Symmetry transformations on Grassmann spaces

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Notation and some classical theorems

H	–	complex (or real) Hilbert space
$P_1(H)$	–	the set of all rank-one projections (i.e. projective space)
$P_n(H)$	–	the set of all rank-n projections (i.e. Grassmann space)

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$\measuredangle(P, Q) = \{\vartheta_1, \ldots, \vartheta_n\}$, where $\frac{\pi}{2} \geq \vartheta_1 \geq \vartheta_2 \geq \cdots \geq \vartheta_n \geq 0$, stands for the set of all **principal angles** between the subspaces $\mathrm{Im}\, P$ and $\mathrm{Im}\, Q$ where $\cos\vartheta_1, \ldots \cos\vartheta_n$ are the n largest singular values of PQ.

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If $n = 1$, then we call ϑ_1 simply the angle.

If $\vartheta_1 = \cdots = \vartheta_n = \frac{\pi}{2}$, then we use the notation $P \perp Q$.

We have

$$\sin\vartheta_1 = \|P - Q\| \qquad (P, Q \in P_n(H), n \in \mathbb{N})$$

and

$$\mathrm{Tr}\, PQ = \sum_{j=1}^{n} \cos^2\vartheta_j = n - \|P - Q\|_{HS}^2 \quad (P, Q \in P_n(H), n \in \mathbb{N})$$

It is easy to see that the following three conditions are equivalent for any map $\phi \colon P_1(H) \to P_1(H)$:

$$\begin{aligned} & \operatorname{Tr} \phi(P)\phi(Q) = \operatorname{Tr} PQ \quad (P, Q \in P_1(H)) \\ \iff \quad & \measuredangle(\phi(P), \phi(Q)) = \measuredangle(P, Q) \quad (P, Q \in P_1(H)) \\ \iff \quad & \|\phi(P) - \phi(Q)\| = \|P - Q\| \quad (P, Q \in P_1(H)). \end{aligned}$$

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Theorem (Wigner, 1932)

*Let $\phi\colon P_1(H) \to P_1(H)$ be a **not necessarily bijective** map which satisfies one of the above conditions. Then ϕ is induced by either a linear or a conjugatelinear isometry $V : H \to H$, i.e.*

$$\phi(P) = VPV^* \qquad (P \in P_1(H)).$$

Originally, this theorem was proven for bijective maps.

Theorem (Uhlhorn, 1963, *Ark. Fysik*)

Assume that $\dim H \geq 3$ *and let* $\phi\colon P_1(H) \to P_1(H)$ *be a* ***bijection*** *which satisfies*

$$\phi(P) \perp \phi(Q) \iff P \perp Q \qquad (P, Q \in P_1(H)). \tag{1}$$

Then ϕ *is induced by a unitary or an antyunitary operator* $U\colon H \to H$*, i.e.*

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Instead of assuming (1), one may pose the following condition on ϕ:

$$\measuredangle(\phi(P), \phi(Q)) = \alpha \iff \measuredangle(P, Q) = \alpha \quad (P, Q \in P_1(H)). \tag{2}$$

– The real case for every $\alpha \in (0, \frac{\pi}{2})$, and the complex case for $\alpha \in (0, \frac{\pi}{4}]$ have been recently solved by Li–Plevnik–Šemrl (2012) and G. (2017≤).
– **Open problem**: the complex case for $\alpha \in (\frac{\pi}{4}, \frac{\pi}{2})$.

Various generalisations of Wigner's theorem

Theorem (Molnár, 2001, *Commun. Math. Phys.*)

Let $\dim H > n$ *and* $\phi\colon P_n(H) \to P_n(H)$ *be a* ***not necessarily bijective*** *transformation that satisfies*

$$\measuredangle(\phi(P), \phi(Q)) = \measuredangle(P, Q) \qquad (P, Q \in P_n(H)).$$

Then either ϕ *is induced by a linear or a conjugatelinear isometry* $V\colon H \to H$*, i.e.*

$$\phi(P) = VPV^* \qquad (P \in P_n(H)),$$

or we have $\dim H = 2n > 2$ *and*

$$\phi(P) = I - VPV^* \qquad (P \in P_n(H)).$$

Theorem (Győry, 2004, *Publ. Math. Debrecen*;
Šemrl, 2004, *Illinois J. Math.*;
G.–Šemrl, 2016, *J. Funct. Anal.*)

