Symmetries on quantum pure states

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- H – complex or real Hilbert space,
- $\|\cdot\|$ – vector-norm
- $P(H)$ – projective space over H, i.e. the space of all **lines** (i.e. one-dimensional subspaces).
- If $0 \neq x \in H$, then let $[x]$ be the generated line.
- The **angle** between lines $[u], [v] \in P(H)$:

$$\measuredangle([u],[v]) := \arccos \frac{|\langle u, v\rangle|}{\|u\| \cdot \|v\|} \in \left[0, \tfrac{\pi}{2}\right].$$

Proposition

*The $(P(H), \measuredangle)$ is a **metric space**.*

Definition

If $U\colon H \to H$ is a bijective linear or conjugate linear isometry, then the bijective transformation

$$\phi_U\colon P(H) \to P(H), \quad \phi_U([v]) = [Uv]$$

*is called a **Wigner symmetry**.*

Clearly, every Wigner symmetry **preserves the angles between lines**:

$$\measuredangle(\phi_U([u]), \phi_U([v])) = \measuredangle([Uu], [Uv]) = \measuredangle([u], [v])\,.$$

Theorem (Wigner, 1932)

Let $\phi \colon P(H) \to P(H)$ be a bijective map which satisfies

$$\measuredangle([u],[v]) = \measuredangle(\phi([u]),\phi([v])) \quad ([u],[v] \in P(H)).$$

Then ϕ is a Wigner symmetry.

$P(H)$ = the set of **quantum pure states**,
$\cos^2(\measuredangle([u],[v]))$ = the **transition probability**.

Symmetries of $\measuredangle$

=

Symmetries of the transition probability

=

Wigner symmetries.

We usually write $[u] \perp [v]$ instead of $\measuredangle([u],[v]) = \frac{\pi}{2}$.

Theorem (Uhlhorn, 1963)

Let $\dim H \geq 3$ *and* $\phi\colon P(H) \to P(H)$ *be a bijective transformation such that we have*

$$[u] \perp [v] \iff \phi([u]) \perp \phi([v]) \quad ([u],[v] \in P(H)).$$

Then ϕ *is a Wigner symmetry.*

If $\dim H \geq 3$, then:

Symmetries of $\perp$

$=$

Symmetries of zero transition probability

$=$

Wigner symmetries.

Several generalizations of Wigner's theorem have been provided on

- on projective spaces
- on Grassmann spaces (e.g. on the space of all n-dimensional subspaces)
- on classes of idempotent operators
- ...

Some of the main contributors: D.F. Almeida, F. Botelho, G. Chevalier, M. Győry, J.E. Jamison, L. Molnár, J. Rätz, L. Rodman, P. Šemrl, C.S. Sharma ...

The core problem of this talk

Let us **fix** an angle $0 < \alpha < \frac{\pi}{2}$.

$$\text{Symmetries of the angle } \alpha$$
$$=$$
$$\text{Symmetries of the transition probability } \cos^2 \alpha$$

The core problem of this talk

Let us **fix** an angle $0 < \alpha < \frac{\pi}{2}$.

$$\text{Symmetries of the angle } \alpha$$
$$=$$
$$\text{Symmetries of the transition probability } \cos^2 \alpha$$
$$\supseteq \quad \textbf{???=???}$$
$$\text{Wigner symmetries}$$

If $H = \mathbb{R}^2$, then this is not true! There are wild maps which preserve the angle α in both directions.

But what about the **pure qubit** case, i.e. when $H = \mathbb{C}^2$?

The answer for real Hilbert spaces

Theorem (C.-K. Li, L. Plevnik, P. Šemrl, 2012)

Let H be a real Hilbert space with $\mathbf{5 \leq \dim H < \infty}$ *and* $\mathbf{0 < \alpha < \frac{\pi}{4}}$. *If $\phi\colon P(H) \to P(H)$ is a bijection such that*

$$\measuredangle([u],[v]) = \alpha \iff \measuredangle(\phi([u]),\phi([v])) = \alpha \quad ([u],[v] \in P(H)),$$

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Theorem (G., 2016)

Let H be a real Hilbert space with $\mathbf{3 \leq \dim H}$ and $\mathbf{0 < \alpha < \frac{\pi}{2}}$. If $\phi\colon P(H) \to P(H)$ is a bijection which satisfies

$$\measuredangle([u],[v]) = \alpha \iff \measuredangle(\phi([u]),\phi([v])) = \alpha \quad ([u],[v] \in P(H)),$$

then ϕ is a Wigner symmetry.

