let A be a closed, convex cone in $\mathbb{R}^N$ and let B be a compact, convex subset of $\mathbb{R}^N$. If $A \cap B = \emptyset$ then there is a vector $v \in \mathbb{R}^N$ such that for all $a \in A$, $b \in B$,*

$$v \cdot a \leq 0 < v \cdot b.$$

Proof. By the Separating Hyperplane Theorem in the notes on Convex Sets, there is a vector $v \in \mathbb{R}^N$ and a number $r \in \mathbb{R}$ such that for all $a \in A$, $b \in B$,

$$v \cdot a < r < v \cdot b.$$

Since $0 \in A$, this implies $r > 0$, hence $v \cdot b > 0$. Moreover, for any $a \in A$ and any $\gamma > 0$, since A is a cone, $v \cdot (\gamma a) < r$, hence $v \cdot a < r/\gamma$, which implies $v \cdot a \leq 0$. ■

In separate notes, titled A Basic Separation Theorem for Cones, I prove a version of Theorem 2 that takes B to be a singleton, $B = \{b\}$. This weaker version is sufficient for many applications (including both the proof of the Karush-Kuhn-Tucker Theorem and Theorem 3 below) and admits a somewhat more elementary, self-contained proof.

4 Farkas's Lemma.

For the case in which B is a singleton, $B = \{b\}$, Theorem 2 implies the following result, called Farkas's Lemma, which reinterprets Theorem 2 as a statement about matrices.

Theorem 3 (Farkas's Lemma)**.** *Let M be an $N \times K$ matrix. Then for any $b \in \mathbb{R}^N$, either*

1. *$Mx = b$ has a solution $x \in \mathbb{R}^K$ or*
2. *there is a $v \in \mathbb{R}^K$ such that $v'M \leq 0$ and $v \cdot b > 0$.*

Proof. Let A be the cone positively spanned by the columns of M. By Theorem 1, A is closed. Farkas's Lemma then follows by Theorem 2. In particular, $Mx = b$ has a solution iff $b \in A$. $v'M \leq 0$ iff $v \cdot a \leq 0$ for every a that is a column of M. ■

5 Support for Pointed Cones.

Theorem 4 (A Supporting Hyperplane Theorem for Pointed Cones)**.** *Let $A \subseteq \mathbb{R}^N$ be a non-degenerate closed, convex, pointed cone. Then it is strictly supported at the origin: there is a $v \in \mathbb{R}^N$ such that if $a \in A$ and $a \neq 0$ then $v \cdot a > 0$.*

Proof. Let A be a subset of a vector space, call it V. V is a vector subspace of some $\mathbb{R}^N$. The proof is by induction on the dimension of V. Since A is non-degenerate, this dimension is at least 1.

If the dimension of V is 1, then A is a half line containing the origin. Take v to be any non-zero point in A and the result follows. Suppose, then, that the theorem holds for any closed, convex, pointed cone contained in any vector space of dimension K or less.

Suppose that V has dimension $K+1$. Because A is pointed, the origin is not interior to A. Therefore, by the standard Supporting Hyperplane Theorem restricted to V, there is a $v^* \in V$, $v^* \neq 0$, such that $v^* \cdot a \geq v^* \cdot 0 = 0$ for all $a \in A$. If $v^* \cdot a > 0$ for all $a \in A$, $a \neq 0$, then I am done.

Otherwise, the set $A_1 = \{a \in A : v^* \cdot a = 0\}$ contains some $a \neq 0$. A_1 is a closed, convex, pointed cone since it is the intersection of A and the set $V_1 = \{x \in V : v^* \cdot x = 0\}$, which is a vector space and hence is a closed, convex cone. Since V has dimension $K+1$, V_1 has dimension K. By the induction hypothesis, since $A_1 \subseteq V_1$, there is a $v_1 \in V_1$ such that if $a \in A_1$ and $a \neq 0$ then $v_1 \cdot a > 0$.

For $t \in \{1, 2, \dots\}$, let

$$v_t = (1/t)v_1 + (1 - 1/t)v^*.$$

I claim that there is a t such that, if $a \in A$ and $a \neq 0$ then $v_t \cdot a > 0$, which proves the result. Equivalently, I claim that if there is a sequence (a_t) in A, $a_t \neq 0$, such that $v_t \cdot a_t \leq 0$ for all t, then A is not closed, and the proof then follows by contraposition.

For any $a_t \in A$ such that $v_t \cdot a_t \leq 0$, and for any $\gamma > 0$, $\gamma a_t \in A$ (since A is a cone) and $v_t \cdot (\gamma a_t) \leq 0$. Therefore, since $a_t \neq 0$ for all t, it is without loss of generality to take $a_t \in A \cap S$ where $S = \{x \in \mathbb{R}^N : \|x\| = 1\}$ is the unit sphere.

It remains to show that if there is a sequence (a_t) in $A \cap S$ such that $v_t \cdot a_t \leq 0$ for all t then A is not closed. Since S is compact, (a_t) has a convergent subsequence (a_{t_k}) converging to, say, $x^* \in S$. The proof follows if $x^* \notin A$.

I claim that $v^* \cdot x^* = 0$. To see this, note the following.

- Since $v_t \cdot a_t \leq 0$ for all t, and $v_t \to v^*$, continuity of inner product implies that $v^* \cdot x^* \leq 0$.
- Since S is compact and inner product is continuous, $v_1 \cdot x$ attains a minimum value on S; call this minimum value M. Then, since $v^* \cdot a_t \geq 0$,

$$v_t \cdot a_t \geq (1/t)M,$$

  which implies, by continuity, that $v^* \cdot x^* \geq 0$.

