Banach-Stone type theorems on spaces of probability measures

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Some history

Theorem (Banach-Stone)

Let K be a compact Hausdorff space and $C(K)$ be the Banach space of all real-valued continuous functions endowed with the usual supremum norm $\|f\|_\infty$. Assume that $T\colon C(K) \to C(K)$ is a linear operator which is onto, and it is an isometry, i.e.

$$\|Tf\|_\infty = \|f\|_\infty \qquad (f \in C(K)).$$

Then T is a product of

- *a composition operator $C_h \in B(C(K))$ with the symbol $h\colon K \to K$ being a homeomorphism and*
- *a multiplication operator $M_\tau \in B(C(K))$ with continuous symbol $\tau\colon K \to \{-1, 1\}$;*

that is we have

$$(Tf)(t) = (M_\tau C_h f)(t) = \tau(t) \cdot f(h(t)) \qquad (t \in K). \tag{1}$$

Note that obviously all operators of the form (1) is an isometry of $C(K)$.

Let us denote by $\mathcal{D}(\mathbb{R})$ the set of all probability distribution functions $f\colon \mathbb{R} \to \mathbb{R}$. The **Kolmogorov-Smirnov distance** between $f, g \in \mathcal{D}(\mathbb{R})$ is defined by

$$\rho(f,g) := \|f-g\|_\infty = \sup_{t\in\mathbb{R}} |f(t)-g(t)|.$$

We call a map $\phi\colon \mathcal{D}(\mathbb{R}) \to \mathcal{D}(\mathbb{R})$ a **Kolmogorov-Smirnov isometry** if the following is satisfied:

$$\rho(\phi(f),\phi(g)) = \rho(f,g) \qquad (f,g \in \mathcal{D}(\mathbb{R})).$$

Note that equivalently, we can consider the set of all Borel probability measures $\mathcal{P}_\mathbb{R}$, and define the notions of Kolmogorov-Smirnov distance and Kolmogorov-Smirnov isometries. For the Kolmogorov-Smirnov distance between μ and $\nu \in \mathcal{P}_\mathbb{R}$ we have

$$\rho(\mu,\nu) = \sup_{t\in\mathbb{R}} \left|\mu\big((-\infty,t]\big) - \nu\big((-\infty,t]\big)\right|.$$

Theorem (Dolniar-Molnár, 2008, JMAA, original phrasing)

Let $\phi\colon \mathcal{D}(\mathbb{R}) \to \mathcal{D}(\mathbb{R})$ be an arbitrary onto Kolmogorov-Smirnov isometry. Then there exists a strictly increasing bijection $\psi\colon \mathbb{R} \to \mathbb{R}$ such that ϕ has one of the following two forms:

(1) $\phi(f)(t) = f(\psi(t)) \quad (t \in \mathbb{R})$, *or*

(2) $\phi(f)(t) = 1 - f(\psi(-t)-) \quad (t \in \mathbb{R})$.

Theorem (Dolniar-Molnár, re-phrasing)

Let $\varphi\colon \mathcal{P}_\mathbb{R} \to \mathcal{P}_\mathbb{R}$ be an arbitrary surjective Kolmogorov-Smirnov isometry. Then there exists a homeomorphism $h\colon \mathbb{R} \to \mathbb{R}$ such that φ is of the following form:

$$\varphi(\mu)(A) = \mu(h^{-1}[A]) \qquad (\mu \in \mathcal{P}_\mathbb{R}, A \in \mathscr{B}_\mathbb{R}),$$

i.e. $\varphi(\mu)$ is the push-forward measure of μ by h.

Note that the relation between h and ψ is that $h = \psi^{-1}$.

The **Lévy distance** between $f, g \in \mathcal{D}(\mathbb{R})$ is defined by

$$L(f,g) := \inf_{\varepsilon>0} \{t \in \mathbb{R} \colon f(t-\varepsilon) - \varepsilon \le g(t) \le f(t+\varepsilon) + \varepsilon\}.$$

We call a map $\phi \colon \mathcal{D}(\mathbb{R}) \to \mathcal{D}(\mathbb{R})$ a **Lévy isometry** if the following is satisfied:

$$L(\phi(f), \phi(g)) = L(f,g) \qquad (f,g \in \mathcal{D}(\mathbb{R})).$$

Equivalently, the **Lévy distance** between $\mu, \nu \in \mathcal{P}_{\mathbb{R}}$ is

$$\begin{aligned} L(\mu,\nu) := \inf_{\varepsilon>0} \{\mu\big((-\infty, t-\varepsilon]\big) - \varepsilon \le \nu\big((-\infty, t]\big) \\ \le \mu\big((-\infty, t+\varepsilon]\big) + \varepsilon \quad \forall t \in \mathbb{R}\} \end{aligned}$$

and we call a map $\varphi \colon \mathcal{P}_{\mathbb{R}} \to \mathcal{P}_{\mathbb{R}}$ a **Lévy isometry** if we have

$$L(\varphi(\mu), \varphi(\nu)) = L(\mu,\nu) \qquad (\mu, \nu \in \mathcal{P}_{\mathbb{R}}).$$

Note that the Lévy distance *metrises the weak convergence*.