The answer for complex Hilbert spaces 1/2

There were no results about the complex case.

Theorem (G., 2016)

Let H be a complex Hilbert space, $3 \leq \dim \boldsymbol{H}$ *and* $0 < \alpha \leq \frac{\pi}{4}$. *Assume that* $\phi \colon P(H) \to P(H)$ *is a bijective map such that*

$$\measuredangle([u],[v]) = \alpha \iff \measuredangle(\phi([u]),\phi([v])) = \alpha \quad ([u],[v] \in P(H))$$

holds. Then ϕ *is a Wigner symmetry.*

The answer for complex Hilbert spaces 1/2

There were no results about the complex case.

Theorem (G., 2016)

Let H be a complex Hilbert space, $3 \leq \dim H$ and $0 < \alpha \leq \frac{\pi}{4}$. Assume that $\phi \colon P(H) \to P(H)$ is a bijective map such that

$$\measuredangle([u],[v]) = \alpha \iff \measuredangle(\phi([u]),\phi([v])) = \alpha \quad ([u],[v] \in P(H))$$

holds. Then ϕ is a Wigner symmetry.

Open problem: What happens if $\alpha \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$?

The answer for complex Hilbert spaces 2/2

Theorem (G., 2016)

Assume that $0 < \alpha < \frac{\pi}{2}$ *and that* $\phi\colon P(\mathbb{C}^2) \to P(\mathbb{C}^2)$ *is a bijective map such that*

$$\measuredangle([u],[v]) = \alpha \iff \measuredangle(\phi([u]),\phi([v])) = \alpha \quad ([u],[v] \in P(\mathbb{C}^2)).$$

Then

(i) *in the* $\alpha \neq \frac{\pi}{4}$ *case* ϕ *is a Wigner symmetry,*

(ii) *in the* $\alpha = \frac{\pi}{4}$ *case there exists a unitary or antiunitary operator* $U\colon \mathbb{C}^2 \to \mathbb{C}^2$ *such that*

$$\phi([v]) \in \left\{[Uv],[Uv]^\perp\right\} \quad ([v] \in P(\mathbb{C}^2)).$$

Ideas in the real case

Lemma

Let $\dim H \geq 3$*, and* $\phi\colon P(H) \to P(H)$ *be a bijection. If there is a sequence of positive angles* $\{\alpha_n\}_{n=1}^{\infty} \subset (0, \frac{\pi}{2})$ *such that* $\lim_{n\to\infty} \alpha_n = 0$ *and* ϕ *preserves these angles in both directions, then* ϕ *is a Wigner symmetry.*

Lemma

Let $\dim H \geq 3$, and $\phi\colon P(H) \to P(H)$ be a bijection. If there is a sequence of positive angles $\{\alpha_n\}_{n=1}^{\infty} \subset (0, \frac{\pi}{2})$ such that $\lim_{n\to\infty} \alpha_n = 0$ and ϕ preserves these angles in both directions, then ϕ is a Wigner symmetry.

The set of unit vectors:

$$S_H := \{x \in H\colon \|x\| = 1\}.$$

Angle between two unit vectors $x, y \in S_H$:

$$\sphericalangle(x, y) = \arccos\langle x, y\rangle \in [0, \pi].$$

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For any $[v] \in P(H)$ we write

$$[v]^{\alpha} := \{[w] \in P(H) \colon \measuredangle([v], [w]) = \alpha\}$$

Let $v \in S_H$ and $w \in S_H$ such that $\phi([v]) = [w]$. We have

$$\phi([v]^\alpha) = \phi([v])^\alpha = [w]^\alpha,$$

$$[v]^\alpha = \{[\cos\alpha \cdot v + \sin\alpha \cdot x] \colon x \in S_{H\ominus[v]}\} \ \equiv \ S_{H\ominus[v]}$$

and

$$[w]^\alpha = \{[\cos\alpha \cdot w + \sin\alpha \cdot u] \colon u \in S_{H\ominus[w]}\} \ \equiv \ S_{H\ominus[w]}.$$

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and

$$[w]^\alpha = \{[\cos\alpha \cdot w + \sin\alpha \cdot u] \colon u \in S_{H\ominus[w]}\} \;\equiv\; S_{H\ominus[w]}.$$

Therefore we have a bijective mapping

$$\psi \colon S_{H\ominus[v]} \to S_{H\ominus[w]}$$

which **implements** the restriction $\phi|_{[v]^\alpha}$

$$\phi([\cos\alpha \cdot v + \sin\alpha \cdot \boldsymbol{x}]) = [\cos\alpha \cdot w + \sin\alpha \cdot \psi(\boldsymbol{x})] \quad (\boldsymbol{x} \in S_{H\ominus[v]}).$$

Let $v \in S_H$ and $w \in S_H$ such that $\phi([v]) = [w]$. We have

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Throughout the proof we have to distinguish 6 different cases!

