Wigner's theorem on quantum mechanical symmetry transformations

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Introduction

- H – a complex (or real) Hilbert space
- $\mathcal{P}_1 := \mathcal{P}_1(\mathcal{H})$ – the set of all rank-one projections
- $\mathsf{P}[\vec{u}]$ – the rank-one projection with range $\mathbb{C} \cdot \vec{u}$

(where $\vec{u} \in H$, $\|\vec{u}\| = 1$)

$$\mathcal{P}_1(\mathcal{H}) \ni \mathsf{P}[\vec{u}] \quad \longleftrightarrow \quad \mathbb{C} \cdot \vec{u} \in \mathrm{Lat}_1(\mathcal{H})$$

- $\mathrm{Tr}\, \mathsf{P}[\vec{u}]\mathsf{P}[\vec{v}]$ – transition probability

An easy calculation gives the following:

$$\begin{aligned} \mathrm{Tr}\, \mathsf{P}[\vec{u}]\mathsf{P}[\vec{v}] &= |\langle \vec{u}, \vec{v} \rangle|^2 \\ &= 1 - \|\mathsf{P}[\vec{u}] - \mathsf{P}[\vec{v}]\|^2 \qquad (\|\vec{u}\| = \|\vec{v}\| = 1) \end{aligned}$$

Theorem (E.P. Wigner, 1932(?!); 1962-1964)

Let $\varphi\colon \mathcal{P}_1(\mathcal{H}) \to \mathcal{P}_1(\mathcal{H})$ *be a map such that*

$$\mathrm{Tr}\left(\mathbf{P}[\vec{u}]\cdot\mathbf{P}[\vec{v}]\right) = \mathrm{Tr}\left(\varphi(\mathbf{P}[\vec{u}])\cdot\varphi(\mathbf{P}[\vec{v}])\right) \qquad (\|\vec{u}\| = \|\vec{v}\| = 1). \tag{Wtr}$$

Then there is a linear or antilinear isometry $\mathbf{W}\colon \mathcal{H} \to \mathcal{H}$ *such that*

$$\varphi(\mathbf{P}[\vec{u}]) = \mathbf{W}\cdot\mathbf{P}[\vec{u}]\cdot\mathbf{W}^* = \mathbf{P}[\mathbf{W}\vec{u}] \qquad (\|\vec{u}\| = 1).$$

- In fact, (Wtr) is an **isometriness** condition:

$$\|\mathbf{P}[\vec{u}] - \mathbf{P}[\vec{v}]\| = \|\varphi(\mathbf{P}[\vec{u}]) - \varphi(\mathbf{P}[\vec{v}])\| \qquad (\|\vec{u}\| = \|\vec{v}\| = 1). \tag{Wiso}$$

- Re-phrasing for **vectors**: If $\phi\colon H \to H$ satisfies

$$|\langle\vec{u},\vec{v}\rangle| = |\langle\phi(\vec{u}),\phi(\vec{v})\rangle| \quad (\vec{u},\vec{v}\in H), \tag{Wvec}$$

then we have

$$\phi(\vec{u}) \in \mathbb{C}\cdot\mathbf{W}\vec{u} \qquad (\vec{u}\in H).$$

Metric resolving sets

Definition

Let (X, d) be a metric space and $D, R \subseteq X$. We say that R is a **resolving set** for D if for any two points $x_1, x_2 \in D$ whenever

$$d(x_1, y) = d(x_2, y) \qquad (\forall\, y \in R)$$

is satisfied, then

$$x_1 = x_2.$$

- Note that R does not have to be a subset of D.

- Throughout this talk we will assume that $\underline{\dim \mathcal{H} = \aleph_0}$. Fix an ONB: $\{\vec{e}_j\}_{j=1}^N$. For $j \in \mathbb{N}$ and $\vec{v} \in H$, $\|\vec{v}\| = 1$ we set

$$v_j := \langle \vec{v}, \vec{e}_j \rangle.$$

- The set

$$D := \{\mathbf{P}[\vec{v}] \colon v_j \neq 0,\ \forall\, j\}$$

is clearly **dense** in $\mathcal{P}_1(\mathcal{H})$ with respect to the operator norm.

Lemma

The set

$$R = \{\mathbf{P}[\vec{e}_j]\}_{j=1}^{\infty} \bigcup \left\{ \mathbf{P}\left[\frac{\vec{e}_j - \vec{e}_{j+1}}{\sqrt{2}}\right], \mathbf{P}\left[\frac{\vec{e}_j + i\,\vec{e}_{j+1}}{\sqrt{2}}\right] \right\}_{j=1}^{\infty}$$

resolves the dense subset D.

