LECTURE 3: GEOMETRY OF LP

1. Terminologies
2. Background knowledge
3. Graphic method
4. Fundamental theorem of LP

Terminologies

- Baseline model:

$$(\text{LP}) \qquad \begin{aligned} &\text{Min } \mathbf{c}^T\mathbf{x} \\ &\text{s. t. } \mathbf{A}\mathbf{x} = \mathbf{b} \\ &\mathbf{x} \geq 0 \end{aligned}$$

- Feasible domain

$$P = \{\mathbf{x} \in \mathbf{R}^n | \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$$

- Feasible solution

$\mathbf{x}$ is a <u>*feasible solution*</u> if $\mathbf{x} \in P$.

- Consistency

When $P \neq \phi$, LP is <u>*consistent.*</u>

Terminologies

- Bounded feasible domain:

  P is $\underline{\textit{bounded}}$ if

  $$\exists\ M > 0 \text{ such that } ||\mathbf{x}|| \leq M,\ \forall\ \mathbf{x} \in P.$$

  In this case, we say “LP has bounded feasible domain.”

- Bounded LP:

  LP is $\underline{\textit{bounded}}$ if

  $$\exists\ M \in R \text{ such that } \mathbf{c}^T\mathbf{x} \geq M\ \forall\ \mathbf{x} \in P.$$

- Question: LP has a bounded feasible domain.

  $\Downarrow\quad\Uparrow?$

  LP is bounded.

Terminologies

- Optimal solution:

  x* is an optimal solution if

$$\mathbf{x}^* \in P \text{ and } \mathbf{c}^T\mathbf{x}^* = \underset{x \in P}{\text{Min}}\, \mathbf{c}^T\mathbf{x}$$

- Optimal solution set

$$P^* = \{\mathbf{x}^* \mid \mathbf{x}^* \text{ is optimal}\}$$

- We say

$$\mathbf{x}^* \quad \underline{\textit{solves}} \text{ LP, if } \mathbf{x}^* \in P^*.$$

Background knowledge

- Observation 1: each equality constraint in the standard form LP is a "hyperplane" in the solution space.

- What does the equation $x_1 - 2x_2 = 30$ represent in the 2-d Euclidean space?

Definition:

For a vector $\mathbf{a} \in \mathbf{R}^n, \mathbf{a} \neq 0$, and a scalar $\beta \in \mathbf{R}$, define

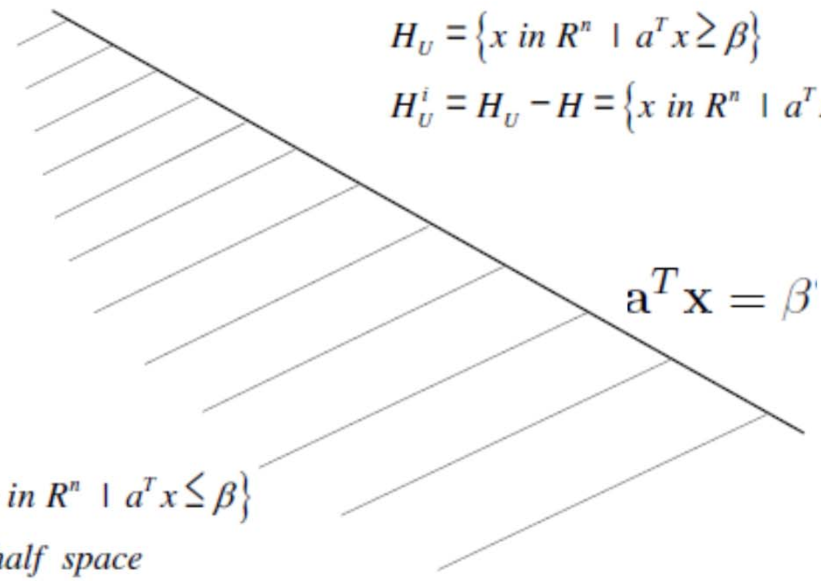
$$H = \{\mathbf{x} \in \mathbf{R}^n | \mathbf{a}^T\mathbf{x} = \beta\} \ \underline{\textit{hyperplane}}$$

Hyperplane

- Geometric representation

$$H_U = \{x \text{ in } R^n \mid a^T x \geq \beta\}$$

$$H_U^i = H_U - H = \{x \text{ in } R^n \mid a^T x > \beta\}$$



$$\mathbf{a}^T\mathbf{x} = \beta$$

$$H_L = \{x \text{ in } R^n \mid a^T x \leq \beta\}$$

closed half space

$$H_L^i = H_L - H = \{x \text{ in } R^n \mid a^T x < \beta\}$$

open half space

H is the <u>*bounding hyperplane*</u> of H_L and H_L^i.

$\mathbf{a}$ is the <u>*normal*</u> of H.

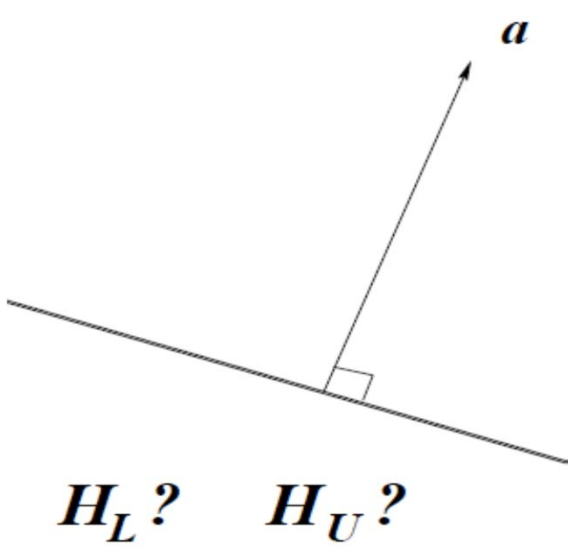
Properties of hyperplanes

- Property 1: The normal vector **a** is orthogonal to all vectors in the hyperplane **H**.