Case 1: when $\dim H \geq 4$ and $0 < \alpha < \frac{\pi}{3}$.

For every $x, y \in S_{H \ominus [v]}$ we have

$$\measuredangle([\cos\alpha \cdot v + \sin\alpha \cdot x], [\cos\alpha \cdot v + \sin\alpha \cdot y])$$
$$= \arccos\left(\left|\cos^2\alpha + \langle x, y\rangle \sin^2\alpha\right|\right).$$

The inequality $-\cos\alpha < \cos 2\alpha$ implies

$$\measuredangle([\cos\alpha \cdot v + \sin\alpha \cdot x], [\cos\alpha \cdot v + \sin\alpha \cdot y]) = \alpha$$
$$\Updownarrow$$
$$\sphericalangle(x, y) = \arccos\left(\frac{\cos\alpha}{1+\cos\alpha}\right) \in \left(\tfrac{\pi}{3}, \arccos\tfrac{1}{3}\right) \subset \left(\tfrac{\pi}{3}, \tfrac{\pi}{2}\right).$$

Therefore $\psi\colon S_{H\ominus[v]} \to S_{H\ominus[w]}$ preserves the above angle between unit vectors in both directions:

$$\measuredangle(x, y) = \arccos\left(\tfrac{\cos\alpha}{1+\cos\alpha}\right) \iff \measuredangle(\psi(x), \psi(y)) = \arccos\left(\tfrac{\cos\alpha}{1+\cos\alpha}\right)$$

Case 1: when $\dim H \geq 4$ and $0 < \alpha < \frac{\pi}{3}$.

This suggests the investigation of angles preservers on unit spheres.

Theorem (U. Everling, 1995)

If $0 < \alpha < \frac{\pi}{2}$, $3 \leq \dim H < \infty$ *and* $\psi\colon S_H \to S_H$ *is a* ***(general) transformation*** *such that*

$$\measuredangle(x, y) = \alpha \Longrightarrow \measuredangle(\psi(x), \psi(y)) = \alpha \quad (x, y \in S_H),$$

then there exists an orhtogonal transformation $R\colon H \to H$ *such that*

$$\psi(x) = Rx \quad (x \in S_H),$$

consequently we have

$$\measuredangle(\psi(x), \psi(y)) = \measuredangle(x, y) \quad (x, y \in S_H).$$

Case 1: when $\dim H \geq 4$ and $0 < \alpha < \frac{\pi}{3}$.

Theorem (G., 2016)

Let $\mathbf{3 \leq \dim H}$ *and* $\mathbf{0 < \alpha < \pi}$*. Assume that* $\psi\colon S_H \to S_H$ *is a* ***bijective transformation*** *such that*

$$\sphericalangle(x, y) = \alpha \iff \sphericalangle(\psi(x), \psi(y)) = \alpha \quad (x, y \in S_H).$$

Then there exists a bijective linear isometry $R\colon H \to H$ *such that*

(i) *if* $\alpha \neq \frac{\pi}{2}$*, then we have*

$$\psi(x) = Rx \quad (x \in S_H),$$

(ii) *if* $\alpha = \frac{\pi}{2}$*, then we have*

$$\psi(x) \in \{-Rx, Rx\} \quad (x \in S_H).$$

Case 1: when $\dim H \geq 4$ and $0 < \alpha < \frac{\pi}{3}$.

Since $0 < \arccos\left(\frac{\cos\alpha}{1+\cos\alpha}\right) < \frac{\pi}{2}$, the map ψ preserves every angle. Therefore $\phi|_{[v]^\alpha}$ preserves every angle $0 < \beta \leq \alpha$ in both directions. But this is true for every $[v] \in P(H)$, whence we get that

$$\phi \text{ preserves every angle } 0 < \beta \leq \alpha \text{ in both directions.}$$

By the before mentioned Lemma we obtain that ϕ is indeed a Wigner symmetry.

■

Case 2: when $\dim H \geq 4$ and $\alpha = \frac{\pi}{3}$.