Theorem (Molnár, 2011, JMAA, original phrasing)

Let $\phi\colon \mathcal{D}(\mathbb{R}) \to \mathcal{D}(\mathbb{R})$ be an arbitrary onto Lévy isometry. Then there exists a number $c \in \mathbb{R}$ such that ϕ has one of the following two forms:

(1) $\phi(f)(t) = f(t+c) \qquad (t \in \mathbb{R})$, *or*

(2) $\phi(f)(t) = 1 - f((-t+c)-) \qquad (t \in \mathbb{R})$.

Theorem (Molnár, re-phrasing)

Let $\varphi\colon \mathcal{P}_{\mathbb{R}} \to \mathcal{P}_{\mathbb{R}}$ be an arbitrary surjective Lévy isometry. Then there exists an isometry $\psi\colon \mathbb{R} \to \mathbb{R}$ such that φ is of the following form:

$$\varphi(\mu)(A) = \mu(\psi^{-1}[A]) \qquad (\mu \in \mathcal{P}_{\mathbb{R}}, A \in \mathscr{B}_{\mathbb{R}}),$$

i.e. $\varphi(\mu)$ is the push-forward measure of μ by the isometry ψ.

Note that in case of (1) ψ is a translation by $-c$, and in case of (2) ψ is a reflection in the point $\frac{c}{2}$.

Let $\mathcal{I}$ denote the set of all non-degenerate intervals of $\mathbb{R}$. The **Kuiper distance** between μ and $\nu \in \mathcal{P}_{\mathbb{R}}$

$$d_{Ku}(\mu,\nu) := \sup_{t\in\mathbb{R}}(f_\mu(t) - f_\nu(t)) + \sup_{t\in\mathbb{R}}(f_\nu(t) - f_\mu(t))$$
$$= \sup\{|\mu(I) - \nu(I)| \colon I \in \mathcal{I}\}.$$

Molnár posed me the problem of characterising all surjective **Kuiper isometries** on the space of all **continuous probability measures**

$$\mathcal{P}_{\mathbb{R}}^c := \left\{\mu \in \mathcal{P}_{\mathbb{R}} \middle| \forall x \in \mathbb{R} \colon \mu(\{x\}) = 0\right\}.$$

For any $x \in \mathbb{R}$ let us define the function

$$r_x \colon \mathbb{R} \setminus \{x\} \to \mathbb{R}, \quad r_x(t) = \frac{1}{t-x}$$

and let

$$r_\infty \colon \mathbb{R} \to \mathbb{R}, \quad r_\infty(t) = t$$

The answer for Molnár's question:

Theorem (G., $\geq$2017, HJM)

Let $\phi\colon \mathcal{P}_{\mathbb{R}}^c \to \mathcal{P}_{\mathbb{R}}^c$ be a surjective Kuiper isometry. Then there exists a homeomorphism $g\colon \mathbb{R} \to \mathbb{R}$ and an $x \in \mathbb{R} \cup \{\infty\}$ such that ϕ has the following form:

$$\phi(\mu)(A) = \mu(g \circ r_x[A]) \qquad (\mu \in \mathcal{P}_{\mathbb{R}}^c, A \in \mathscr{B}_{\mathbb{R}}).$$

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The problem on $\mathcal{P}_{\mathbb{R}}$ was more difficult, the answer reads as follows:

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We emphasise again that the reverse directions are easy to verify in the previous theorems.

Our idea with T. Titkos was to generalize Molnár's theorem for the Lévy-Prokhorov metric. Let X be a real and separable Banach space and $\mathscr{B}_X$ be the set of Borel sets on X. The symbol $\mathcal{P}_X$ stands for the set of all Borel probability measures on X. The so-called **Lévy-Prokhorov distance** metrises the topology of weak convergence, and it is given by

$$\pi(\mu,\nu) := \inf\left\{\varepsilon > 0 \,\middle|\, \forall\, A \in \mathscr{B}_X \colon \mu(A) \leq \nu(A^\varepsilon) + \varepsilon\right\},$$

where

$$A^\varepsilon := \bigcup_{x\in A} B_\varepsilon(x) \qquad \text{and} \qquad B_\varepsilon(x) := \left\{z \in X \,\middle|\, \|x - z\| < \varepsilon\right\}.$$

Note that in case when $X = \mathbb{R}$, the Lévy and the Lévy-Prokhorov metrics are different.

Theorem (G. & Titkos, ≥2017)

Let $(X, \|\cdot\|)$ be a separable real Banach space and $\varphi\colon \mathcal{P}_X \to \mathcal{P}_X$ be a surjective Lévy–Prokhorov isometry. Then there exists a surjective affine isometry $\psi\colon X \to X$ which implements φ, i.e. we have

$$(\varphi(\mu))\,(A) = \mu(\psi^{-1}[A]) \qquad (\forall\, A \in \mathscr{B}_X),$$

where $\psi^{-1}[A]$ denotes the inverse-image set $\{\psi^{-1}(a) \,|\, a \in A\}$.

