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NECESSARY AND SUFFICIENT CONDITION FOR THE GLOBAL STABILITY OF A DELAYED DISCRETE-TIME SINGLE NEURON MODEL

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ABSTRACT. We consider the global asymptotic stability of the trivial fixed point of the difference equation $x_{n+1} = mx_n - \alpha\varphi(x_{n-1})$, where $(\alpha, m) \in \mathbb{R}^2$ and φ is a real function satisfying the discrete Yorke condition: $\min\{0,x\} \le \varphi(x) \le \max\{0,x\}$ for all $x \in \mathbb{R}$. If φ is bounded then $(\alpha, m) \in [|m|-1, 1] \times [-1, 1]$, $(\alpha, m) \ne (0,-1), (0,1)$ is necessary for the global stability of 0. We prove that if $\varphi(x) \equiv \tanh(x)$, then this condition is sufficient as well.

1. Introduction.

Consider the difference equation given by

$$x_{n+1} = mx_n - \alpha\varphi(x_{n-1}), \tag{1.1}$$

where $(\alpha, m) \in \mathbb{R}^2$ and φ is a real function which satisfies the discrete Yorke condition

$$\min\{0,x\} \le \varphi(x) \le \max\{0,x\} \text{ for all } x \in \mathbb{R}. \tag{1.2}$$

Note that (1.2) implies that φ is continuous at 0, $\varphi(0) = 0$ and $0 \le \liminf_{x\to 0} \frac{\varphi(x)-\varphi(0)}{x} \le \limsup_{x\to 0} \frac{\varphi(x)-\varphi(0)}{x} \le 1$. The situation is depicted on Figure 1.

Equation (1.1) can be interpreted as a discrete-time single neuron model with delay or as a discrete-time version of the Krisztin–Walther equation [16], as well. For a comprehensive description of the global dynamics of the delayed, continuous Krisztin–Walther equation see papers of Cao, Krisztin and Walther [3, 14, 15, 16] and the monograph of Krisztin, Walther and Wu [17].

We consider (1.1) as the two dimensional map

$$F_{\alpha,m}\colon (x,y) \mapsto (y, my - \alpha\varphi(x)) \tag{1.3}$$

that has a fixed point at the origin. We shall investigate the global asymptotic stability (GAS) of this fixed point. Besides of that from a mathematical point of view, global stability of a unique equilibrium point is always a fundamental topic, in neural networks it is also important in solving optimization and signal processing problems.

There is a vast number of papers giving sufficient conditions for global stability of more complicated models of neural networks (see in [4, 10, 21, 34, 35] and the references therein) and of more general difference equations (see in [13, 25, 26, 27, 19]), without attempting

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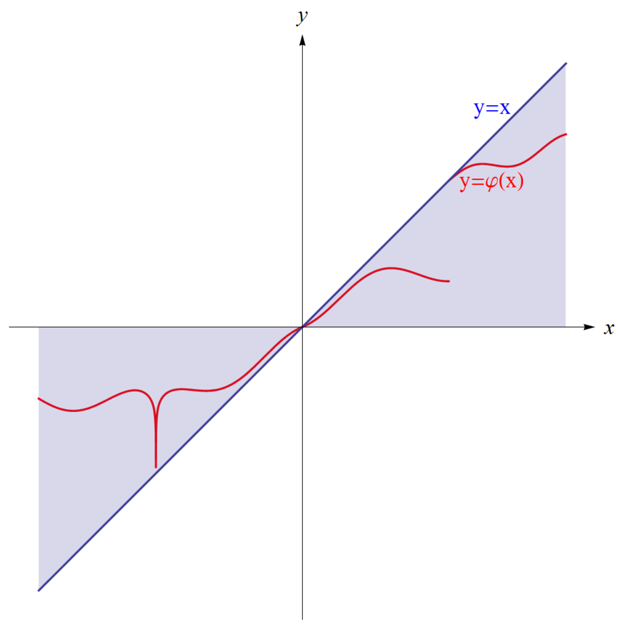

FIGURE 1. The map $\varphi(x)$ satisfies the discrete Yorke condition.

to be comprehensive; but to the best of our knowledge, none of those are claimed to be necessary.

The main result of the paper is Theorem 4.2, in which we give a necessary and sufficient condition for the global stability of our model difference equation

$$x_{n+1} = mx_n - \alpha \tanh(x_{n-1}), \tag{1.4}$$

with $(\alpha, m) \in \mathbb{R}^2$ as above. The tanh function is one of the most common examples for a sigmoid-type feedback function occurring in neural network models (see e.g. the books of Haykin [11] and Wu [33]).