- Proof:

$$\forall\ \mathbf{y}, \mathbf{z} \in H,$$

$$\begin{aligned} \mathbf{a}^T(\mathbf{y} - \mathbf{z}) &= \mathbf{a}^T\mathbf{y} - \mathbf{a}^T\mathbf{z} \\ &= \beta - \beta = 0. \end{aligned}$$



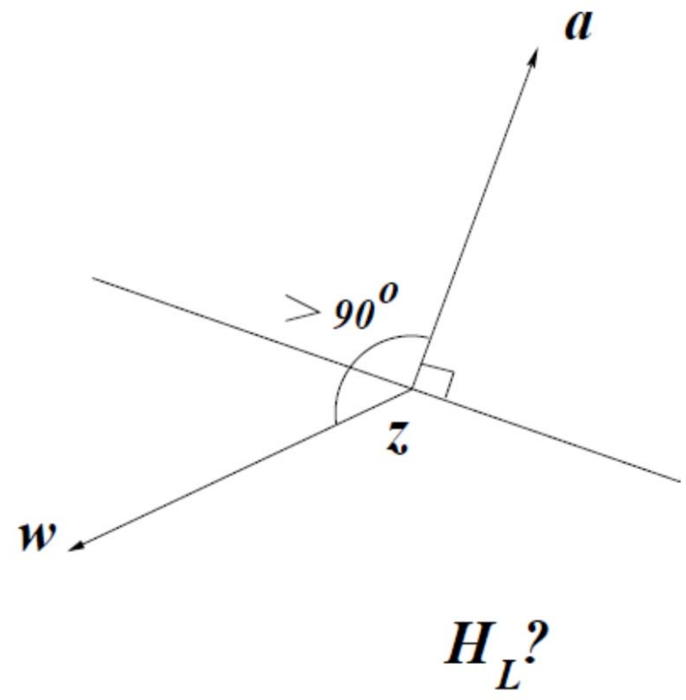
Properties of hyperplane

- Property 2: The normal vector is directed toward the upper half space.

- Proof:

For any $\mathbf{z} \in H, \mathbf{w} \in H_L^i$,

$$\mathbf{a}^T(\mathbf{w}-\mathbf{z}) = \mathbf{a}^T\mathbf{w}-\mathbf{a}^T\mathbf{z}$$
$$< \beta - \beta = 0.$$



Properties of feasible solution set

- Definition:

  A polyhedral set or polyhedron is a set formed by the intersection of a finite number of a closed half spaces. If it is nonempty and bounded, it is a polytope.

- Property 3:

  The feasible domain of a standard form LP

$$P = \{x \in R^n | Ax = b, x \geq 0\}$$

is a polyhedral set.

Properties of optimal solutions

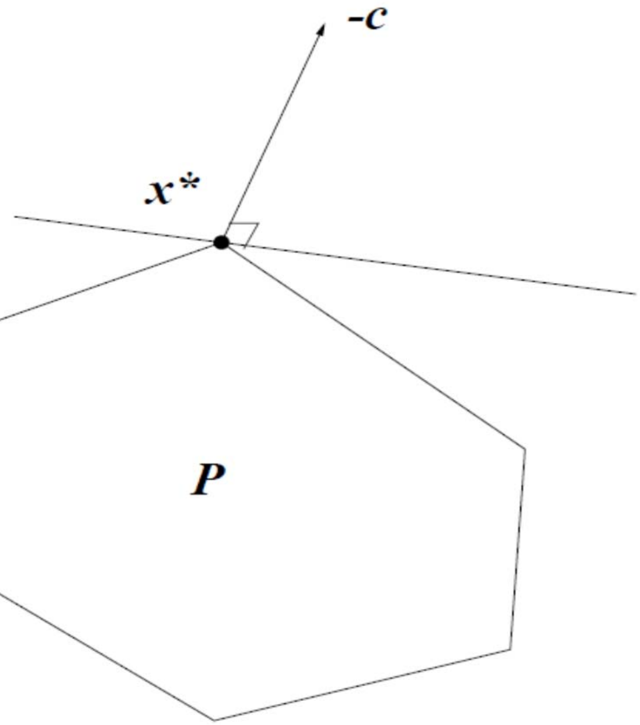
- Property 4:

If $P \neq \emptyset$ and $\exists \beta \in \mathrm{R}$ such that

$$P \subset H_L := \{\mathrm{x} \in \mathrm{R}^n \mid -\mathrm{c}^T\mathrm{x} \leq \beta\},$$

then $\underset{x \in P}{\mathrm{Min}}\, \mathrm{c}^T\mathrm{x} \geq -\beta$

Moreover, if $\mathrm{x}^* \in P \bigcap H$ then $\mathrm{x}^* \in P^*$.



