

Non-determinism

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Computation vs. Reasoning

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The human brain is not like that (we think). Thinking/reasoning does not work this way.

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There are also non-deterministic machines. Below, we provide two alternative definitions for non-deterministic Turing machines.

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Definition: Non-deterministic TM

The transition function for a non-deterministic TM:

$$\delta: \Sigma \times \Gamma \times S \rightarrow \mathcal{P}(\{\leftarrow, \cdot, \rightarrow\} \times \Gamma \times \{\leftarrow, \cdot, \rightarrow\} \times S) \setminus \{\emptyset\}.$$

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For a given configuration, it is not necessary to specify a single succeeding configuration. Instead, a set of possible succeeding configurations is provided (each element of the set described by the transition function represents a possible succeeding configuration).

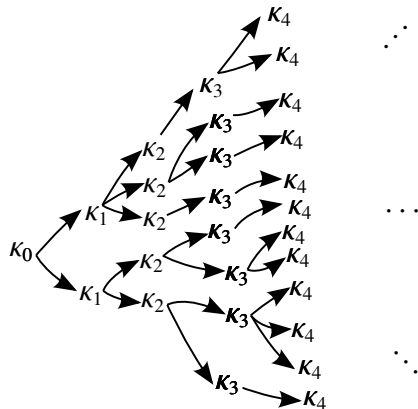
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Thus, the run for the input ω is not determined (in other words, non-deterministic), i.e., from the initial configuration $\kappa_0(\omega)$, multiple possible configurations can be reached. Thus, a tree rooted at $\kappa_0(\omega)$ describes the possible runs of the machine.

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That is, rejecting ω is equivalent to all runs on ω leading to a REJECT state.

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This tape is read-only, and the head can only move to the right.

The transition function is defined the same way as in the deterministic case, and the run is deterministic. That is, ω and τ (the content of the witness tape) uniquely determine a configuration sequence:

$$\kappa_0 = \kappa_0(\omega, \tau) \rightarrow \kappa_1 \rightarrow \kappa_2 \rightarrow \dots$$

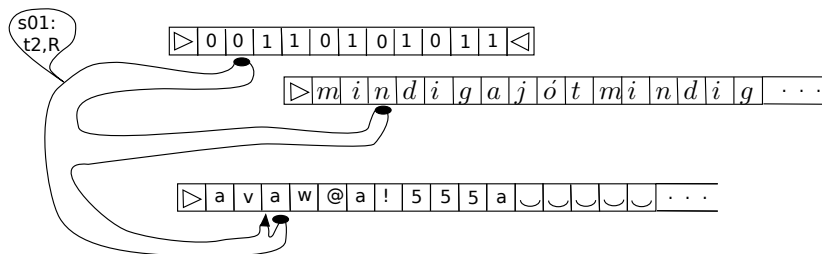
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During a witness tape τ , we may reach a REJECT state on the ω input. This does not necessarily mean that the input is incorrect. If $\omega \in L$, it means that τ is a bad choice/unconvincing witness. That's why in the non-deterministic case, we often give the name NOT-CONVINCED to the REJECT state. Rejection occurs when every τ witness tape content leads to the NOT-CONVINCED state.

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Theorem

L is decidable by I-non-deterministic TM if and only if it is decidable by II-non-deterministic TM.

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We won't prove the theorem here, but an interested student can easily do so.

Break



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Definition: $NTIME(\omega; T)$

If $\omega \in L$ then $NTIME(\omega; T)$ is defined as

$$\min\{TIME(\omega, \tau; T) : \text{where } \tau \text{ is such that } T \text{ reaches the ACCEPT state on } \omega\}.$$

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In other words, among accepting runs, the *most brilliant* witness tape content determines the time limit.

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We could have introduced the above definitions based on the first version of non-determinism as well.

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The complexities of T and \tilde{T} are the same.

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 \cup & & \cup & & \cup & & \cup & & \cup \\
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So our new machine will definitely halt in $2n + t(n)$ time ($t(n) + 2n$ and $t(n)$ are of the same order of magnitude). It performs the computations of T if they fit into $t(n)$ time.

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The ACCEPT/REJECT state of T halts our machine. So our new machine will use $1 + s(n)$ space ($s(n)$ and $s(n) + 1$ are of the same order of magnitude).