Let $n \in \mathbb{N}$, $\dim H > 2n$ *and* $\phi\colon P_n(H) \to P_n(H)$ *be a* ***bijective*** *transformation that satisfies*

$$\phi(P) \perp \phi(Q) \iff P \perp Q \qquad (P, Q \in P_n(H)).$$

Then ϕ *is induced by a unitary or an antiunitary operator* $U\colon H \to H$, *i.e.*

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Theorem (Botelho–Jamison–Molnár, 2013, *J. Funct. Anal.*; G.–Šemrl, 2016, *J. Funct. Anal.*)

Let $n \in \mathbb{N}$, $\dim H > n$ *and* $\phi\colon P_n(H) \to P_n(H)$ *be a* ***bijective*** *transformation that satisfies*

$$\|\phi(P) - \phi(Q)\| = \|P - Q\| \qquad (P, Q \in P_n(H)).$$

Then either ϕ *is induced by a unitary or an antiunitary operator* $U\colon H \to H$, *i.e.*

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Conjecture: a similar conclusion should hold for non-bijective maps *in infinite dimension* (but of course with V instead of U!).

Theorem (G., 2017, *J. Funct. Anal.*)

Let $\dim H > n$ *and* $\varphi\colon P_n(H) \to P_n(H)$ *be a* ***not necessarily bijective*** *map which preserves the transition probability, that is*

$$\mathrm{Tr}\, \varphi(P)\varphi(Q) = \mathrm{Tr}\, PQ \qquad (P, Q \in P_n(H)).$$

Then either φ *is induced by a linear or conjugatelinear isometry* $V\colon H \to H$*, i.e.*

$$\varphi(P) = VPV^* \qquad (P \in P_n(H)),$$

or we have $\dim H = 2n > 2$ *and*

$$\varphi(P) = I - VPV^* \qquad (P \in P_n(H)).$$

This theorem describes all isometries with respect to the Hilbert–Schmidt norm on $P_n(H)$.

Open problem: What is the general form of (bijective) isometries on $P_n(H)$ with respect to an arbitrary unitarily invariant norm?

Proof of the $2n$-dimensional case

by utilising Chow's fundamental theorem of the geometry of Grassmannians

H	–	a $2n$-dimensional Hilbert space.
$F_s(H)$	–	the set of all selfadjoint, finite-rank operators.

Lemma (Molnár)

*If φ satisfies the conditions of our Theorem, then it has a unique **real-linear extension** $\Phi\colon F_s(H) \to F_s(H)$ which is injective and satisfies*

$$\operatorname{Tr} \Phi(A)\Phi(B) = \operatorname{Tr} AB \quad (A, B \in F_s(H)).$$

- Φ is a homeomorphism.
- By the domain invariance theorem φ is a **homeomorphism** as well.

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Definition

*We call two rank-n projections P and Q **adjacent** if $\dim(\operatorname{Im} P \cap \operatorname{Im} Q) = n - 1$, or equivalently, if $\operatorname{rank}(P - Q) = 2$, and in this case we use the notation $P \overset{a}{\sim} Q$.*

- $P \overset{a}{\sim} Q \Longrightarrow P \neq Q$.
- $P \overset{a}{\sim} Q \iff \vartheta_1 > 0,\ \vartheta_2 = \cdots = \vartheta_n = 0$.

Theorem (Chow, 1949, *Ann. Math. (2)*)

Let $\dim H = 2n$ *and* $\phi\colon P_n(H) \to P_n(H)$ *be a* ***continuous bijection*** *which preserves adjacency in* ***both directions***, *i.e.*

$$\phi(P) \overset{a}{\sim} \phi(Q) \iff P \overset{a}{\sim} Q \quad (P, Q \in P_n(H)).$$

Then there exists a linear or conjugatelinear bijection $A\colon H \to H$ *such that either*

$$\mathrm{Im}\,\phi(P) = A(\mathrm{Im}\,P) \quad (P \in P_n(H)),$$

or

$$\mathrm{Im}\,\phi(P) = (A(\mathrm{Im}\,P))^{\perp} \quad (P \in P_n(H)).$$

- Chow proved a more general result known today as **Chow's fundamental theorem of the geometry of Grassmannians**.