Example

- Give the following LP

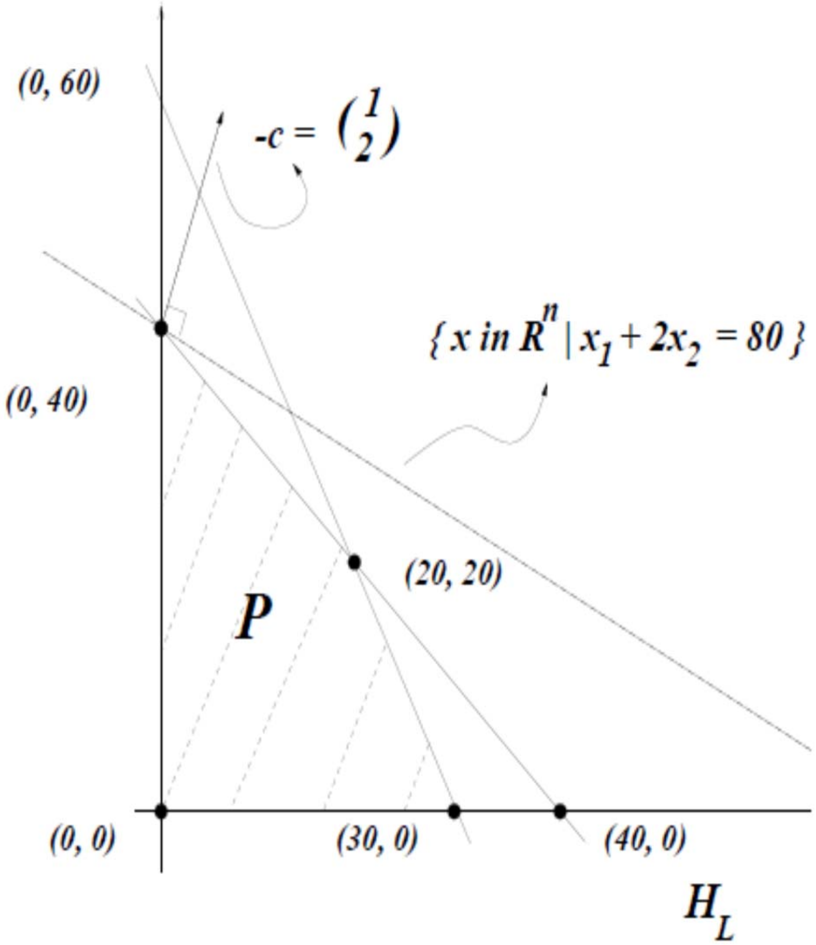
$$
\begin{array}{llllll}
\text{Minimize} & -x_1 & - & 2x_2 & & \\
\text{s. t.} & x_1 & + & x_2 & \leq 40 \\
 & 2x_1 & + & x_2 & \leq 60 \\
 & x_1, & & x_2, & \geq 0
\end{array}
$$

- Covert to standard form

$$
\mathbf{c} = \begin{pmatrix} -1 \\ -2 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{pmatrix}, \quad \mathrm{b} = \begin{pmatrix} 40 \\ 60 \end{pmatrix}
$$

$$
\begin{array}{llllllllll}
\text{Minimize} & -x_1 & - & 2x_2 & & & & & \\
\text{s. t.} & x_1 & + & x_2 & + & x_3 & & & = 40 \\
 & 2x_1 & + & x_2 & & & + & x_4 & = 60 \\
 & x_1, & & x_2, & & x_3, & & x_4 & \geq 0
\end{array}
$$

Graphic Solution



Since $\underset{x \in P}{\text{Min}}\, \mathbf{c}^T \mathbf{x} \geq -80$

also $-x_1 - 2x_2 = -80$ at $\binom{0}{40}$

Hence $\begin{cases} x_1 = 0 \\ x_2 = 40 \end{cases}$ is an optimal solution.

Graphic Method

Step 1: Draw the feasible domain P.
(If $P = \emptyset$, STOP! No solution.)

Step 2: Use $-\mathbf{c}$ as normal vector at each vertex to see if $P \in H_L := \{\mathbf{x} \in \mathbf{R}^n | -\mathbf{c}^T\mathbf{x} \leq \beta\}$ for some $\beta \in \mathbf{R}$.

1. If the answer is "YES", we find an optimal solution.
2. If all answers are "NO", the problem is unbounded below.

Pros and Cons

- Advantages:

  - Geometrically simple.

- Disadvantages

  - Algebraically difficult

    How many vertices are there?

    How to identify each vertex?

Any better way?

- Simplex method

A way to generate and manage the vertices of the feasible solution set, which is a polyhedral set.

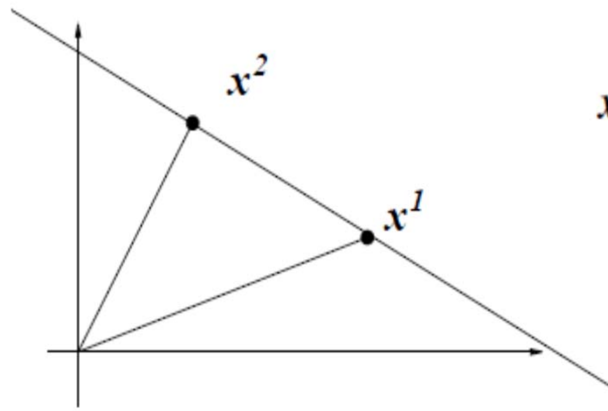
Background knowledge

- Definition: Let $\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^p \in \mathbf{R}^n$, $\lambda_1, \lambda_2, \ldots, \lambda_p \in \mathbf{R}$, and

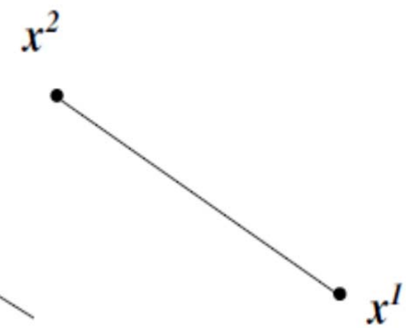
$$\mathbf{x} = \sum_{i=1}^{p} \lambda_i \mathbf{x}^i = \lambda_1 \mathbf{x}^1 + \lambda_2 \mathbf{x}^2 + \cdots + \lambda_p \mathbf{x}^p$$

  we say **x** is a linear combination of $\{\mathbf{x}^1, \ldots, \mathbf{x}^p\}$.
- If $\sum_{i=1}^{p} \lambda_i = 1$, we say x is an affine combination of $\{\mathbf{x}^1, \ldots, \mathbf{x}^p\}$.
- If $\lambda_i \geq 0$, we say x is a conic combination of $\{\mathbf{x}^1, \ldots, \mathbf{x}^p\}$.
- If $\sum_{i=1}^{p} \lambda_i = 1, \lambda_i \geq 0$, we say x is a convex combination of $\{\mathbf{x}^1, \ldots, \mathbf{x}^p\}$.