Restricting the parameter range to $m \in (0,1)$ makes equation (1.4) to be of Clark type [5]. There are several conjectures on the global stability of such systems. The one by El-Morshedy and Liz [8] states that if $f \in C^3([0,\infty),(0,\infty))$, $f'(x) < 0$ and $(Sf)(x) < 0$ holds for all $x > 0$, where $(Sf)(x)$ is the Schwarzian derivative of f at x, then the unique equilibrium of

$$x_{n+1} = mx_n + f(x_{n-k}) \text{ with } m \in (0,1) \text{ and } k \in \mathbb{N}$$

is GAS whenever it is LAS (locally asymptotically stable). This conjecture has been recently proved to be false having $k \geq 3$ (see [12]), but the problem is still open for $k = 1,2$. It is easy to verify that $f(x) \equiv -\alpha \tanh(x)$ fulfills the above assumptions (formally after a translation), thus Theorem 4.2 solves the problem in this particular case of (1.4) and $k = 1$.

Our proof is a combination of analytical and computer-aided tools and is based on a technique presented in our previous work [2]. The term computer-aided refers to that we do our calculations using a computer program that gives validated results, every possible numerical error is controlled. This allows us to prove *mathematical theorems* from the obtained outputs. For more information about computer-aided proofs and rigorous numerics, the reader is referred to Moore [22, 23], Alefeld [1], Tucker [30, 31], and Nedialkov et al. [24]. We shall use graph representations, whereas we model our function on a grid, resulting in a directed graph. This concept has been utilized both in rigorous and non-rigorous computations for analyzing maps by Dellnitz, Hohmann and Junge [6, 7], Galias [9], Luzzatto and Pilarczyk [20] and for studying the attractor of a differential equation by Wilczak [32].

The proof consists of the following three steps:

- Step 1: Construct (by elementary argument) a compact region (S) which is independent of the parameters and contains a positive invariant and globally attracting subset for all (α, m). This is done in Section 3.
- Step 2: Construct a uniform neighborhood (U) of the trivial fixed point which belongs to the basin of attraction of the fixed point for all parameter pairs. We achieve this by linearization and by using the (1:4 resonant) normal form of the Neimark–Sacker bifurcation. It can be done analytically, however, some calculations and estimations are aided by symbolic calculations via *Wolfram Mathematica*. The details may be found in the first part of Section 4.
- Step 3: Show that every trajectory starting from the compact set constructed in Step 1 eventually enters the neighborhood constructed in Step 2. One may do this by using graph representations of the map and interval arithmetic tools, as seen in the second part of Section 4.

Our further research interests in the topic include the application of our method for higher dimensional maps. This should involve a center manifold reduction, that gives birth to new technical challenges. However, the major obstacle is the increasing size of the graph representations that slows down the computations considerably. Having a higher dimensional parameter range has similar consequences. A method for automatized generation of the compact attracting set S, together with a recipe-like algorithm for finding a neighborhood U, which belongs to the basin of attraction of the fixed point is among our plans as well.

The article has the following structure. Section 2 contains the definitions and notations used in this paper. In Section 3, after making some elementary observations and recalling the earlier results, we construct two invariant and attracting sets with a compact intersection S, having $(\alpha, m) \in [0,1]^2$. To do the latter, we assume that besides (1.2), equation (1.1) fulfills the following hypotheses

(H1) φ is bounded, e.g. there exists $M_\varphi > 0$, such that for all $x \in \mathbb{R}$, $|\varphi(x)| < M_\varphi$ holds,
(H2) φ is continuous and $\min\{0, x\} < \varphi(x) < \max\{0, x\}$ holds for all $x \neq 0$.

Note that (H1) and (H2) are satisfied by $\tanh(x)$. In Section 4, we turn our attention to the model equation (1.4) and give a necessary and sufficient condition for the global asymptotic stability of its trivial fixed point.