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Assume $s(n)$ is a nice space function. Let T be an arbitrary Turing machine. Let w be an arbitrary n -length input.

First, simulate the machine W exhibiting the niceness of $s(n)$. Then, overwrite the used cells with an empty symbol and put a SO-FAR character behind them.

After that, start the simulation of the T machine. If we read the SO-FAR character, we stop with a LOT-OF-MEMORY state.

The ACCEPT/REJECT state of T halts our machine. So our new machine will use $1 + s(n)$ space ($s(n)$ and $s(n) + 1$ are of the same order of magnitude).

The new machine performs the computations of T if they fit into $s(n)$ space. Furthermore there will be two halting configurations on any input (LOT-OF-MEMORY \equiv REJECT \equiv NON-CONVINCED).

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Due to the time complexity condition, if $\omega \in L$, there will be a run leading to the ACCEPT state that terminates before the BUZZ state. The simulating machine *detects* this.

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Run T . However, at the end of the computation, *keep to ourselves* the result and before announcing the ACCEPT/REJECT states, erase the work tape in $s(n)$ length (due to niceness, this can be easily done).

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Move every head/hand to the left. After that, reach the halting state corresponding to the computed result.

We have two different halting configurations (*photo* of the machine), and of course, our machine computes the same thing as the original.

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During the simulation of W , all other work tapes with heads/hands continuously move to the right until the BUZZ state. Then, beyond the tapes used by W , exactly $t(n)$ cells are marked on the other tapes.

Break



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Our goal is to understand the following inclusion chain:

$$\mathcal{L} \subset \mathcal{NL} \stackrel{(2)}{\subset} \mathcal{P} \subset \mathcal{NP} \stackrel{(1)}{\subset} \mathcal{PSPACE} \subset \mathcal{NPSPACE} \stackrel{(2)}{\subset} \mathcal{EXP} \subset \mathcal{NEXP}.$$

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Below, we will prove these. However, our goal is not to provide the shortest justification, but to summarize the results and introduce the methods for later use.

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Triviality B

- (i) $TIME(t(n)) \subset NTIME(t(n)).$
- (ii) $SPACE(s(n)) \subset NSPACE(s(n)).$

Indeed, determinism can be seen as a special case of non-determinism.

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For this, we keep the working tapes needed for the description of T and add one that plays the role of the witness tape and another that plays the role of a clock ($t(n)$ is a nice time function).

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For this, we keep the working tapes needed for the description of T and add one that plays the role of the witness tape and another that plays the role of a clock ($t(n)$ is a nice time function).

Of course, the new machine does not possess the *genius*/guessing property of non-deterministic machines.

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We assume that the length of our input is n .

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// We don't need more characters because a time-limited machine cannot read more.

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On the tapes corresponding to the working tapes of the T Turing machine, we simulate the run of T for the first witness for $t(n)$ time. The simulation either ends in the ACCEPT or the NON-AGREE state, or the time runs out/we run out of $t(n)$ time. Even the latter is considered as rejecting the tested witness (NON-AGREE state).

If the simulation reaches the ACCEPT state, we also accept the input, and \tilde{T} stops. If it reaches the NON-AGREE state, then on the tape playing the role of the witness tape, we overwrite its content with the next possible $t(n)$ -length witness beginning. We clear the contents of the other tapes. We repeat the Simulation phase.

If the next witness start cannot be generated because we have tested all possible witness beginnings (the witnesses are exhausted), then we stop with the DISAGREE state.

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Specifically, the containment marked with (1) is implied.

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Reduced Configuration

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A *reduced configuration* for an l-non-deterministic Turing machine with space requirement $s(n)$ on input ω contains the following components:

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$$|V| \leq \alpha_T \cdot (n+1) \cdot \beta_T^{s(n)}.$$

From a configuration κ , we can easily obtain the corresponding reduced configuration $\rho = \text{red}(\kappa)$. If the ω input is known, then conversely, it is also true: ω and the reduced configuration ρ determine the full configuration $\kappa = \text{conf}(\omega, \rho)$.

The Graph

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Let T be a l-non-deterministic Turing machine and ω an input. Then $\vec{G}_{\omega, T}$ is the graph of reduced configurations associated with (T, ω) . This is a directed graph where the set of vertices is the above V set, and \vec{uv} exists if and only if the transition function allows the configuration $\text{konf}(\omega, v)$ after the configuration $\text{konf}(\omega, u)$.