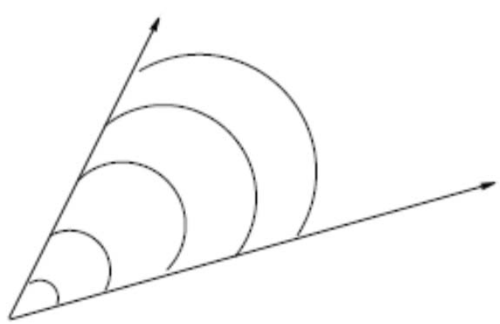
Sets generated by different combinations of two points



Affine combination



Convex combination



Conical combination

Affine set, convex set, and cone

- Definition: Let S be a subset of $\mathbf{R}^n$.

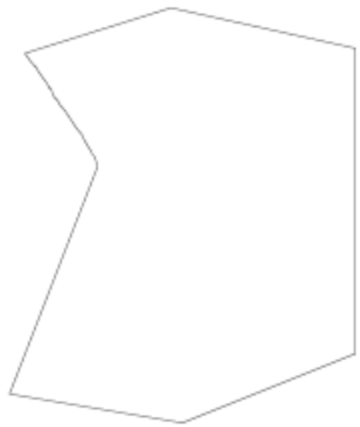
If the affine combination of any two points of S falls in S, then S is an affine set.

If the convex combination of any two points of S falls in S, then S is a convex set.

If $\lambda \mathbf{x} \in S$ for all $\mathbf{x} \in S$ and $\lambda \geq 0$, then S is a cone.

Example

- Which one is convex? Which one is affine?



$$H = \{\mathbf{x} \in \mathbf{R}^n | \mathbf{a}^T\mathbf{x} = \beta\}$$

$$H_L = \{\mathbf{x} \in \mathbf{R}^n | \mathbf{a}^T\mathbf{x} \leq \beta\}$$

$$\{\mathbf{x} \in \mathbf{R}^n | \mathbf{A}\mathbf{x} = \mathbf{b}\}$$

$$P = \{\mathbf{x} \in \mathbf{R}^n | \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq 0\}$$

C ~~A~~   ~~C~~ ~~A~~

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Example

- What's the geometric meaning of the feasible domain ?

$$P = \{\mathbf{x} \in \mathbf{R}^n | \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$$

1. P is a polyhedral set.
2. P is a convex set.
3. P is the intersection of *m* hyperplanes and the cone of the first orthant.
4. "Ax = b and x ≥ 0" means that the rhs vector b falls in the cone generated by the columns of constraint matrix A.

$$\mathbf{A} = (\mathbf{A}_1 | \mathbf{A}_2 | \cdots | \mathbf{A}_n)$$

$$A_j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix}$$

$$\mathbf{A}\mathbf{x} = (\mathbf{A}_1 | \mathbf{A}_2 | \cdots | \mathbf{A}_n) \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ w_n \end{pmatrix} = \sum_{j=1}^{n} x_j \mathbf{A}_j \in \mathbf{R}^{m}$$

Example - continue

5. Actually, the set $A_c = \{y \in R^m | y = Ax, x \in R^n, x \geq 0\}$

is a convex cone generated by the columns of matrix A.

Interior and boundary points

- Given a set, what's the difference betwee

  an interior point and a boundary point?

- Definition: Given a set $S \subset \mathbf{R}^n$, a point $\mathbf{x} \in S$ is an interior point of S, if ,

  $\exists\, \epsilon > 0$ such that the ball $B = \{\mathbf{y} \in \mathbf{R}^n |\ \|\mathbf{x} - \mathbf{y}\| \leq \epsilon\} \subset S.$

  Otherwise, **x** is a boundary point of S.
- We denote that

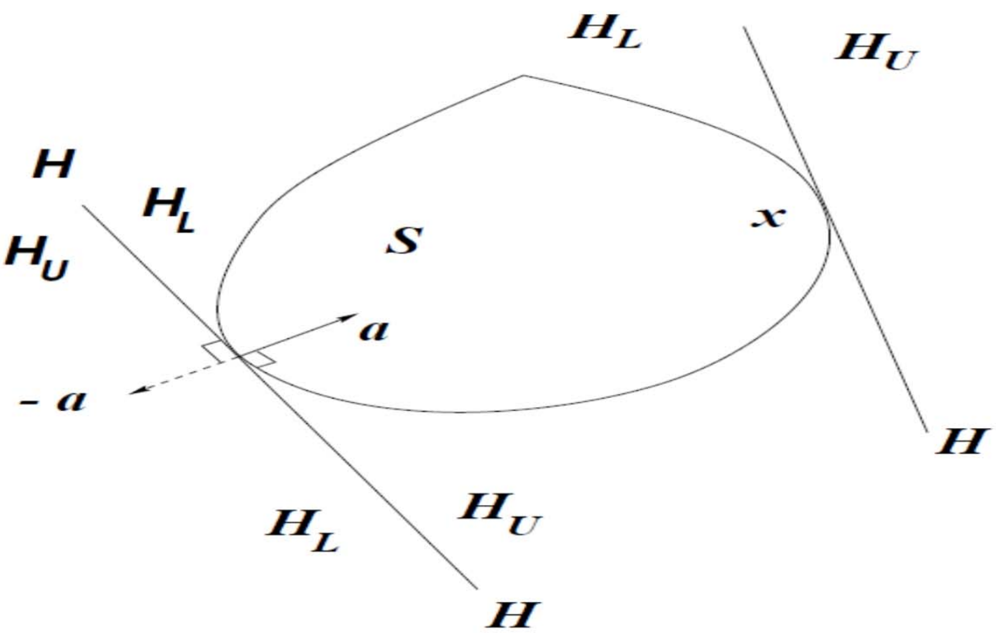
$$int(S) = \{\ \mathbf{x} \text{ is an interior point of } S\ \}$$

$$bdry(S) = \{\ \mathbf{x} \text{ is an boundary point of } S\ \}$$

Boundary points of convex sets

- What's special about boundary points of a convex set?
- Separation Theorem:

$S \subset \mathbf{R}^n$ is convex, then $\forall \mathbf{x} \in bdry(S), \exists$ a hyperplane H such that $\mathbf{x} \in H$ and either $S \subseteq H_L$ or $S \subseteq H_U$.