2. **Definitions and notations.**

Let us define some notations that we shall use in this paper. We denote by $\mathbb{N}, \mathbb{N}_0, \mathbb{R}$ and $\mathbb{C}$ the set of positive integers, non-negative integers, reals and complex numbers respectively. The open ball in the maximum norm with radius $\delta > 0$ around $0 \in \mathbb{R}^n$ is denoted by K_δ. The open disk on the complex plain with radius $\delta > 0$ is denoted by $\mathrm{B}_\delta = \{z \in \mathbb{C} : \; |z| < \delta\}$, where $|z|$ denotes the absolute value of $z \in \mathbb{C}$. For $R \subseteq \mathbb{R}^2$, let $\mathrm{bd}(R)$ and $\mathrm{cl}(R)$ denote the topological boundary and the closure of the set R, respectively. It is unambiguous whether a vector in a formula is a row or a column vector, therefore we omit the usage of the transpose. For $\xi = (\xi_1, \xi_2) \in \mathbb{C}^2$ and $\zeta = (\zeta_1, \zeta_2) \in \mathbb{C}^2$ let $\langle \xi, \zeta \rangle$ denote the scalar product of them defined by $\langle \xi, \zeta \rangle = \overline{\xi_1}\zeta_1 + \overline{\xi_2}\zeta_2$. For a bounded function $\psi \colon \mathbb{R} \to \mathbb{R}$, let $M_\psi = \sup_{x \in \mathbb{R}} |\psi(x)|$. For a real function $\gamma \colon \mathbb{R} \to \mathbb{R}$ and for $c \in \mathbb{R}$, let $\delta_{\inf}(\gamma; c) := \liminf_{x \to c} \frac{\gamma(x) - \gamma(c)}{x - c}$ and $\delta_{\sup}(\gamma; c) := \limsup_{x \to c} \frac{\gamma(x) - \gamma(c)}{x - c}$.

Given a number or set X, by $[X]$ we denote an interval enclosure of X. With the usage of this notation, we emphasize always, that even though we might obtain $[X]$ from a computation, $X \subseteq [X]$ is always satisfied. Any subsequent computations will result in validated results due to the proper usage of interval analysis.

Consider the continuous map $f\colon \mathscr{D}_f \subseteq \mathbb{R}^2 \to \mathbb{R}^2$. For $k \in \mathbb{N}_0$, f^k denotes the k-fold composition of f, i.e., $f^{k+1}(x) = f(f^k(x))$, and $f^0(x) = x$.

Definition 2.1. The point $x^* \in \mathscr{D}_f$ is called a *fixed point* of f if $f(x^*) = x^*$. The point $q \in \mathscr{D}_f$ is a *non-wandering point* of f if for every neighborhood U of q and for any $M \geq 0$, there exists an integer $m \geq M$ such that $f^m(U \cap \mathscr{D}_f) \cap U \cap \mathscr{D}_f \neq \emptyset$.

A fixed point $x^* \in \mathscr{D}_f$ of f is called *locally stable* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\|x - x^*\| < \delta$ implies $\|f^k(x) - x^*\| < \varepsilon$ for all $k \in \mathbb{N}$, where $\|.\|$ denotes the Euclidean norm. We say that the fixed point x^* *attracts* the region $U \subseteq \mathscr{D}_f$ if for all points $u \in U$, $\|f^k(u) - x^*\| \to 0$ as $k \to \infty$. The fixed point x^* is *globally attracting* if it attracts all of $\mathscr{D}_f$, and it is *globally asymptotically stable* (GAS) if it is locally stable and globally attracting.

We shall associate directed graphs with f. The vertices of these graphs are sets and the edges correspond to transitions between them. These graphs reflect the behavior of the map, if for every point (x,y) and its image $f(x,y)$, it is satisfied that there is an edge going from any vertex containing (x,y) to any vertex containing $f(x,y)$. We give the necessary definitions here, the reader is referred to [2] for a more detailed overview of graph representations.

Definition 2.2. $\mathscr{P}$ is called a *partition* of $\mathscr{D} \subseteq \mathbb{R}^2$ if it is a collection of closed subsets of $\mathbb{R}^2$ such that $|\mathscr{P}| := \cup_{p \in \mathscr{P}} p = \mathscr{D}$ and $\forall p_1, p_2 \in \mathscr{P}\colon\ p_1 \cap p_2 \subseteq \mathrm{bd}(p_1) \cup \mathrm{bd}(p_2)$. We define the *diameter* of the partition $\mathscr{P}$ by

$$\mathrm{diam}(\mathscr{P}) = \sup_{p \in \mathscr{P}} \sup_{x,y \in p} \|x - y\|.$$