For any $x \in S_{H\ominus[v]}$ we use the notation

$$x^{(\gamma)} = \{z \in S_{H\ominus[v]} \colon \sphericalangle(x, z) = \gamma\}$$

Let $\beta = \arccos\left(\frac{1}{3}\right)$. We get the following for $\psi \colon S_{H\ominus[v]} \to S_{H\ominus[w]}$:

$$\sphericalangle(x, y) \in \{\beta, \pi\} \iff \sphericalangle(\psi(x), \psi(y)) \in \{\beta, \pi\}\,.$$

Case 2: when $\dim H \geq 4$ and $\alpha = \frac{\pi}{3}$.

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$$\sphericalangle(x, y) \in \{\beta, \pi\} \iff \sphericalangle(\psi(x), \psi(y)) \in \{\beta, \pi\}\,.$$

Since

$$\left(x^{(\beta)} \cup x^{(\pi)}\right) \cap \left(y^{(\beta)} \cup y^{(\pi)}\right) = \emptyset \text{ and } y \in x^{(\beta)} \cup x^{(\pi)} \iff y = -x,$$

we obtain that ψ preserves the angle π in both directions, and thus β in both directions. Therefore ψ preserves every angle.

∎

Case 3: when $\dim H \geq 4$ and $\frac{\pi}{3} < \alpha < \frac{\pi}{2}$.

Let

$$\beta_1 = \arccos\left(1 - \frac{1}{1+\cos\alpha}\right) \in \left(\arccos\tfrac{1}{3}, \tfrac{\pi}{2}\right)$$

and

$$\beta_2 = \arccos\left(1 - \frac{1}{1-\cos\alpha}\right) \in \left(\tfrac{\pi}{2}, \pi\right).$$

Similarly as before, we obtain that

$$\sphericalangle(x,y) \in \{\beta_1, \beta_2\} \iff \sphericalangle(\psi(x), \psi(y)) \in \{\beta_1, \beta_2\}.$$

We would like to show that there is an angle which is preserved by ψ in both directions.

Case 3: when $\dim H \geq 4$ and $\frac{\pi}{3} < \alpha < \frac{\pi}{2}$.

By investigating the cardinality

$$\#\left((x^{(\beta_1)} \cup x^{(\beta_2)}) \cap (y^{(\beta_1)} \cup y^{(\beta_2)})\right)$$

by a quite long computation we can show that ψ preserves either the angle $2\beta_1$ or $2\pi - \beta_1 - \beta_2$ in both directions, depending on what is the exact value of α. Then we can apply the improved version of Everling's theorem.

■

The remaining three cases deal with $P(\mathbb{R}^3)$.

The idea in $P(\mathbb{C}^2)$

Bloch's representation:

$$\rho\colon P(\mathbb{C}^2) \to S_{\mathbb{R}^3},$$

$$\rho\left([(\cos\theta, e^{i\nu}\sin\theta)]\right) = (\sin 2\theta\cos\nu, \sin 2\theta\sin\nu, \cos 2\theta).$$

Moreover, we have

$$\sphericalangle\left(\rho([u]), \rho([v])\right) = 2\cdot\measuredangle([u],[v]) \quad ([u],[v]\in P(H)).$$

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Moreover, we have

$$\sphericalangle\left(\rho([u]), \rho([v])\right) = 2\cdot\measuredangle([u],[v]) \quad ([u],[v]\in P(H)).$$

Therefore we easily obtain that

$$\rho\circ\phi\circ\rho^{-1}\colon S_{\mathbb{R}^3}\to S_{\mathbb{R}^3}$$

preserves the angle 2α in both directions.

- If $\alpha\neq\frac{\pi}{4}$, then $\rho\circ\phi\circ\rho^{-1}$ preserves every angle and thus ϕ is a Wigner symmetry.
- If $\alpha=\frac{\pi}{4}$, then for every $[u],[v]\in P(\mathbb{C}^2)$ we obtain

$$\measuredangle(\phi([u]),\phi([v]))\in\left\{\measuredangle([u],[v]), \tfrac{\pi}{2}-\measuredangle([u],[v])\right\}.$$

Ideas in the general complex case

(i.e. when $\dim H \geq 3$)

The following is a characterization of some angles in terms of α and β:

Lemma

Assume that $0 < \alpha \leq \beta \leq \frac{\pi}{2}$, *and* $v, w \in H$ *with* $\|v\| = \|w\| = 1$ *and* $\gamma := \measuredangle([v],[w]) \neq 0$. *Then we have*

$$[v]^\alpha \cap [w]^\beta = \emptyset \iff \gamma < \beta - \alpha \text{ or } \gamma > \alpha + \beta,$$

$$\#([v]^\alpha \cap [w]^\beta) = 1 \iff \begin{array}{c} \gamma = \beta - \alpha > 0, \text{ or } \gamma = \alpha + \beta < \frac{\pi}{2}, \\ \text{or } \dim H = 3 \text{ and } \alpha = \beta = \frac{\pi}{2}. \end{array}$$

Moreover, if $\gamma = \frac{\pi}{2} = \alpha + \beta$, *then* $[v]^\alpha \cap [w]^\beta = \{[\cos\alpha \cdot v + \lambda \sin\alpha \cdot w] \colon \lambda \in \mathbb{T}\}$.

Lemma

Assume that $0 < \alpha < \beta < \frac{\pi}{2}$, *and* $\phi\colon P(H) \to P(H)$ *is a bijection. If* ϕ *preserves the angles* α *and* β *in both directions, then*

(i) ϕ *also preserves the angles* $\beta - \alpha$ *and* $\alpha + \beta$ *in both directions if* $\alpha + \beta < \frac{\pi}{2}$,

(ii) ϕ *also preserves the angle* $\beta - \alpha$ *in both directions if* $\alpha + \beta \geq \frac{\pi}{2}$,

Lemma

Let $\phi\colon P(H) \to P(H)$ *be a bijection. If* ϕ *preserves the angle* $\alpha \in (0, \frac{\pi}{4})$ *in both directions, then* ϕ *shares the following property:*

$$\measuredangle([v],[w]) = j\alpha \iff \measuredangle(\phi([v]),\phi([w])) = j\alpha$$

$$\left([v],[w] \in P(H),\ 2 \leq j < \frac{\pi}{2\alpha}\right).$$

Lemma

Let $\phi\colon P(H) \to P(H)$ be a bijection. If there is a sequence of positive angles $\{\alpha_n\}_{n=1}^{\infty} \subset (0, \frac{\pi}{2})$ such that $\lim_{n\to\infty} \alpha_n = 0$ and ϕ preserves these angles in both directions, then ϕ is a Wigner symmetry.

Lemma

Let $0 < \alpha < \frac{\pi}{4}$, $[v], [w] \in P(H)$ *be two different lines with* $\measuredangle([v], [w]) =: \gamma \in (0, 2\alpha)$. *Then*

$$\mathrm{diam}_{\measuredangle}\left([v]^{\alpha} \cap [w]^{\alpha}\right) = 2 \cdot \arccos \sqrt{\frac{\cos^2 \alpha - \sin^2\left(\frac{\gamma}{2}\right)}{\cos^2\left(\frac{\gamma}{2}\right) - \sin^2\left(\frac{\gamma}{2}\right)}}.$$

Lemma

Let $0 < \alpha < \frac{\pi}{4}$, $[v], [w] \in P(H)$ *be two different lines with* $\measuredangle([v],[w]) =: \gamma \in (0, 2\alpha)$. *Then*

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Moreover,

- *there exists exactly one pair of lines in* $[v]^\alpha \cap [w]^\alpha$ *for which the angle is exactly this diameter;*
- *if* $0 < \beta < \mathrm{diam}_{\measuredangle}\left([v]^\alpha \cap [w]^\alpha\right)$, *then there exists three different lines* $[u_0], [u_1], [u_2] \in [v]^\alpha \cap [w]^\alpha$ *with* $\measuredangle([u_0],[u_1]) = \measuredangle([u_0],[u_2]) = \beta$.

We point out that for $\alpha = \frac{\pi}{4}$ we have

$$\operatorname{diam}_{\measuredangle}\left([v]^\alpha \cap [w]^\alpha\right) = \operatorname{diam}_{\measuredangle}\left([v]^\alpha \cap [w]^\alpha \cap P_{[v],[w]}\right) = \frac{\pi}{2}$$

for any two different lines $[v], [w] \in P(H)$, where

$$P_{[v],[w]} = \left\{[u] \in P(H) \colon u \in S_{[v]+[w]}\right\}.$$

This can be verified by using Bloch's representation.

Case 1: when $0 < \alpha < \frac{\pi}{4}$.

The following equation has a unique solution γ_0:

$$\alpha = 2 \cdot \arccos \sqrt{\frac{\cos^2 \alpha - \sin^2\left(\frac{\gamma}{2}\right)}{\cos^2\left(\frac{\gamma}{2}\right) - \sin^2\left(\frac{\gamma}{2}\right)}}$$

Actually, it can be shown that $\alpha < \gamma_0 < 2\alpha$.