Sketch of the proof

Let

$$\mathcal{F}_X := \left\{ \sum_{i \in I} \lambda_i \delta_{x_i} \,\middle|\, \#I < \aleph_0,\ \sum_{i \in I} \lambda_i = 1,\ \lambda_i > 0,\, x_i \in X\ (\forall\, i \in I) \right\}$$

be the set of all **finitely supported measures**, and Δ_X stand for the set of all **Dirac measures**.
The closed **support** of $\mu \in \mathcal{P}_X$ will be denoted by S_μ.

There will be four major steps:

1. The action on Δ_X;
2. Isolated atoms on the vertices of the convex hull of the support;
3. The story beyond vertices;
4. The action on $\mathcal{F}_X$ and $\mathcal{P}_X$.

Proposition

Assume that $\mu, \nu \in \mathcal{P}_X$. Then the following are equivalent:

(i) $\pi(\mu, \nu) = 1$,

(ii) $\underline{\mathrm{d}}(S_\mu, S_\nu) := \inf \{d(x, y) \,|\, x \in S_\mu,\ y \in S_\nu\} \geq 1$,

(iii) $S_\nu \cap S_\mu^1 = \emptyset$.

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(iii) $S_\nu \cap S_\mu^1 = \emptyset$.

We define the **unit distance set** of a set of measures $\mathscr{A} \subseteq \mathcal{P}_X$ by

$$\mathscr{A}^{\mathbb{u}} = \left\{\nu \in \mathcal{P}_X \,\middle|\, \forall\, \mu \in \mathscr{A} : \pi(\mu, \nu) = 1\right\}.$$

Observe that $\mathscr{A}^{\mathbb{u}}$ depends only on the shape of S_μ's ($\mu \in \mathscr{A}$).

Proposition

Suppose that $\mu \in \mathcal{P}_X$. Then the following are equivalent:

(i) $(\{\mu\}^{\mathbb{u}})^{\mathbb{u}} = \{\mu\}$,

(ii) *there exists an $x \in X$ such that $\mu = \delta_x$.*

1. The action on Δ_X

Lemma

There exists a surjective affine isometry $\psi \colon X \to X$ *such that*

$$\varphi(\delta_x) = \delta_{\psi(x)} \qquad (\forall\, x \in X).$$

Proof. Since

$$\varphi\big((\{\mu\}^{\mathbb{u}})^{\mathbb{u}} \big) = \big(\{\varphi(\mu)\}^{\mathbb{u}}\big)^{\mathbb{u}} \qquad (\forall\, \mu \in \mathcal{P}_X),$$

there exists a bijective map $\psi \colon X \to X$ such that

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Observe that

$$\pi(\delta_{x_1}, \delta_{x_2}) = \min\{1, \|x_1 - x_2\|\} \qquad (\forall\, x_1, x_2 \in X).$$

This is possible only if ψ is an isometry. ■

1. The action on Δ_X

From now on we may and do assume without loss of generality that

$$\varphi(\delta_x) = \delta_x \qquad (\forall\, x \in X).$$

Our aim will be to show that φ acts identically on the whole of $\mathcal{P}_X$. We define the following continuous function for each $\mu \in \mathcal{P}_X$:

$$\begin{aligned} W_\mu \colon X \to [0,1], \quad W_\mu(x) &:= \pi(\delta_x, \mu) \\ &= \inf\{\varepsilon > 0 \,|\, 1 \le \mu(B_\varepsilon(x)) + \varepsilon\} \\ (\text{if } \mu \notin \Delta_X\ ) \quad &= \min\left\{\varepsilon > 0 \,|\, 1 \le \mu(\overline{B_\varepsilon(x)}) + \varepsilon\right\} \end{aligned}$$

which will be called the **witness function** of μ. Clearly, we have

$$W_\mu(x) = W_{\varphi(\mu)}(x) \quad (\forall\, x \in X).$$

It is natural to expect that the shape of the witness function carries some information about the measure.

2. Isolated atoms on the vertices of the convex hull

Our aim: Let $\mu \in \mathcal{F}_X$ and $\hat{x}$ be a vertex of $\mathrm{conv}(S_\mu)$. Is $\hat{x}$ an isolated atom of $\varphi(\mu)$? If so, do we have $\mu(\{\hat{x}\}) = (\varphi(\mu))(\{\hat{x}\})$?

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Proposition

Let $\mu \in \mathcal{F}_X \setminus \Delta_X$, $K = \mathrm{conv}(S_\mu)$, and assume that $\hat{x}$ is a vertex of K. Set $\hat{\lambda} := \mu(\{\hat{x}\}) \in (0,1)$. Then for every $\vartheta \in \mathcal{P}_X$ with $S_\vartheta \subseteq K$ the following two conditions are equivalent:

(i) $\vartheta = \hat{\lambda} \cdot \delta_{\hat{x}} + (1-\hat{\lambda}) \cdot \widetilde{\vartheta}$ *where* $\widetilde{\vartheta} \in \mathcal{P}_X$ *with* $S_{\widetilde{\vartheta}} \subseteq K \setminus B_r(\hat{x})$ *for some* $r > 0$,