Question

- Can you now see that if an LP (in two or three dimensions) has a finite optimal solution, then one vertex of P is optimal ?

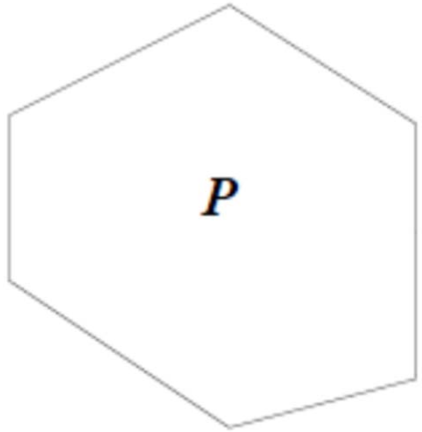
- Hint: Consider the supporting hyperplane

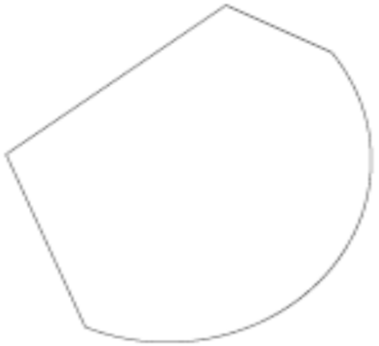
$$H = \{\mathbf{x} \in \mathbf{R}^n | -\mathbf{c}^T\mathbf{x} = \beta\}$$

- How about higher dimensional case?

  - This leads to the Fundamental Theorem of LP.

Are all boundary points the same?





- Some sits on the shoulders of others, and some don't.
- Definition: **x** is an extreme point of a convex set S if **x** cannot be expressed as a convex combination of other points in S.

Geometrical meaning of extreme points

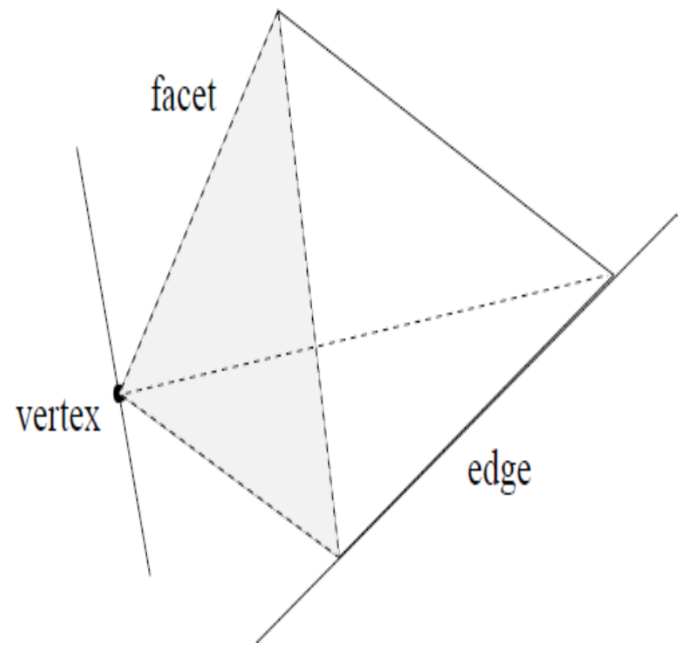
- Definition:

Let P be a convex polyhedron and H be a supporting hyperplane of P, then $F = P \bigcap H$ defines a *face* of P.

When $\dim(F) = 0$, it is a *vertex*

$\dim(F) = 1$, it is an *edge*

$\dim(F) = \dim(P) - 1$, a *facet*



- Theorem:

Let P be a convex polyhedron, $\mathbf{x} \in P$ is a vertex if and only if $\mathbf{x}$ is an extreme point of P.

Representation of extreme points

- For the feasible domain P of an LP, its vertices are the extreme points. How can we take this advantage to generate and manage all vertices?

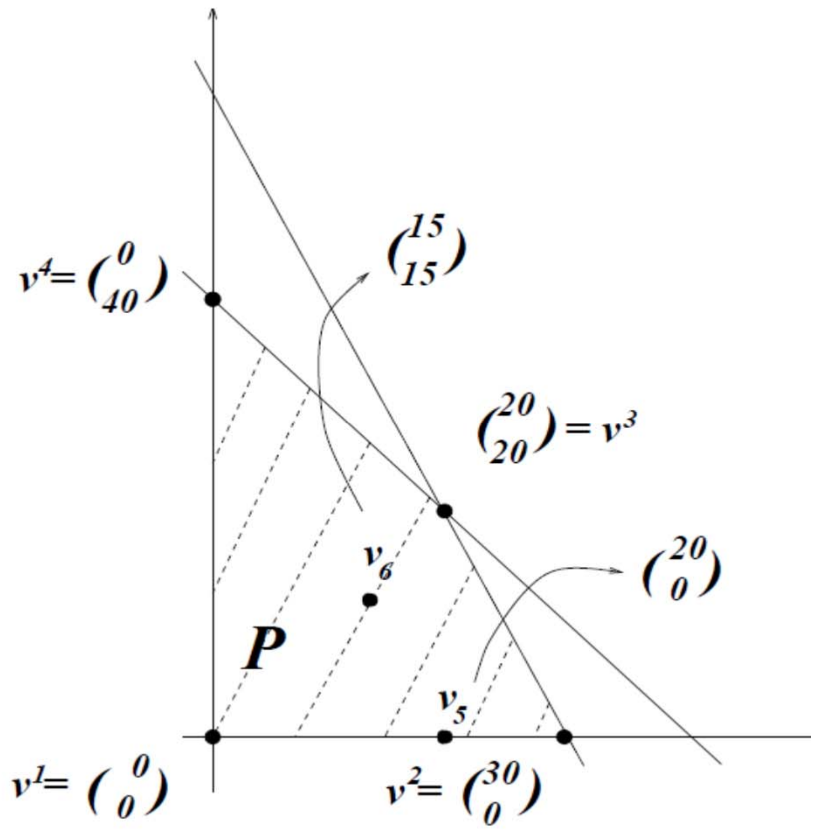
$\mathbf{x}$ is an extreme point of P, then $\mathbf{x}$ is of course a feasible solution of

$$\begin{cases} \mathbf{A}\mathbf{x} = \mathbf{b} \\ \mathbf{x} \geq 0 \end{cases}$$

But what's special of being an extreme point? (in terms of feasible solution).

