Divide and conquer

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Sorting

Multiplication

The basic idea of divide and conquer

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• Your given a data. The problem is such that for any subset of the data it makes sense to propose the problem.

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• From the solutions for the parts we try to figure out the answer for the whole data. The mathematical essence is that very often combining the two (or more) partial solutions can be done efficiently.

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• The best way to understand the scheme is to see examples.

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Sorting problem

Given *n* real numbers: x_1, x_2, \ldots, x_n . Find their ordered sequence.

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We are lookinf for such a π permutation of the index set $[n] = \{1, 2, ..., n\}$ that $x_{\pi 1} \le x_{\pi 2} \le ... \le x_{\pi n}$.

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Our model is a comparison based computation. I.e. our only elementary step is comparison. When we analyze the algorithm we just count how many comparisons we make.

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- In the case of larger input size consider the number $x_1, x_2, \ldots, x_{\lceil n/2 \rceil}$ and the numbers $x_{\lceil n/2 \rceil+1}, x_{\lceil n/2 \rceil}+2, \ldots, x_n$ separately. Sort them using the idea recursively.

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• Let $y_1 \leq y_2 \leq \ldots \leq y_{\lceil n/2 \rceil}$ be the sorted list of $x_1, x_2, \ldots, x_{\lceil n/2 \rceil}$. Let $z_1 \leq z_2 \leq \ldots \leq z_{\lfloor n/2 \rfloor}$ be the sorted list of $x_{\lceil n/2 \rceil+1}, x_{\lceil n/2 \rceil} + 2, \ldots, x_n$.

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• Combine the two sorted lists into one. Combining the sorted list of the y's and z's are called merging the two sorted lists. The heart of the divide and conquer sorting algorithm is the way of merging.

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Sorting



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Theorem

Let
$$y_1 \leq y_2 \leq \ldots \leq y_k$$
 and $z_1 \leq z_2 \leq \ldots \leq z_\ell$ two ordered sequence. With at most

$$k + \ell - 1$$

comparisons we can merge them.

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Proof: See recitation session.

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Merge sort

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Merge sort

(Case of small inputs) If $n \leq 2$, then sort them.

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Merge sort

(Case of small inputs) If $n \leq 2$, then sort them.

(Divide and use recursion) If $n \ge 3$, then Sort $x_1, x_2, \ldots, x_{\lceil n/2 \rceil}$ and $x_{\lceil n/2 \rceil + 1}, x_{\lceil n/2 \rceil} + 2, \ldots, x_n$ separately using Merge sort.

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Merge sort

(Case of small inputs) If $n \leq 2$, then sort them.

(Divide and use recursion) If $n \ge 3$, then Sort $x_1, x_2, \ldots, x_{\lceil n/2 \rceil}$ and $x_{\lceil n/2 \rceil+1}, x_{\lceil n/2 \rceil} + 2, \ldots, x_n$ separately using Merge sort. (Merge) Merge the two sequences using the algorithm from the

recitation session.

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Sorting

The analysis of the algorithm

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The analysis of the algorithm

• Let S(n) the minimal number of comparison that is enough to handle all inputs of size n.

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Sorting

The analysis of the algorithm

- Let S(n) the minimal number of comparison that is enough to handle all inputs of size n.
- We know that S(1) = 0 and S(2) = 1.

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Sorting

Multiplication

The analysis of the algorithm (cont'd)

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Sorting

The analysis of the algorithm (cont'd)

Theorem

$$S(n) \leq S\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + S\left(\left\lceil \frac{n}{2} \right\rceil\right) + n - 1.$$

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The analysis of the algorithm: Conquer

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Sorting

The analysis of the algorithm: Conquer

Lemma If $n = 2^k$ $S(n) \le (k-1)2^k + 1 = n(\log_2 n - 1) + 1.$

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Sorting

The analysis of the algorithm: Conquer

Lemma If $n = 2^k$ $S(n) \le (k-1)2^k + 1 = n(\log_2 n - 1) + 1.$

Theorem

$$S(n) < 2n \log_2 n + 1 = \mathcal{O}(n \log n).$$

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$$S(n) < 2n \log_2 n + 1 = \mathcal{O}(n \log n).$$

• This is the optimal order of magnitude (see recitation).