(ii) *there exist a number* $0 < \rho \leq 1 - \hat{\lambda}$ *and a half-line* $\mathfrak{e}$ *starting from* $\hat{x}$ *such that the restriction* $W_\vartheta|_\mathfrak{e}$ *is of the following form:*

$$W_\vartheta|_\mathfrak{e}(x) = \begin{cases} 1 & \text{if } \|x-\hat{x}\| \geq 1, \\ \|x-\hat{x}\| & \text{if } 1-\hat{\lambda} < \|x-\hat{x}\| < 1, \\ 1-\hat{\lambda} & \text{if } 1-\hat{\lambda}-\rho \leq \|x-\hat{x}\| \leq 1-\hat{\lambda}. \end{cases}$$

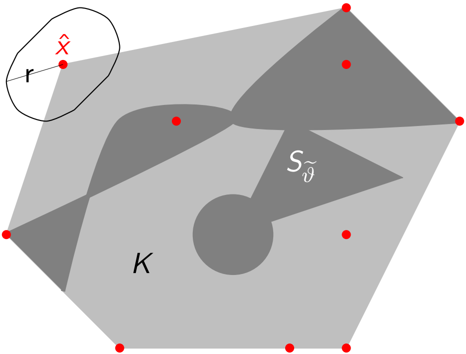
In particular, we have that if $\mu \in \mathcal{F}_X, \#S_\mu \leq 2$, $\nu \in \mathcal{P}_X$ and $W_\mu \equiv W_\nu$, then $\mu = \nu$.

Illustrating the proof of (i)$\Longrightarrow$(ii):

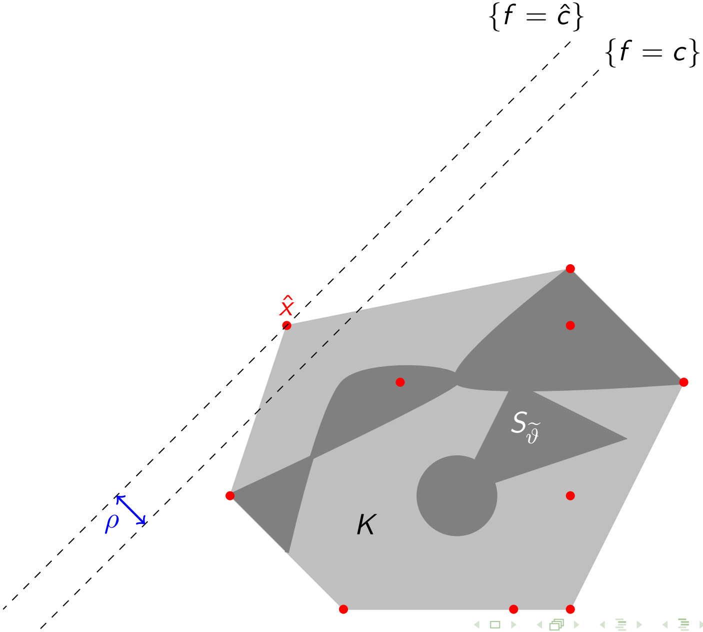
We concentrate on the finite dimensional subspace spanned by K.

The support S_μ consists of the **red** points in K.

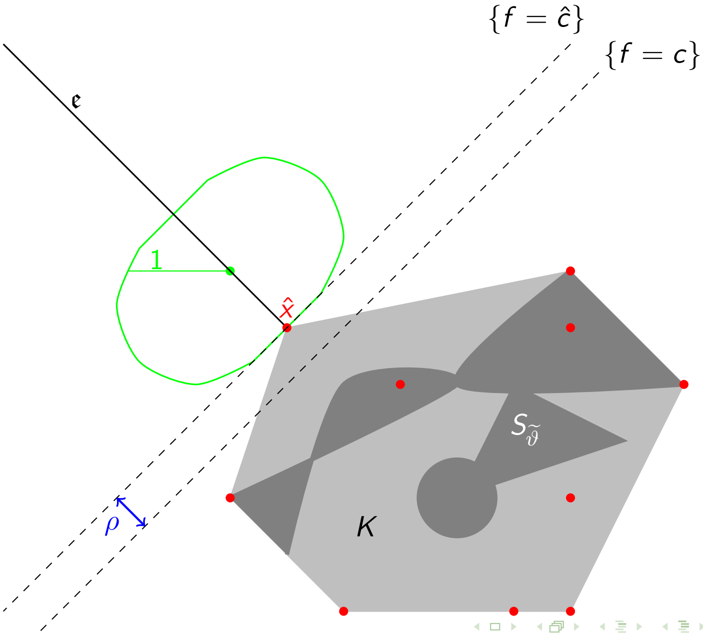
The support S_ϑ is the union of $\{\hat{x}\}$ and $S_{\widetilde{\vartheta}}$.