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Note that for a deterministic machine, the above concepts can also be introduced. In this case, every vertex in the defined directed graph would have outdegree 1.

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The graph defined above can be calculated efficiently.

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The given space is sufficient to encode a constant number of reduced configuration codes on the work tape.

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- If not, we move on to the next label. If yes, we copy it to the output tape, followed by a ':'. After that, we write down the sequence of out-neighbors.

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- In the second block of the work tape, we also begin the enumeration of reduced configurations. If the codes of x and y reduced configurations appear on the work tape, we decide whether there is an edge from x to y . The input tape contains ω , and the T Turing machine — with a finite description — is known to us. The implementation of this subtask is again simple.

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- If we get an edge, we copy y to the output tape. If we don't get an edge, we immediately move on to the next y search.
- If y is exhausted, we move on to the next x search. If the x 's are also exhausted, then we have computed the code of $\vec{G}_{\omega, T}$.

Proof of the Lemma (Continued)

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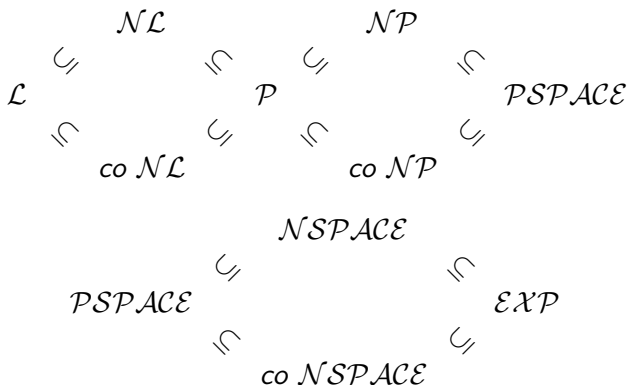
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We will discuss these relationships later if time allows.

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The question of whether „The inclusion $\mathcal{P} \subseteq \mathcal{NP}$ is proper, or an equality holds?“ is considered a central problem in 21st-century mathematics.

Break



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Our tests can certainly be carried out in polynomial time. If all of these tests pass, then the witness proves that there is a Hamiltonian cycle in the input graph. On the other hand, for every graph with a Hamiltonian cycle, a witness can be found. Thus, we have that

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As with the previously unexplained reasoning, it can be shown that among the conclusions of the proof, there is also one that can be handled with combinable coefficients.

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Therefore,

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LP-TESTING problem (continued)

Currently, there are several algorithms for linear programming that run in polynomial time. That is,

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- The first polynomial algorithm was given by Kachian in 1979.

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This implies that

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The Agrawal-Kayal-Saxena primality test leads to the following theorem:

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We do not discuss the encoding of the input. However, we note that in the above sense, v , the number of vertices, can also be considered the size of the input (instead of the length of the code).

PERFECT-MATCHING-TESTING (continued)

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First, we describe a nondeterministic algorithm. We use the second interpretation of nondeterminism, deciding acceptance with the help of the content of a witness tape. The content of the witness tape will be a set M of pairs of vertices.

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The machine T tests whether the pairs of vertices are connected by an edge, and each vertex appears in exactly one pair. If the answer is yes for both tests, then we reach the ACCEPT state. If the witness fails any of the tests, the machine transitions to the REJECT state.

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For the existence of a perfect matching, it is easy to provide a proving witness. If there is no perfect matching, then each witness will fail.

The tests can be easily performed in polynomial time. Thus, we have that

PERFECT-MATCHING-TESTING $\in \mathcal{NP}$.

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Let the content of the witness tape be a set T . For $\omega \equiv G$ graph and $\tau \equiv T$ set, the machine determines the components of $G - T$, counts the components with odd size (number of vertices), and compares this number with $|T|$.

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The proof of the polynomial realizability of the machine is left to the reader.

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Thus, we obtain the following

$$\text{PERFECT-MATCHING-TESTING} \in \text{co}\mathcal{NP}.$$

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This (the quite complex) algorithm leads to a stronger statement than the previous two results:

$$\text{PERFECT-MATCHING-TESTING} \in \mathcal{P}.$$

DIRECTED-REACHABILITY

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The breadth-first search algorithm implies that

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If we don't reach t , we check if we have taken v steps. If yes, we transition to the REJECT/NON-CONVINCED state.