- Observe that $R \cap D = \emptyset$.

Proof of Wigner's theorem

(in the separable infinite dimensional case)

- By (Wvec) the system $\{\vec{f_j}\}_{j=1}^{\infty}$ of unit vectors which satisfy

$$\mathbf{P}[\vec{f_j}] = \varphi(\mathbf{P}[\vec{e_j}]) \qquad (\forall\, j)$$

is clearly an ONS. Define

$$\mathcal{H}' := \vee\{\vec{f_j}\}_{j=1}^{\infty}.$$

- **ϕ's range is contained in $\mathcal{P}_1(\mathcal{H}')$:** If we have

$$\varphi(\mathbf{P}[\vec{v}]) = \mathbf{P}[\vec{w}]$$

with some unit vectors $\vec{v}, \vec{w} \in \mathcal{H}$, then by (Wvec) we infer

$$|v_j| = |\langle \vec{w}, \vec{f_j} \rangle| \qquad (\forall\, j).$$

Moreover, from Parseval's identity $\vec{w} \in \mathcal{H}'$, and therefore indeed

$$\varphi(\mathcal{P}_1(\mathcal{H})) \subseteq \mathcal{P}_1(\mathcal{H}').$$

• **We would like to modify φ so that each $\mathbf{P}[\vec{e}_j]$ is fixed.**
Therefore we define the following linear isometry:

$$\mathbf{V}\colon \mathcal{H} \to \mathcal{H}' \subseteq \mathcal{H}, \quad \mathbf{V}\vec{e}_j = \vec{f}_j \qquad (\forall\, j).$$

The map $\varphi_1(\cdot) := \mathbf{V}^*\varphi(\cdot)\mathbf{V}$ obviously satisfies (Wtr). Moreover

$$\begin{aligned}\varphi_1(\mathbf{P}[\vec{e}_j]) &= \mathbf{V}^*\varphi(\mathbf{P}[\vec{e}_j])\mathbf{V} \\ &= \mathbf{V}^*\mathbf{P}[\vec{f}_j]\mathbf{V} = \mathbf{P}[\mathbf{V}^*\vec{f}_j] \\ &= \mathbf{P}[\vec{e}_j] \qquad\qquad (\forall\, j \in \mathbb{N}).\end{aligned}$$

• Moreover, if $\varphi_1(\mathbf{P}[\vec{v}]) = \mathbf{P}[\vec{w}]$, then by (Wvec) we also have

$$|v_j| = |w_j| \qquad (\forall\, j \in \mathbb{N}),$$

in particular, we obtain

$$\varphi_1(D) = D.$$

- Thus there exist $|\delta_{j+1}| = |\varepsilon_{j+1}| = 1$ such that

$$\varphi_1\left(\mathsf{P}\left[\frac{\vec{e}_j - \vec{e}_{j+1}}{\sqrt{2}}\right]\right) = \mathsf{P}\left[\frac{\vec{e}_j - \delta_{j+1}\vec{e}_{j+1}}{\sqrt{2}}\right]$$
$$\varphi_1\left(\mathsf{P}\left[\frac{\vec{e}_j + i\vec{e}_{j+1})}{\sqrt{2}}\right]\right) = \mathsf{P}\left[\frac{\vec{e}_j + i\varepsilon_{j+1}\vec{e}_{j+1}}{\sqrt{2}}\right] \qquad (\forall\, j \in \mathbb{N}).$$

Applying (Wvec) for the above yields

$$\sqrt{2} = |1 + i\delta_{j+1}\overline{\varepsilon_{j+1}}| \qquad (\forall\, j \in \mathbb{N}),$$

and consequently,

$$\delta_{j+1} \in \{-\varepsilon_{j+1}, \varepsilon_{j+1}\} \qquad (\forall\, j \in \mathbb{N}).$$

• We would like to modify φ_1 so that more elements of R are fixed. Therefore we define $\varphi_2(\cdot) := \mathbf{U}\varphi_1(\cdot)\mathbf{U}^*$, where

- if $\varepsilon_2 = \delta_2$, then $\mathbf{U}$ is the unitary operator with

$$\mathbf{U}\vec{e}_j = \begin{cases} \vec{e}_1 & \text{if } j = 1 \\ \prod_{k=2}^{j} \overline{\delta_k} \cdot \vec{e}_j & \text{if } j > 1 \end{cases}$$