Definition

We call P and $Q \in P_n(H)$ ***orthogonal adjacent*** *if $P \overset{a}{\sim} Q$ and $\vartheta_1 = \frac{\pi}{2}$. In this case we use the notation $P \overset{\perp a}{\sim} Q$.*

Definition

$P, Q \in P_n(H)$ are said to be ***non-orthogonal adjacent*** *if $P \overset{a}{\sim} Q$ and $\vartheta_1 < \frac{\pi}{2}$. We use the notation $P \overset{\not\perp a}{\sim} Q$.*

Definition

Let $k \in \mathbb{N}$, M be a subspace and $P, Q \in P_k(M)$. We define the set

$$\mathcal{A}^{(\mathbf{k})}_{\mathbf{P},\mathbf{Q}} = \{R \in P_k(M) \colon P + Q - R \in P_k(M)\}$$
$$= \left\{R \in P_k(M) \colon \exists\ T \in P_k(M) \text{ such that } \frac{P+Q}{2} = \frac{R+T}{2}\right\}$$

Main Lemma

Suppose that $P, Q \in P_n(H)$. Then $\mathcal{A}_{P,Q}^{(n)}$ is a one-dimensional (real) manifold if and only if $P \overset{\not{a}}{\sim} Q$.

Assume for a moment that we have proven the above lemma. Then we can utilise it as follows:

- $\varphi(\mathcal{A}_{P,Q}^{(n)}) = \Phi(\mathcal{A}_{P,Q}^{(n)}) = \mathcal{A}_{\Phi(P),\Phi(Q)}^{(n)} = \mathcal{A}_{\varphi(P),\varphi(Q)}^{(n)}$.
- $P \overset{\not{a}}{\sim} Q \iff \mathcal{A}_{P,Q}^{(n)}$ is a one-dimensional manifold $\iff \mathcal{A}_{\varphi(P),\varphi(Q)}^{(n)}$ is a one-dim manifold $\iff \varphi(P) \overset{\not{a}}{\sim} \varphi(Q)$.
- Since rank is lower semicontinuous on $F_s(H)$ and φ is a homeomorphism, we obtain $P \overset{a}{\sim} Q \iff \varphi(P) \overset{a}{\sim} \varphi(Q)$.
- Finally, Chow's theorem completes the proof of the $2n$-dimensional case. ∎

Proof of the Main Lemma

CASE 0, if $P = Q$**:** then $\mathcal{A}^{(n)}_{P,Q} = \{P\}$.

CASE 1, if $P \overset{a}{\sim} Q$**:** Then P and Q can be represented by the following block-matrices where $\mathfrak{p}, \mathfrak{q} \in P_1(\mathbb{C}^2)$:

$$P = \begin{pmatrix} I_{n-1} & 0 & 0 \\ 0 & \mathfrak{p} & 0 \\ 0 & 0 & 0_{n-1} \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} I_{n-1} & 0 & 0 \\ 0 & \mathfrak{q} & 0 \\ 0 & 0 & 0_{n-1} \end{pmatrix}.$$

Suppose that $R \in \mathcal{A}^{(n)}_{P,Q}$ and set $S = P + Q - R \in P_n(H)$.

$$R = \begin{pmatrix} I_{n-1} & 0 & 0 \\ 0 & \mathfrak{r} & 0 \\ 0 & 0 & 0_{n-1} \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} I_{n-1} & 0 & 0 \\ 0 & \mathfrak{s} & 0 \\ 0 & 0 & 0_{n-1} \end{pmatrix},$$

where $\mathfrak{r}, \mathfrak{s} \in P_1(M_2)$, whence we easily conclude the following:

$$\mathcal{A}^{(n)}_{P,Q} = \left\{ \begin{pmatrix} I_{n-1} & 0 & 0 \\ 0 & \mathfrak{t} & 0 \\ 0 & 0 & 0_{n-1} \end{pmatrix} : \mathfrak{t} \in \mathcal{A}^{(1)}_{\mathfrak{p},\mathfrak{q}} \right\}.$$