2. Isolated atoms on the vertices of the convex hull



2. Recall that $W_\vartheta(x) = \min\{\varepsilon > 0 \,|\, 1 \leq \vartheta(\overline{B_\varepsilon(x)}) + \varepsilon\}$



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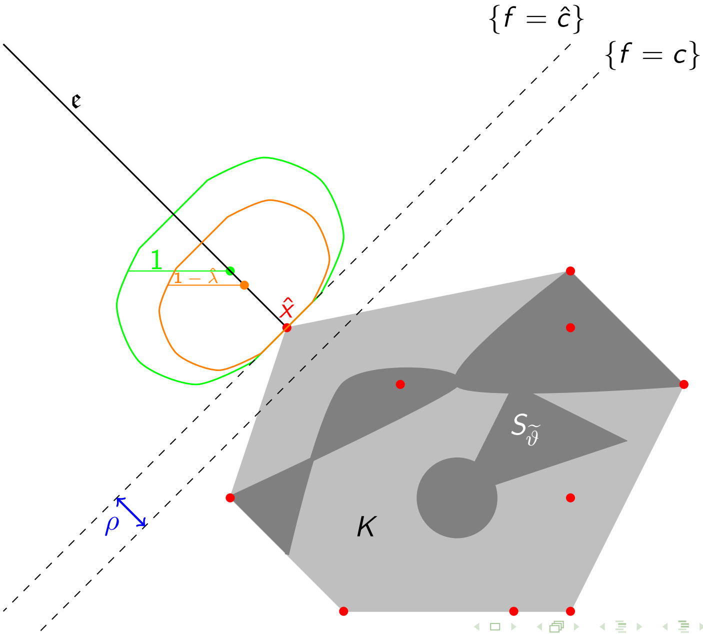

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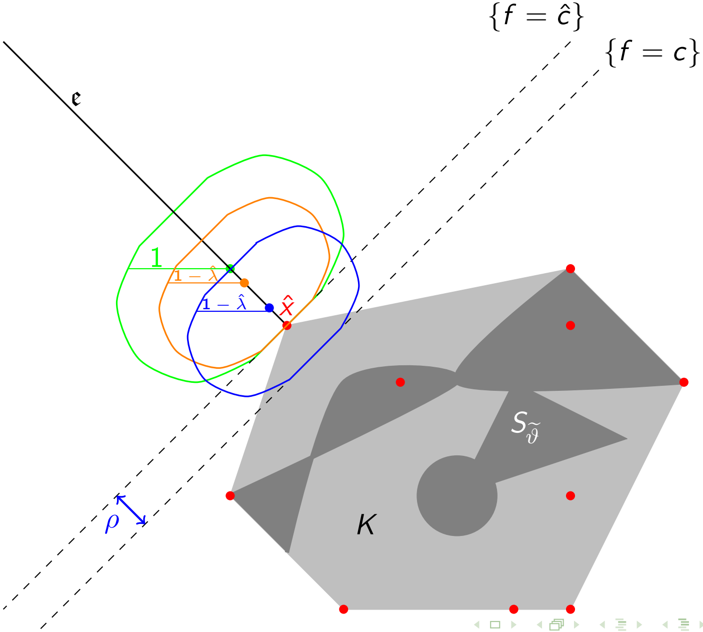

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The proof of the reverse direction is similar. ■

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Corollary (Our aim)

Let $\mu \in \mathcal{F}_X$ be arbitrary. Then we have $S_{\varphi(\mu)} \subseteq K$, moreover, $\hat{x}$ is an isolated atom of $\varphi(\mu)$ with $(\varphi(\mu))(\{\hat{x}\}) = \hat{\lambda}$.

Proof: Since

$$\pi(\delta_x, \mu) = 1 \qquad (x \in X \setminus K^1),$$

the φ-invariance of the witness function gives

$$W_{\varphi(\mu)}(x) = \pi(\delta_x, \varphi(\mu)) = 1 \qquad (x \in X \setminus K^1),$$

and hence we conclude

$$S_{\varphi(\mu)} \cap B_1(x) = \emptyset \qquad (x \in X \setminus K^1).$$

Consequently,

$$S_{\varphi(\mu)} \subseteq X \setminus (X \setminus K^1)^1 \subseteq K$$

and an application of (i) $\iff$ (ii) completes the proof. ■

3. The story beyond vertices

Our aim: Suppose that $m \in \mathbb{N}$ pieces of atoms of a measure $\vartheta \in \mathcal{P}_X$ have been already detected. We would like to get information of the remaining part of ϑ in terms of the Lévy–Prokhorov distances between ϑ and some measures which are supported on at most $m+1$ points.

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For an $s > 0$ we define the s-**Lévy–Prokhorov distance** by

$$\pi_s = \pi_{s,\|\cdot\|} \colon \; \mathcal{P}_X \times \mathcal{P}_X \to [0,1]$$
$$\pi_s(\mu,\nu) := \inf\{\varepsilon > 0 \,|\, \forall\, A \in \mathscr{B}_X \colon s \cdot \mu(A) \leq s \cdot \nu(A^\varepsilon) + \varepsilon\}$$

and the s-**witness function** of $\mu \in \mathcal{P}_X$ by

$$W_{s,\mu} \colon X \to \mathbb{R}, \quad W_{s,\mu}(x) := \pi_s(\delta_x, \mu).$$

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and the s-**witness function** of $\mu \in \mathcal{P}_X$ by

$$W_{s,\mu} \colon X \to \mathbb{R}, \quad W_{s,\mu}(x) := \pi_s(\delta_x, \mu).$$

It is not too hard to see that for every $\mu, \nu \in \mathcal{P}_X$ we have

$$\pi_{s,\|\cdot\|}(\mu,\nu) = s \cdot \pi_{1,\frac{1}{s}\|\cdot\|}(\mu,\nu),$$

in particular π_s is a distance.