Learning from example

$$\begin{aligned} \text{Minimize} \quad & x_1 - 2x_2 \\ \text{subject to} \quad & x_1 + x_2 + x_3 = 40 \\ & 2x_1 + x_2 + x_4 = 60 \\ & x_1, x_2, x_3, x_4 \geq 0. \end{aligned}$$



What's special?

- Vertices

$$v^1 = \begin{pmatrix} 0 \\ 0 \\ 40 \\ 60 \end{pmatrix}, v^2 = \begin{pmatrix} 30 \\ 0 \\ 10 \\ 0 \end{pmatrix}, v^3 = \begin{pmatrix} 20 \\ 20 \\ 0 \\ 0 \end{pmatrix}, v^4 = \begin{pmatrix} 0 \\ 40 \\ 0 \\ 20 \end{pmatrix}.$$

- Edge

$$v^5 = \begin{pmatrix} 20 \\ 0 \\ 20 \\ 20 \end{pmatrix} \leftarrow \text{one zero } x_i$$

Interior

$$v^6 = \begin{pmatrix} 15 \\ 15 \\ 10 \\ 15 \end{pmatrix} \leftarrow \text{no zero } x_i$$

$$n = 4,\ m = 2,\ n - m = 2$$

Observations

- Ax = b has n variables in m linear equations.
- When $n > m$, we only need to consider m variables in m equations for solving a system of linear equations.
- An extreme point of P is obtained by setting $n - m$ variables to be zero and solving the remaining m variables in m equations.
- the columns of A corresponding to the non-zero (positive) variables better be linear independent!

Example

- System of equations

$$\begin{cases} x_1 + x_2 + x_3 & & = 40 \\ 2x_1 + x_2 & + x_4 & = 60 \\ x_1, x_2, x_3, x_4 \geq 0. \end{cases}$$

- Linear independence of the columns

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} x_1 + \begin{pmatrix} 1 \\ 1 \end{pmatrix} x_2 + \begin{pmatrix} 1 \\ 0 \end{pmatrix} x_3 + \begin{pmatrix} 0 \\ 1 \end{pmatrix} x_4 = \begin{pmatrix} 40 \\ 60 \end{pmatrix}$$

Finding extreme points

- Theorem:

  A point **x** $\in P = \{\mathbf{x} \in \mathbf{R}^n | \mathbf{A}\mathbf{x} = b, \mathbf{x} \geq 0\}$ is an extreme point of P if and only if the columns of A corresponding to the positive components of x are linearly independent.
- Proof:

  Without loss of generality, we may assume that the first *p* components of x are positive and rest are zero, i.e.,

$$\mathbf{x} = \begin{pmatrix} \bar{\mathbf{x}} \\ \mathbf{0} \end{pmatrix} \text{ where } \bar{\mathbf{x}} = \begin{pmatrix} x_1 \\ \vdots \\ x_p \end{pmatrix} > \mathbf{0}$$

also denote the first p columns of $\mathbf{A}$ by $\bar{\mathbf{A}}$, then
$\mathbf{A}\mathbf{x} = \bar{\mathbf{A}}\bar{\mathbf{x}} = \mathrm{b}$.

Proof - continue

Suppose that the columns of $\bar{\mathbf{A}}$ are not linearly independent, then $\exists\ \bar{\mathrm{w}} \neq \mathbf{0}$ such htat $\bar{\mathbf{A}}\bar{\mathrm{w}} = 0$.
Notice that for ϵ is small enough
$\bar{\mathrm{x}} \pm \epsilon\bar{\mathrm{w}} \geq \mathbf{0}$ and $\bar{\mathbf{A}}(\bar{\mathrm{x}} \pm \epsilon\bar{\mathrm{w}}) = \bar{\mathbf{A}}\bar{\mathrm{x}} = \mathrm{b}$

Hence

$$\mathbf{y}^1 = \begin{pmatrix} \bar{\mathrm{x}} + \epsilon\bar{\mathrm{w}} \\ \mathbf{0} \end{pmatrix} \in P$$

$$\mathbf{y}^2 = \begin{pmatrix} \bar{\mathrm{x}} - \epsilon\bar{\mathrm{w}} \\ \mathbf{0} \end{pmatrix} \in P$$

and $\mathbf{x} = \frac{1}{2}\mathbf{y}^1 + \frac{1}{2}\mathbf{y}^2$, *i.e.* $\mathbf{x}$ can not be a vertex (extreme point) of P.
Thus, $\mathbf{x}$ is an exterme point $\Rightarrow$ columns of $\bar{\mathbf{A}}$ are linearly independent.

Proof - continue

Suppose that x is not an extreme point, then

$x = \lambda y^1 + (1-\lambda)y^2$ for some

$y^1, y^2 \in P,\ y^1 \neq y^2$ and $0 < \lambda < 1,$

Since $y^1 \geq 0, y^2 \geq 0$ and $0 < \lambda < 1.$

the last $n-p$ components of y^1 must be zero, *i.e.*

$$y^1 = \begin{pmatrix} \bar{y}^1 \\ 0 \end{pmatrix}$$

Now

$$x - y^1 = \begin{pmatrix} \bar{x} - \bar{y}^1 \\ 0 \end{pmatrix} \neq 0$$

and $A(x - y^1) = Ax - Ay^1 = b - b = 0$

$\Rightarrow$ columns of A are linearly dependent.