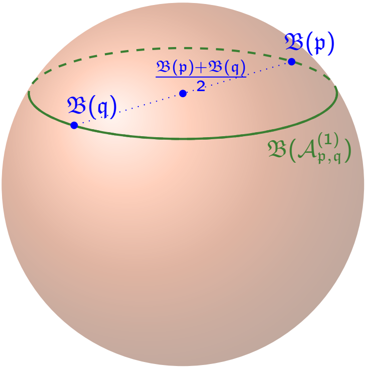
In particular, $\mathcal{A}^{(n)}_{P,Q}$ and $\mathcal{A}^{(1)}_{\mathfrak{p},\mathfrak{q}}$ **are homeomorphic**.

$\mathbb{S}^2$ – the unit sphere of $\mathbb{R}^3$.

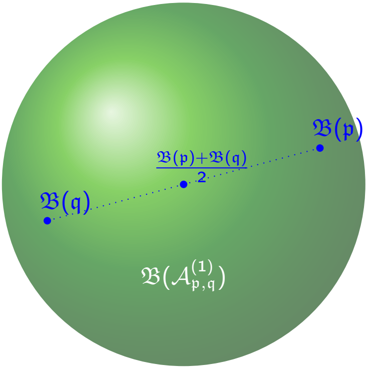
$\mathfrak{B}$ – the Bloch representation of $\operatorname{conv} P_1(\mathbb{C}^2)$ which is **affine**.

By the following observations we conclude this case:

$$\mathfrak{B}(\mathcal{A}^{(1)}_{\mathfrak{p},\mathfrak{q}}) = \left\{ x \in \mathbb{S}^2 : \exists\, y \in \mathbb{S}^2 \text{ s. t. } \frac{\mathfrak{B}(\mathfrak{p})+\mathfrak{B}(\mathfrak{q})}{2} = \frac{x+y}{2} \right\}$$



if $\mathfrak{p} \not\perp \mathfrak{q}$, i.e. $P \overset{\not\perp_a}{\sim} Q$



if $\mathfrak{p} \perp \mathfrak{q}$, i.e. $P \overset{\perp_a}{\sim} Q$

CASE 2, if $P \neq Q$ and $P \overset{a}{\nsim} Q$: We can write

$$P = \begin{pmatrix} \mathfrak{p}_1 & 0 & \dots & 0 \\ 0 & \mathfrak{p}_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathfrak{p}_n \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} \mathfrak{q}_1 & 0 & \dots & 0 \\ 0 & \mathfrak{q}_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathfrak{q}_n \end{pmatrix}$$

where $\mathfrak{p}_j, \mathfrak{q}_j \in P_1(\mathbb{C}^2)$, $j = 1, \dots n$. But observe the following:

$$\left\{ \begin{pmatrix} \mathfrak{t}_1 & 0 & \dots & 0 \\ 0 & \mathfrak{t}_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathfrak{t}_n \end{pmatrix} : \mathfrak{t}_j \in \mathcal{A}^{(1)}_{\mathfrak{p}_j, \mathfrak{q}_j} \right\} \subset \mathcal{A}^{(n)}_{P,Q}.$$

This completes the proof. ■

An important observation

- Clearly, every rank-one projection $\mathfrak{p} \in P_1(H)$ can be expressed as a real-linear combination of $n+1$ rank-n projections, moreover, if this linear combination is $\mathfrak{p} = \sum_{j=1}^{n+1} t_j P_j$, then taking the trace of both sides gives $\sum_{j=1}^{n+1} t_j = \frac{1}{n}$.
- Therefore in case when $\varphi(P) = VPV^*$ $(P \in P_n(H))$, we have

$$\Phi(\mathfrak{p}) = V\mathfrak{p}V^* \qquad (\mathfrak{p} \in P_1(H)),$$

- and in case when $\varphi(P) = I - VPV^*$ $(P \in P_n(H))$, then we have

$$\begin{aligned}\Phi(\mathfrak{p}) &= \sum_{j=1}^{n+1} t_j \Phi(P_j) = V\left(\sum_{j=1}^{n+1} t_j (I_{2n} - P_j)\right) V^* \\ &= \frac{1}{n} I_{2n} - V\mathfrak{p}V^* \qquad (\mathfrak{p} \in P_1(H)),\end{aligned}$$

Proof of the at least $(2n+1)$-dimensional case

Neither φ nor Φ has to be bijective! Nevertheless, they are injective.