Lemma

Let $s > 0$, $\mu \in \mathcal{F}_X \setminus \Delta_X$. Set $K = \mathrm{conv} S_\mu$. Assume that $\hat{x}$ is a vertex of K and set $\hat{\lambda} := \mu(\{\hat{x}\}) \in (0,1)$. Then for every $\vartheta \in \mathcal{P}_X$ with $S_\vartheta \subseteq K$ the following two conditions are equivalent:

(i) $\vartheta = \hat{\lambda} \cdot \delta_{\hat{x}} + (1 - \hat{\lambda}) \cdot \widetilde{\vartheta}$ *where* $\widetilde{\vartheta} \in \mathcal{P}_X$ *with* $S_{\widetilde{\vartheta}} \subseteq K \setminus B_r(\hat{x})$ *for some* $r > 0$,

(ii) *there exist a number* $0 < \rho \leq s(1 - \hat{\lambda})$ *and a half-line* $\mathfrak{e}$ *starting from* $\hat{x}$ *such that* $W_{s,\vartheta}|_{\mathfrak{e}}$ *has the following form:*

$$W_{s,\vartheta}|_{\mathfrak{e}}(x) = \begin{cases} s & \text{if} \quad \|x - \hat{x}\| \geq s, \\ \|x - \hat{x}\| & \text{if} \quad s(1-\hat{\lambda}) < \|x - \hat{x}\| < s, \\ s(1-\hat{\lambda}) & \text{if} \quad s(1-\hat{\lambda}) - \rho \leq \|x - \hat{x}\| \leq \\ & \qquad\qquad \leq s(1-\hat{\lambda}). \end{cases}$$

As a consequence we have that if $\mu \in \mathcal{F}_X, \#S_\mu \leq 2$, $\nu \in \mathcal{P}_X$ and $W_{s,\mu} \equiv W_{s,\nu}$, then $\mu = \nu$.

3. The story beyond vertices

Assume that $m \in \mathbb{N}$ pieces of atoms of $\vartheta \in \mathcal{P}_X$ have been already detected. Let us fix a point $x \in X$. We are interested in $W_{\widetilde{w},\widetilde{\vartheta}}(x)$.

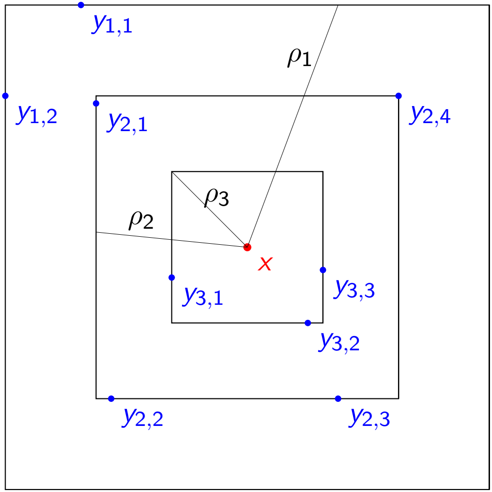

Figure: An example when $X = \mathbb{R}^2$ with the ℓ^∞-norm.

The main lemma (Part 1)

Let $x \in X$ *and* $\{y_{j,l} \,|\, 1 \le j \le k, 1 \le l \le d_j\} \subset X$ *be some pairwise different points such that for every* $1 \le j \le k$ *we have*
$\rho_j := \|x - y_{j,1}\| = \|x - y_{j,l}\| \quad (\forall\, 1 \le l \le d_j)$.
Assume that $\rho_j > \rho_{j+1} > 0 \quad (\forall\, 1 \le j \le k-1)$. *We set*

$$w_{j,l} := \vartheta(\{y_{j,l}\}) > 0 \quad (\forall\, 1 \le j \le k, 1 \le l \le d_j),$$

$$w_j := \sum_{l=1}^{d_j} w_{j,l} = \vartheta(\{y_{j,1}, \dots, y_{j,d_j}\}) \quad (\forall\, 1 \le j \le k),$$

$$\widetilde{w} := 1 - \sum_{j=1}^{k} w_j,$$

$$\eta_r := \sum_{j=1}^{r} \sum_{l=1}^{d_j} w_{j,l} \cdot \delta_{y_{j,l}} + \left(1 - \sum_{j=1}^{r} w_j\right) \cdot \delta_x \in \mathcal{F}_X \quad (\forall\, 0 \le r \le k).$$

The main lemma (Part 2)

Furthermore, denote by $\widetilde{\vartheta} \in \mathcal{P}_X$ the measure which satisfies

$$\vartheta = \sum_{j=1}^{k} \sum_{l=1}^{d_j} w_{j,l} \cdot \delta_{y_{j,l}} + \widetilde{w} \cdot \widetilde{\vartheta}.$$