- if $\varepsilon_2 = -\delta_2$, then $\mathbf{U}$ is the antiunitary operator with

$$\mathbf{U}\vec{e}_j = \begin{cases} \vec{e}_1 & \text{if } j = 1 \\ \prod_{k=2}^{j} \delta_k \cdot \vec{e}_j & \text{if } j > 1 \end{cases}.$$

- It is obvious that for every $j \in \mathbb{N}$ we have

$$\varphi_2(\mathsf{P}[\vec{e}_j]) = \mathsf{P}[\vec{e}_j],$$

therefore, if $\varphi_2(\mathsf{P}[\vec{v}]) = \mathsf{P}[\vec{w}]$ with $\|\vec{v}\| = \|\vec{w}\| = 1$, then

$$|v_j| = |w_j|.$$

In particular, we have $\varphi_2(D) = D$. Moreover,

$$\varphi_2\left(\mathsf{P}\left[\frac{\vec{e}_j - \vec{e}_{j+1}}{\sqrt{2}}\right]\right) = \mathsf{P}\left[\frac{\vec{e}_j - \vec{e}_{j+1}}{\sqrt{2}}\right] \qquad (\forall\, j \in \mathbb{N}),$$

$$\varphi_2\left(\mathsf{P}\left[\frac{\vec{e}_1 + i\vec{e}_2}{\sqrt{2}}\right]\right) = \mathsf{P}\left[\frac{\vec{e}_1 + i\vec{e}_2}{\sqrt{2}}\right],$$

and for every $j > 1$

$$\varphi_2\left(\mathsf{P}\left[\frac{\vec{e}_j + i\vec{e}_{j+1}}{\sqrt{2}}\right]\right) \in \left\{\mathsf{P}\left[\frac{\vec{e}_j - i\vec{e}_{j+1}}{\sqrt{2}}\right], \mathsf{P}\left[\frac{\vec{e}_j + i\vec{e}_{j+1}}{\sqrt{2}}\right]\right\}.$$

- Our goal is to show that $\varphi_2\big|_D$ is the identity map. In order to show this, by (Wiso), it is enough to see that $\varphi_2\big|_R$ is the identity transformation. (Recall that $R \cap D = \emptyset$.)

- Now, assume that there exists an index $j > 1$ for which

$$\varphi_2\left(\mathbf{P}\left[\frac{\vec{e}_j + i\vec{e}_{j+1}}{\sqrt{2}}\right]\right) = \mathbf{P}\left[\frac{\vec{e}_j - i\vec{e}_{j+1}}{\sqrt{2}}\right]$$

holds. We may assume that this j is the first such index.

- **Claim:** Assuming the above we have

$$\varphi_2(\mathbf{P}[v_{j-1}\vec{e}_{j-1} + t\vec{e}_j + v_{j+1}\vec{e}_{j+1}]) = \mathbf{P}[v_{j-1}\vec{e}_{j-1} + t\vec{e}_j + \overline{v_{j+1}}\vec{e}_{j+1}]$$

holds for every $t > 0, v_{j-1} \neq 0, v_{j+1} \neq 0, |v_{j-1}|^2 + t^2 + |v_{j+1}|^2 = 1.$

Proof: it is a rather easy calculation. □

- Now, let

$$\vec{x} = \frac{-1}{2}\vec{e}_{j-1} + \frac{1}{2}\vec{e}_j + \frac{1}{\sqrt{2}}\vec{e}_{j+1}, \quad \vec{y} = \frac{i}{2}\vec{e}_{j-1} + \frac{1}{2}\vec{e}_j + \frac{i}{\sqrt{2}}\vec{e}_{j+1}.$$

Claim and (Wvec) implies

$$\sqrt{2}/4 = |i/4 + 1/4 - i/2| = |i/4 + 1/4 + i/2| = \sqrt{10}/4,$$

which is a contradiction. Therefore φ_2 is the identity mapping on R, hence on D, and therefore on $\mathcal{P}_1$. Then we easily calculate φ:

$$\varphi(\mathbf{P}[\vec{u}]) = \mathbf{W}\mathbf{P}[\vec{u}]\mathbf{W}^*$$

where $\mathbf{W} = \mathbf{V}\mathbf{U}^*$. $\square$

Remarks

- The finite dimensional case can be proved in a very similar way, even with some simplifications.
- The non-separable case can be proven as a consequence of the separable case (technical).
- A similar, but somewhat simpler proof works for the real case.

References 1/2

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Thank You for Your Kind Attention