Thus , columns of $\bar{A}$ are linearly independent

$\Rightarrow$ x is an extreme point.

Managing extreme points algebraically

- Let A be an *m* by *n* matrix with $m \le n$, we say A has full rank (full row rank) if A has *m* linearly independent columns.
- In this, we can rearrange

$$\mathbf{x} = \begin{pmatrix} \mathbf{x}_B \\ \mathbf{x}_N \end{pmatrix} \begin{matrix} \leftarrow \text{basic variables} \\ \leftarrow \text{non-basic variables} \end{matrix}$$

$$\begin{matrix} \mathbf{A} & = & (& \mathbf{B} & | & \mathbf{N} &) \\ & & & \uparrow & & \uparrow & \\ & & & \text{basis} & & \text{non} - \text{basis} & \end{matrix}$$

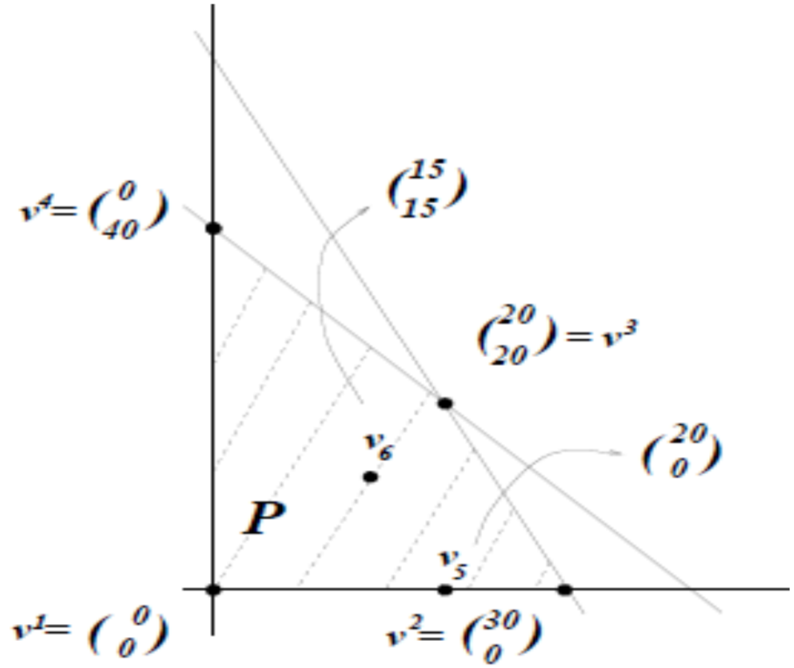
- Definition: (basic solution and basic feasible solution)

If we set $\mathbf{x}_N = 0$ and solve $\mathbf{x}_B$ for $\mathbf{A}\mathbf{x} = \mathbf{B}\mathbf{x}_B = \mathbf{b}$ then $\mathbf{x}$ is a <u>*basic solution*</u> (bs).

Furthermore, if $\mathbf{x}_B \geq 0$, then $\mathbf{x}$ is a <u>*basic feasible solution*</u> (bfs).

Example of basic and basic feasible solutions

$$
\begin{aligned}
\text{Minimize} \quad & x_1 - 2x_2 \\
\text{subject to} \quad & x_1 + x_2 + x_3 = 40 \\
& 2x_1 + x_2 + x_4 = 60 \\
& x_1, x_2, x_3, x_4 \geq 0.
\end{aligned}
$$



$$v^1 = \begin{pmatrix} 0 \\ 0 \\ 40 \\ 60 \end{pmatrix}, v^2 = \begin{pmatrix} 30 \\ 0 \\ 10 \\ 0 \end{pmatrix}, v^3 = \begin{pmatrix} 20 \\ 20 \\ 0 \\ 0 \end{pmatrix}, v^4 = \begin{pmatrix} 0 \\ 40 \\ 0 \\ 20 \end{pmatrix} \text{bfs}$$

$$v^7 = \begin{pmatrix} 40 \\ 0 \\ 0 \\ -20 \end{pmatrix}, v^8 = \begin{pmatrix} 0 \\ 60 \\ -20 \\ 0 \end{pmatrix} \text{bs}$$

$$v^5 = \begin{pmatrix} 20 \\ 0 \\ 20 \\ 20 \end{pmatrix}, v^6 = \begin{pmatrix} 15 \\ 15 \\ 10 \\ 15 \end{pmatrix}$$

Further results

- Observation: When A does not have full rank, then either

  (1) Ax = b has no solution and hence $P = \emptyset$, or

  (2) some constraints are redundant.

  For the second case, after removing the redundant constraints, new A has full rank.

- Corollary: A point **x** in P is an extreme point of P if and only if **x** is a bfs corresponding to some basis B.

- Corollary: The polyhedron P has only a finite number of extreme points.

  <u>Proof:</u> # of ways to choose m linearly independent columns from n columns $\leq C(n, m) = \frac{n!}{m!(n-m)!}$.

Are there many vertices for LP?

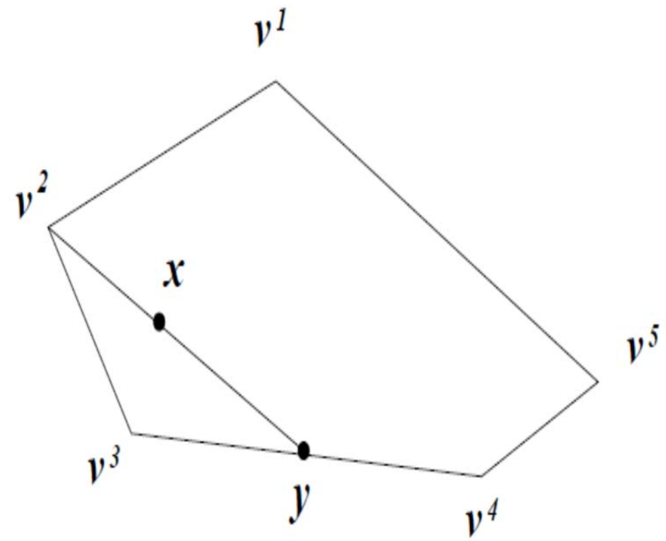
- Yes!

$$C(n,m) = \frac{n!}{m!(n-m)!}$$

- This is not a small number, when n and m become large. Please try it out by taking *n* = 100 and *m* = 50.