Then the $\widetilde{w}$-witness function of $\widetilde{\vartheta}$ can be expressed in terms of the Lévy–Prokhorov distances of ϑ and η_r's in the following way:

$$W_{\widetilde{w},\widetilde{\vartheta}}(x) = \begin{cases} \pi(\delta_x, \vartheta) & \text{if } x \text{ is not } (P_1) \\ \pi(\eta_r, \vartheta) & \text{if } x \text{ is } (P_r) \text{ but not } (P_{r+1}) \\ & \text{with some } 1 \le r < k \\ \pi(\eta_k, \vartheta) & \text{if } x \text{ is } (P_k) \end{cases}$$

where for every $1 \le r \le k$ the property (P_r) means

$$\pi(\eta_{r-1}, \vartheta) \le \rho_r. \tag{P_r}$$

4. The action on $\mathcal{F}_X$ and $\mathcal{P}_X$

Let us assume that we showed that $\phi(\mu) = \mu$ was satisfied for every $\mu \in \mathcal{F}_X$, i.e. that $\phi|\mathcal{F}_X$ was the identity map. Since $\mathcal{F}_X$ is dense in $\mathcal{P}_X$ with respect to the Lévy-Prokhorov metric and ϕ is continuous with respect to the Lévy-Prokhorov metric, we could conclude that ϕ is the identity map.

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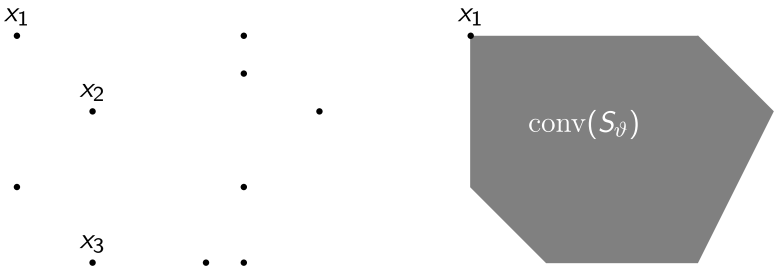
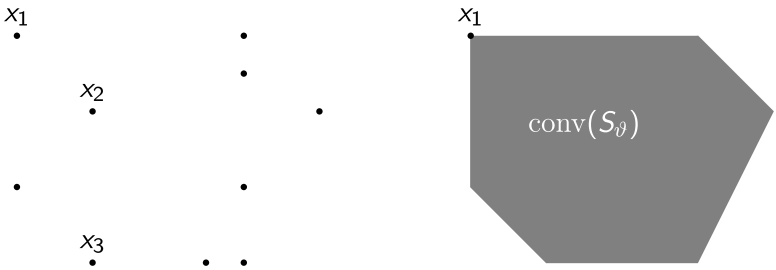
Our aim: We will consider a $\mu \in \mathcal{F}_X$ and show that $\varphi(\mu) = \mu$.

We will use the notation $\vartheta = \varphi(\mu)$.

4. The action on $\mathcal{F}_X$ and $\mathcal{P}_X$



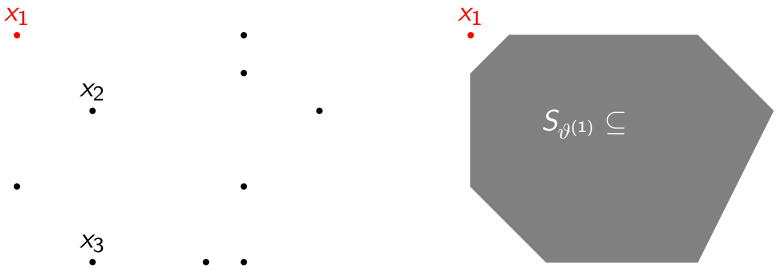
We know what is $\mathrm{conv}(S_\vartheta)$ and that $W_\mu \equiv W_\vartheta$.



We know what is $\mathrm{conv}(S_\vartheta)$ and that $W_\mu \equiv W_\vartheta$.

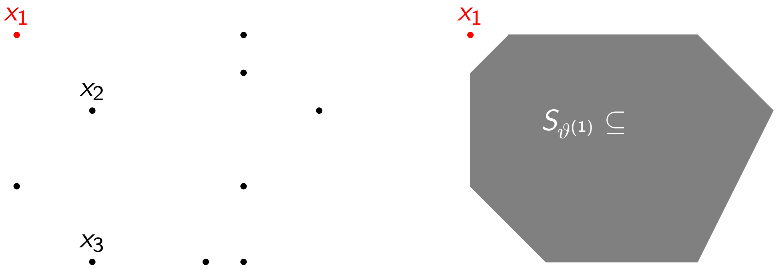
Therefore we can detect an isolated atom of ϑ at x_1 which has the same weight as μ has at x_1.

4. The action on $\mathcal{F}_X$ and $\mathcal{P}_X$



$\vartheta = \vartheta(\{x_1\}) \cdot \delta_{x-1} + (1 - \vartheta(\{x_1\})) \cdot \vartheta^{(1)}$.
We compute the $(1 - \vartheta(\{x_1\}))$–witness function of $\vartheta^{(1)}$.