If we haven't taken so many steps, we nondeterministically choose a vertex. We check if we can move from the previous vertex to this one via an edge. If not, we again transition to the REJECT state. If yes, we erase the previous vertex (!).

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Thus, we have that

$$\text{DIRECTED-REACHABILITY} \in \mathcal{NL}.$$

DIRECTED-REACHABILITY (continued)

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We present another deterministic algorithm, which uses space very economically:

$$\begin{aligned}\text{DIRECTED-REACHABILITY} &\in \bigcup_{\alpha \in \mathbb{N}} \mathcal{SPACE}(\alpha \log_2^2 n) \\ &= \mathcal{SPACE}(\log^2 n).\end{aligned}$$

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This is proven by the following recursive algorithm.

DIRECTED-REACHABILITY (continued)

Directed-Reachability:

$\text{BOUNDED-DIRECTED-REACHABILITY}(x, y, 2^\ell)$

DIRECTED-REACHABILITY (continued)

Directed-Reachability:

BOUNDED-DIRECTED-REACHABILITY($x, y, 2^\ell$)

// Given vertices x and y , it tests whether there is a walk between them of at most 2^ℓ steps. We can think of this walk as a *lazy* walk. At each step, we have two options: either move to a neighbor or stay put. In a lazy walk, we can assume that the length is exactly 2^ℓ .

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If $\ell = 0$, then test if $x = y$ or \overrightarrow{xy} is an edge. If the test succeeds, halt with the ACCEPT state, otherwise halt with the REJECT state.

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// k is the middle point of the lazy walk, meaning there is a lazy walk of $2^{\ell-1}$ steps from x to k and from k to y . We try all possibilities.

DIRECTED-REACHABILITY (continued)

Directed-Reachability:

BOUNDED-DIRECTED-REACHABILITY($x, y, 2^\ell$) (continued)

(1) BOUNDED-DIRECTED-REACHABILITY($x, k, 2^{\ell-1}$)

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The above algorithm, when run with $\ell = \lceil \log_2 |V(G)| \rceil$, solves reachability. The algorithm is implemented on a Savitch Turing machine in a space-efficient way.

DIRECTED-REACHABILITY (continued)

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Savitch's Theorem

The recursive algorithm described above can be implemented on a Turing machine in such a way that at any configuration, at most ℓ blocks are used on the work tape, where each block has a length of at most $\mathcal{O}(\log |V|)$ (enough to store a finite number of vertices).

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We provide only ideas for the exact implementation:

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- For a middle vertex w , $p_{\text{left}}(w) = (\text{does a lazy walk of length } 2^{\ell-1} \text{ lead from } u \text{ to } w)$,

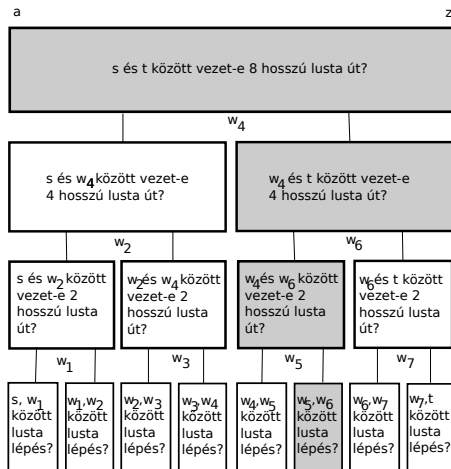
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- For a middle vertex w , $p_{\text{left}}(w)=$ (does a lazy walk of length $2^{\ell-1}$ lead from u to w), and $p_{\text{right}}(w)=$ (does a lazy walk of length $2^{\ell-1}$ lead from w to v) are the two subproblems of the p problem.

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- The two tasks are *siblings* to each other.

DIRECTED-REACHABILITY (Continuation)



The content of the work tape always represents a task (node in the tree) along with the path to the root. One example is highlighted with shading.

DIRECTED-REACHABILITY (Continuation)

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Furthermore, updating the path can be solved with a Turing machine.

The details of the complete implementation go beyond the scope of the course.

This is the End!

Thank you for your attention!