- Let $P, Q \in P_n(H)$, $P \perp Q$.
- We also have $\varphi(P) \perp \varphi(Q)$, by the trace preservation.
- For any $R \in P_n(H)$ we have $R \leq P + Q \Longrightarrow P + Q - R \in P_n(H) \Longrightarrow \Phi(P + Q - R) = \varphi(P) + \varphi(Q) - \varphi(R) \in P_n(H) \Longrightarrow \varphi(R) \leq \varphi(P) + \varphi(Q)$.
- We have either

$$\Phi(\mathfrak{p}) \in P_1(H) \quad (\mathfrak{p} \in P_1(H), \mathfrak{p} \leq P + Q),$$

  or

$$\mathrm{Im}\, \Phi(\mathfrak{p}) = \mathrm{Im}\, \varphi(P) \oplus \mathrm{Im}\, \varphi(Q) \quad (\mathfrak{p} \in P_1(H), \mathfrak{p} \leq P + Q).$$

- Assume for a moment that we have the latter case. Then if we replace Q by another $Q' \in P_n(H)$ such that $Q' \perp P$, we obtain

$$\mathrm{Im}\, \Phi(\mathfrak{p}) = \mathrm{Im}\, \varphi(P) \oplus \mathrm{Im}\, \varphi(Q') \quad (\mathfrak{p} \in P_1(H), \mathfrak{p} \leq P + Q'),$$

  and considering a $\mathfrak{p} \leq P$ implies $\mathrm{Im}\, \varphi(Q) = \mathrm{Im}\, \varphi(Q')$, a contradiction.
- Therefore we have $\Phi(P_1(H)) \subseteq P_1(H)$ and the optimal version of Wigner's theorem completes the proof. $\blacksquare$

F. Botelho, J. Jamison and L. Molnár, Surjective isometries on Grassmann spaces, *J. Funct. Anal.* **265** (2013), 2226–2238.

W.-L. Chow, On the geometry of algebraic homogeneous spaces, *Ann. Math. (2)* **50** (1949), 32-67.

Gy.P. Gehér, Symmetries of projective spaces and spheres, *submitted*

Gy.P. Gehér, Wigner's theorem on Grassmann spaces, *J. Funct. Anal.* (2017), to appear.

Gy.P. Gehér and P. Šemrl, Isometries of Grassmann spaces, *J. Funct. Anal.* **270** (2016), 1585–1601.

M. Győry, Transformations on the set of all n-dimensional subspaces of a Hilbert space preserving orthogonality, *Publ. Math. Debrecen* **65** (2004), 233-242.

C.-K. Li, L. Plevnik, P. Šemrl, Preservers of matrix pairs with a fixed inner product value, *Oper. Matrices* **6** (2012), 433–464.

- L. Molnár, Maps on the n-dimensional subspaces of a Hilbert space preserving principal angles, *Proc. Amer. Math. Soc.* **136** (2008), 3205–3209.
- L. Molnár, Transformations on the set of all n-dimensional subspaces of a Hilbert space preserving principal angles, *Commun. Math. Phys.* **217** (2001), 409–421.
- P. Šemrl, Orthogonality preserving transformations on the set of n-dimensional subspaces of a Hilbert space, *Illinois J. Math.* **48** (2004), 567–573.
- U. Uhlhorn, Representation of symmetry transformations in quantum mechanics, *Ark. Fysik* **23** (1963), 307–340.
- E.P. Wigner, *Gruppentheorie und ihre Anwendung auf die Quantenmechanik der Atomspektrum*, Fredrik Vieweg und Sohn, 1931.

Thank you for your attention