4. The action on $\mathcal{F}_X$ and $\mathcal{P}_X$



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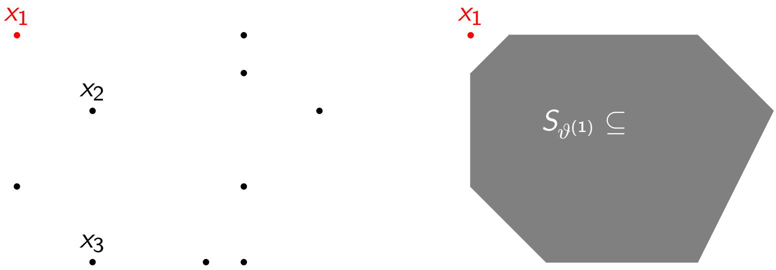
If $\#S_\mu$ was 3, then we are done. Moreover, we infer

$$\varphi(\mu) = \mu \qquad (\mu \in \mathcal{F}_X, \#S_\mu \leq 3).$$

4. The action on $\mathcal{F}_X$ and $\mathcal{P}_X$



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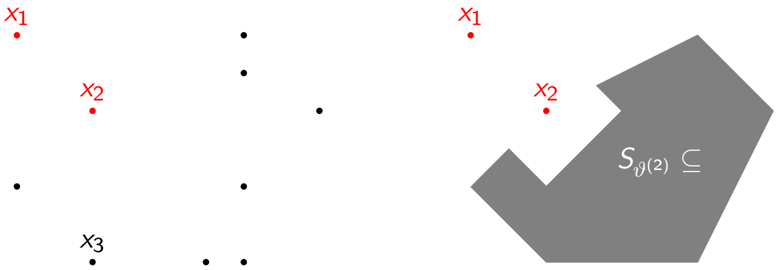
If $\#S_\mu$ was 3, then we are done. Moreover, we infer

$$\varphi(\mu) = \mu \qquad (\mu \in \mathcal{F}_X, \#S_\mu \leq 3).$$

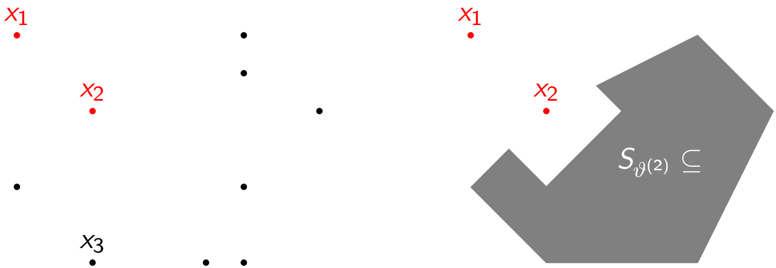
Otherwise, x_2 must be an isolated atom of $\vartheta^{(1)}$, and hence of ϑ.

4. The action on $\mathcal{F}_X$ and $\mathcal{P}_X$



$\vartheta = \vartheta(\{x_1\}) \cdot \delta_{x_1} + \vartheta(\{x_2\}) \cdot \delta_{x_2} + (1 - \vartheta(\{x_1\}) - \vartheta(\{x_2\})) \cdot \vartheta^{(2)}$.
We compute the $(1 - \vartheta(\{x_1\}) - \vartheta(\{x_2\}))$–witness function of $\vartheta^{(2)}$.

4. The action on $\mathcal{F}_X$ and $\mathcal{P}_X$



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If $\#S_\mu$ was 4, then we are done. Moreover, we infer

$$\varphi(\mu) = \mu \qquad (\mu \in \mathcal{F}_X, \#S_\mu \leq 4).$$

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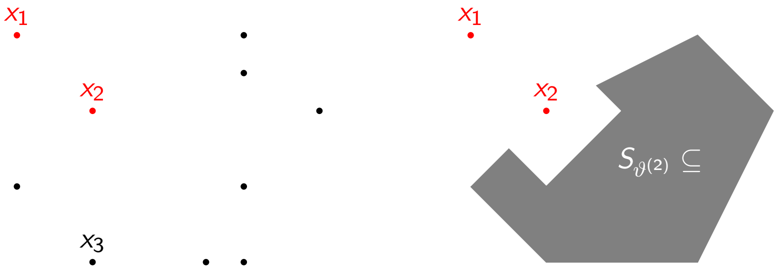

$\vartheta = \vartheta(\{x_1\}) \cdot \delta_{x_1} + \vartheta(\{x_2\}) \cdot \delta_{x_2} + (1 - \vartheta(\{x_1\}) - \vartheta(\{x_2\})) \cdot \vartheta^{(2)}$.
We compute the $(1 - \vartheta(\{x_1\}) - \vartheta(\{x_2\}))$–witness function of $\vartheta^{(2)}$.

If $\#S_\mu$ was 4, then we are done. Moreover, we infer

$$\varphi(\mu) = \mu \qquad (\mu \in \mathcal{F}_X, \#S_\mu \leq 4).$$

Otherwise, x_3 must be an isolated atom of $\vartheta^{(2)}$, and hence of ϑ.
. . . ■

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Thank you for your attention