Combining the above inequalities, $v^* \cdot x^* = 0$.

Therefore, if $x^* \in A$ then $x^* \in A_1$. It remains, therefore, to show that $x^* \notin A_1$. For any $a \in A_1$, $a \neq 0$, and for any t, $v_t \cdot a > 0$ (since, for $a \in A_1$, $a \neq 0$, $v_1 \cdot a > 0$ and $v^* \cdot a = 0$). Therefore, for all t, since $v_t \cdot a_t \leq 0$ and $a_t \neq 0$, $a_t \notin A_1$. Since $a_t \neq 0$, this implies $v^* \cdot a_t > 0$. Since $v_t \cdot a_t \leq 0$, it follows that $v_1 \cdot a_t < 0$. But then, by continuity, $v_1 \cdot x^* \leq 0$, which implies, since $x^* \neq 0$, that $x^* \notin A_1$, and I am done. ■

Remark 2. As in the proof of Theorem 4, let V be a vector space containing A. By the Supporting Hyperplane Theorem, there is a $v \in V$ such that $v \cdot a \geq 0$ for all $a \in A$. Let A^* be the set of all such v. A^* is called the *dual cone* of A. It is easy to verify that the interior of A^* (relative to V) is exactly the set of $v \in A^*$ for which $v \cdot a > 0$ for all $a \in A$, $a \neq 0$. Theorem 4 is thus equivalent to showing that if A is a closed, convex, pointed cone then A^* has a non-empty (relative) interior. □

Remark 3. A partial converse to Theorem 4 is that if A is a non-degenerate cone that is strictly supported at 0 then A is pointed. The argument is by contraposition. Suppose A is not pointed. Then there is an $a \in A$, $a \neq 0$, such that $-a \in A$. Let v be any vector that supports A at the origin. Then $v \cdot a \geq 0$ and $v \cdot (-a) \geq 0$, implying $v \cdot a = 0$: v does not *strictly* support A at the origin. □

Remark 4. If A is not closed then things become complicated. I give two examples.

Let $A \subseteq \mathbb{R}^2$ be the union of the open upper half plane $\{x \in \mathbb{R}^2 : x_2 > 0\}$ and the closed half line $\{x \in \mathbb{R}^2 : x_1 \geq 0 \text{ and } x_2 = 0\}$ (i.e., the non-negative x_1 axis). This is a convex, pointed cone that is not closed. The only vectors that support A

at 0 are $v = (0, 1)$ or any vector collinear with v. But v does not strictly support A. For example, $(1, 0) \in A$ but $(0, 1) \cdot (1, 0) = 0$.

On the other hand, let $\hat{A} \subseteq \mathbb{R}^2$ be the union of the open half plane $\{x \in \mathbb{R}^2 : x_2 > 0\}$ and the point $\{0\}$. Then this convex, pointed cone is strictly supported at the origin, even though it is not closed. □

6 Separation for Pointed Cones.

Theorem 4 implies the following useful separation result.

Theorem 5 (A Separating Hyperplane Theorem for Pointed Cones)**.** *Let A be a closed, convex, pointed cone and let B be a non-empty convex, compact set that is disjoint from A. Then there is a $v \in \mathbb{R}^N$ such that $v \cdot a < 0$ for all $a \in A$, $a \neq 0$, and $v \cdot b > 0$ for all $b \in B$.*

Proof. By the Separating Hyperplane Theorem, there is a $v^* \in \mathbb{R}^N$ and an $r \in \mathbb{R}$ such that for all $a \in A$, $b \in B$,

$$v^* \cdot a < r < v^* \cdot b.$$

As in the proof of Theorem 2, this implies $r > 0$ and

$$v^* \cdot a \leq 0 < v^* \cdot b.$$

This is close to, but somewhat weaker than, what I need.

By Theorem 4, A is strictly supported at the origin. Therefore, by taking the negative of the supporting vector, there is a $v_A \in \mathbb{R}^N$ such that $v_A \cdot a < 0$ for any $a \in A$, $a \neq 0$.

If $v_A \cdot b > 0$ for every $b \in B$ then set $v = v_A$ and I am done. It is possible, however, that $v_A \cdot b \leq 0$ for some $b \in B$. Since B is compact and inner product is continuous, there is a $b^* \in B$ that minimizes $v_A \cdot b$. Assume, therefore, that $v_A \cdot b^* \leq 0$.

For $\theta \in (0, 1)$, let

$$v_\theta = \theta v_A + (1 - \theta) v^*.$$

For any $\theta > 0$, and any $a \in A$, $a \neq 0$, $v_\theta \cdot a < 0$ (since $v^* \cdot a \leq 0$ and $v_A \cdot a < 0$). If $v_A \cdot b^* = 0$ then, since $v^* \cdot b > 0$, $v = v_\theta$ will satisfy the theorem for any $\theta \in (0, 1)$. Otherwise, suppose $v_A \cdot b^* < 0$. Then $v = v_\theta$ will satisfy the theorem for any $\theta \in (0, r/(r - v_A \cdot b^*))$, since then

$$v_\theta \cdot b = \theta(v_A \cdot b) + (1 - \theta)(v^* \cdot b) > \theta(v_A \cdot b^*) + (1 - \theta) r > 0.$$

■