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Actually, it can be shown that $\alpha < \gamma_0 < 2\alpha$.
Next, by the previous lemma we infer

$$\measuredangle([v],[w]) = \gamma_0 \iff \mathrm{diam}_{\measuredangle}\left([v]^\alpha \cap [w]^\alpha\right) = \alpha$$

$$\iff \#\left([u]^\alpha \cap [v]^\alpha \cap [w]^\alpha\right) \leq 1 \ \ (\forall\, [u] \in [v]^\alpha \cap [w]^\alpha)$$

$$\iff \#\left([x]^\alpha \cap \phi([v])^\alpha \cap \phi([w])^\alpha\right) \leq 1 \ \ (\forall\, [x] \in \phi([v])^\alpha \cap \phi([w])^\alpha)$$

$$\iff \mathrm{diam}_{\measuredangle}\left(\phi([v])^\alpha \cap \phi([w])^\alpha\right) = \alpha \iff \measuredangle(\phi([v]),\phi([w])) = \gamma_0,$$

i.e. ϕ preserves the angle γ_0 in both directions.

Case 1: when $0 < \alpha < \frac{\pi}{4}$.

But ϕ also preserves the angle 2α in both directions. Therefore, it also preserves the angles $\gamma_0 - \alpha$ and $2\alpha - \gamma_0$ in both directions. Now, let $\alpha_1 := \alpha$ and $\alpha_2 := \min(\gamma_0 - \alpha, 2\alpha - \gamma_0)$. Clearly, we have $0 < \alpha_2 \leq \frac{\alpha_1}{2}$. A rather straightforward induction gives us a sequence of angles $\{\alpha_n\}_{n=1}^{\infty}$ with $0 < \alpha_n \leq \frac{\alpha_{n-1}}{2}$ $(n \geq 2)$ such that all of them is preserved in both directions. Therefore ϕ is a Wigner symmetry.

∎

Case 2: when $\alpha = \frac{\pi}{4}$.

If $[v] \perp [w]$, then

$$[v]^{\pi/4} \cap [w]^{\pi/4} = \left\{ \left[\sqrt{\tfrac{1}{2}} \cdot v + \lambda \sqrt{\tfrac{1}{2}} \cdot w \right] : \lambda \in \mathbb{T} \right\}.$$

For arbitrary $\lambda, \mu \in \mathbb{T}$ we compute the following:

$$\measuredangle \left(\left[\sqrt{\tfrac{1}{2}} \cdot v + \lambda \sqrt{\tfrac{1}{2}} \cdot w \right], \left[\sqrt{\tfrac{1}{2}} \cdot v + \mu \sqrt{\tfrac{1}{2}} \cdot w \right] \right) = \arccos \frac{|1 + \lambda \overline{\mu}|}{2}.$$

Therefore we conclude

$$\# \left([u]^{\pi/4} \cap [v]^{\pi/4} \cap [w]^{\pi/4} \right) = 2 \qquad \left([u] \in [v]^{\pi/4} \cap [w]^{\pi/4} \right).$$

Case 2: when $\alpha = \frac{\pi}{4}$.

Actually, we can show that

$$[v] \perp [w]$$

$$\Updownarrow$$

$$\#\left([u]^{\pi/4} \cap [v]^{\pi/4} \cap [w]^{\pi/4}\right) = 2 \qquad \left([u] \in [v]^{\pi/4} \cap [w]^{\pi/4}\right),$$

thus we can show that ϕ preserves orthogonality in both directions. Therefore by Uhlhorn's theorem, ϕ has to be a Wigner symmetry.

■

Final Remarks

- Every theorem remains true in non-complete inner product spaces as well. But we need Uhlhorn's theorem in this more general case (surprisingly, this was not published before).
- **Open question**: What happens in the general complex case, when $\frac{\pi}{4} < \alpha < \frac{\pi}{2}$? Conjecture: They are all Wigner symmetries.
- **Open question**: What can be ψ in Everling's theorem if we assume that $\frac{\pi}{2} < \alpha < \pi$?
- **Open question**: If $\dim H < \infty$ can we relax the condition of bijectivity of ϕ? What happens if we assume only that $\measuredangle([v],[w]) = \alpha \implies \measuredangle(\phi([v]),\phi([w])) = \alpha$? (Optimal version in the finite dimensional case?)

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Thank you for your kind attention

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''Preservers Everywhere''

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