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Break



The basic problem

The problem of multiplication

Given two *n*-digit numbers $x_1x_2...x_n$ and $y_1y_2...y_n$ (i.e. x_i 's and y_j 's are elements of $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$). Compute the digits of their product.

Sorting




• Why we consider the multiplication?

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Why?

- Why we consider the multiplication?
- \bullet Summation, subtraction, dividing/multiplying by 2 are much easier.

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Why?

- Why we consider the multiplication?
- Summation, subtraction, dividing/multiplying by 2 are much easier. The well-known algorithm uses O(n) operations on digits.
- This is optimal.

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The problem

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The problem

• The well-known algorithm for multiplication takes $O(n^2)$ operations on digits.

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The problem

• The well-known algorithm for multiplication takes $\mathcal{O}(n^2)$ operations on digits.

• In the XXth century, at the beginning of the 60's Kolmogorov conjectured that one can't do better. He proposed this as a problem to work on at his world-famous seminar.

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The problem

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• In the XXth century, at the beginning of the 60's Kolmogorov conjectured that one can't do better. He proposed this as a problem to work on at his world-famous seminar.

• One of the participant, Karatsuba refuted the conjecture.

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The naive "divide"

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• The naive application of the Divide and conquer technique suggests the following. Think about the two input numbers x, and y as

$$x_{\text{first}} \cdot 10^{\nu} + x_{\text{second}}, \quad \text{and} \quad y_{\text{first}} \cdot 10^{\nu} + y_{\text{second}}.$$

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• We need four multiplications on half-size numbers, and additional simple operations (summations).

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• The product of x and y

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- We need four multiplications on half-size numbers, and additional simple operations (summations).
- Important to note that multiplication with 10^{ν} and $10^{2\nu}$ do not use any digit operations.

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• Introduce the function N(n): the maximal number of digit operations is needed to multiply two *n*-digit numbers. (The minimal guarantee for *n*-digits numbers.)

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 \bullet Using the ${\cal O}$ notation is laziness. The hidden constant can be determined, or can be easily bounded by 100.

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The basic idea

Sorting



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• The "solution" of the inequality is $N(n) = O(n^2)$. The bound matches the complexity the algorithm from elementary school.

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The first idea of Karatsuba

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The first idea of Karatsuba

• The computing the square of a given number has the same complexity as multiplication.

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The basic idea

Sorting

The first idea of Karatsuba

- The computing the square of a given number has the same complexity as multiplication.
- Indeed

$$x \cdot y = \frac{(x+y)^2 - x^2 - y^2}{2},$$

hence multiplication of two numbers can be substituted by three computations of squares and "easy" side computations.

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hence multiplication of two numbers can be substituted by three computations of squares and "easy" side computations.

• Now on we consider the problem of squaring a number.

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Divide and conquer for squaring

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The basic idea

Sorting

Divide and conquer for squaring

• The formula

$$(x_{\textit{first}} \cdot 10^{\nu} + x_{\textit{second}})^2 = x_{\textit{first}}^2 \cdot 10^{2\nu} + x_{\textit{second}}^2 + 2x_{\textit{first}}x_{\textit{second}} \cdot 10^{\nu}$$

is most promising. We need two squaring and one multiplication on half-size numbers.

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The second idea

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The second idea

• The second idea is the same as the first:

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The second idea

• The second idea is the same as the first:

$$(x_{first} \cdot 10^{\nu} + x_{second})^2 = x_{first}^2 \cdot 10^{2\nu} + x_{second}^2 + ((x_{first} + x_{second})^2 - x_{first}^2 - x_{second}^2) \cdot 10^{\nu}$$

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The analysis

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The analysis

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Final remark

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- Karatsuba's algorithm is far being optimal.
- There is "new" algorithm and its analysis gives that in $\mathcal{O}(n \log n)$ steps one can multiply two *n*-digit numbers.

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- D. Harvey, J. van der Hoeven, Integer multiplication in time $\mathcal{O}(n \log n)$, Annals of Mathematics, 193(2), 2021, 563–617.

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This is the end!

Thank you for your attention!

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