What do extreme points bring us?

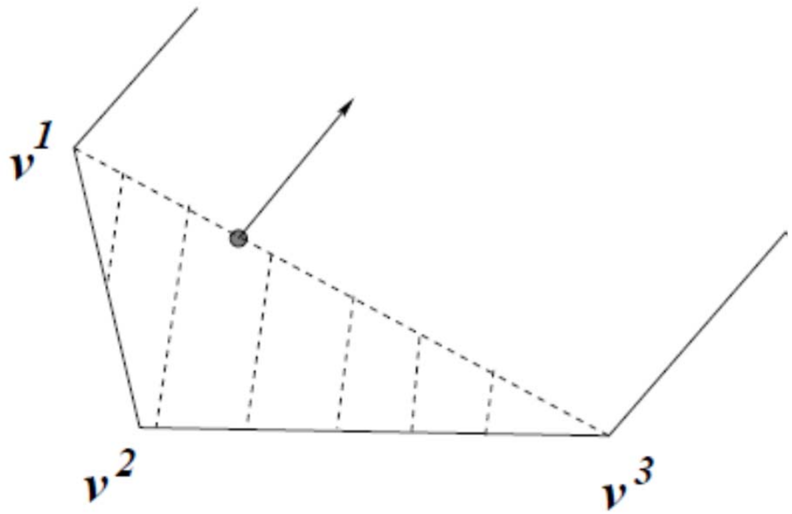
- Observation:
  When $P = \{\mathbf{x} \in \mathbf{R}^n | \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq 0\}$
  is a nonempty polytope, then
  any point in P can be represented
  as a convex combination of the
  extreme points of P.



Question: Can it be more general?

Extremal direction for unboundedness

- When P is unbounded, we need a direction leading to infinity.



- Definition:
- A vector $\mathbf{d}\ (\neq 0) \in \mathbf{R}^n$ is an extremal direction of P, if

$$\{\mathbf{x} \in \mathbf{R}^n | \mathbf{x} = \mathbf{x}^0 + \lambda \mathbf{d},\ \lambda \geq 0\} \subset P$$

for all $\mathbf{x}^0 \in P$..

- Observations:

(1) P is unbounded $\Leftrightarrow$ P has an extremal direction.

(2) $\mathbf{d}\ (\neq \mathbf{0})$ is an extremal direction of P $\Leftrightarrow$ $\mathbf{Ad} = 0$ and $\mathbf{d} \geq \mathbf{0}$

Resolution theorem

- Theorem: Let $V = \{v^i \in \mathbf{R}^n | i \in I\}$ be a set of all exterme points of P, I is a finite index set, then $\forall\ \mathbf{x} \in P$, we have
$$\mathbf{x} = \sum_{i \in I} \lambda_i v^i + \mathbf{d}$$
where
$$\sum_{i \in I} \lambda_i = 1,\ \ \lambda_i \geq 0\ \forall i \in I.$$
and either $\mathbf{d} = \mathbf{0}$ or
$\mathbf{d}$ is an extermal direction of P.

- We can also write
$\mathbf{x} = \sum_{i \in I} \lambda_i v^i + s\,\mathrm{d}$, for some $s \geq 0$.

Implications of resolution theorem

- Corollary:
  If P is bounded (a polytope) , then any **x** in P can be expressed as a convex combination of its extreme points.

- Corollary:
  If P is nonempty, then it has at leas one extreme point.

Note that $\mathbf{x} = \sum_{i \in I} \lambda_i v^i + s\,\mathrm{d}$ implies that the objective value of x is determined by the objective values of extreme points and extremal direction.

Fundamental theorem of LP

- Theorem: For a standard form LP, if its feasible domain P is nonempty, then the optimal objective value of $\mathbf{z} = \mathbf{c}^T\mathbf{x}$ over P is either unbounded below, or it is attained at (at least) an extreme point of P.

- Proof:

By the resolution theorem, there are two cases:

Case 1:

P has an extremal direction $\mathbf{d}$ such that $\mathbf{c}^T\mathbf{d} < \mathbf{0}$. Hence P is unbounded and $\mathbf{z} \rightarrow -\infty$ along $\mathbf{d}$.

Proof - continue

- Case 2: P does not have any extremal direction $\mathbf{d}$ such that $\mathbf{c}^T\mathbf{d} < 0$, then $\forall\ \mathbf{x} \in P$, either $\mathbf{x} = \sum_{i\in I} \lambda_i v^i$ with $\sum_{i\in I} \lambda_i = 1,\ \lambda_i \geq 0$, or $\mathbf{x} = \sum_{i\in I} \lambda_i v^i + \bar{\mathbf{d}}$ with $\mathbf{c}^T\bar{\mathbf{d}} \geq \mathbf{0}$.

- In both cases,

$$
\begin{aligned}
\mathbf{c}^T\mathbf{x} &= \mathbf{c}^T[\textstyle\sum_{i\in I} \lambda_i v^i](+\mathbf{c}^T\bar{\mathbf{d}}) \\
&\geq \textstyle\sum_{i\in I} \lambda_i(\mathbf{c}^T v^i) \\
&\geq \min_{i\in I}\{\mathbf{c}^T v^i\}(\textstyle\sum_{i\in I} \lambda_i) \\
&= \min_{i\in I}\{\mathbf{c}^T v^i\} \\
&= \mathbf{c}^T v^{\min}.
\end{aligned}
$$

Hence the minimum of z is attained at one extreme point!