Let $f\colon \mathscr{D}_f \subseteq \mathbb{R}^2 \to \mathbb{R}^2$, $\mathscr{D} \subseteq \mathscr{D}_f$, and $\mathscr{P}$ be a partition of $\mathscr{D}$. We say that the directed graph $\mathscr{G}(\mathscr{V}, \mathscr{E})$ is a *graph representation of f on $\mathscr{D}$ with respect to $\mathscr{P}$*, if there exists a bijection $\iota\colon \mathscr{V} \to \mathscr{P}$ such that the following implication is true for all $u, v \in \mathscr{V}$:

$$f(\iota(u) \cap \mathscr{D}) \cap \iota(v) \cap \mathscr{D} \neq \emptyset \Rightarrow (u,v) \in \mathscr{E}.$$

We take the liberty to handle the elements of the partition as vertices and vice versa, omitting the usage of ι.

Let us now define the following 2-dimensional map, corresponding to equation (1.1)

$$F\colon \mathbb{R}^2 \to \mathbb{R}^2,\ F(x,y) = F_{\alpha,m}(x,y) = (y, my - \alpha\varphi(x)). \tag{2.1}$$

For $(x,y) \in \mathbb{R}^2$ and $k \in \mathbb{N}_0$, we shall use notation $(x_k, y_k) = F^k(x,y)$.

3. **Preliminaries and earlier results.** Even though φ is not assumed to be differentiable at 0, one may characterize the local stability of the origin by the following *generalized multipliers* of the map F at $(0,0)$

$$\mu_{1,2}(\lambda) = \mu_{1,2}(\alpha, m; \lambda) = \tfrac{m \pm \sqrt{m^2 - 4\lambda\alpha}}{2},$$

where $\lambda \in [\delta_{\inf}(\varphi;0), \delta_{\sup}(\varphi;0)]$ is such that there exists a sequence $x_k \to 0$ for which $\frac{\varphi(x_k) - \varphi(0)}{x_k} = \frac{\varphi(x_k)}{x_k} \to \lambda$, as $k \to \infty$. Recall from our initial observations, that (1.2) implies $\lambda \in [0,1]$. It is easy to see, that $\max|\mu_{1,2}(\lambda)| \leq 1$ is satisfied only if $m \in [-2,2]$ and $\lambda\alpha \in [|m| - 1, 1]$ hold and we have equality if and only if $\lambda\alpha = 1$ or $\lambda\alpha = |m| - 1$. Consequently, the global stability of the zero solution may hold only if both $\delta_{\inf}(\varphi;0)\alpha \in [|m|-1,1]$ and $\delta_{\sup}(\varphi;0)\alpha \in [|m|-1,1]$. Let us define the following parameter range

$$(\alpha,m) \in \mathscr{R}(m) := \mathrm{cl}(\mathscr{R}_0(m)) \setminus \{(0,-1),(0,1)\},$$

where $\mathscr{R}_0(m)$ is the open set $(|m|-1,1)\times(-1,1)$. These regions are depicted on Figure 2.

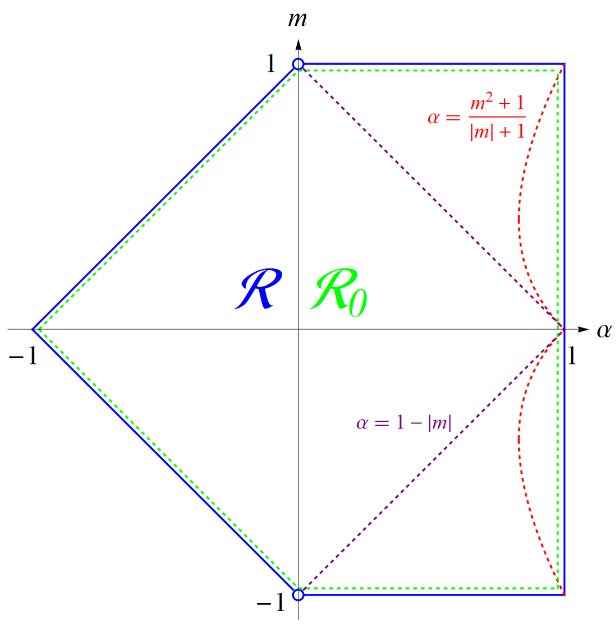

FIGURE 2. The solid blue and dashed green lines represent the sets $\mathscr{R}(m)$ and $\mathscr{R}_0(m)$, respectively. The dashed red and purple lines correspond to the curves $\alpha = \frac{m^2+1}{|m|+1}$ and $\alpha = 1-|m|$, respectively.

Lemma 3.1. *Assume that φ satisfies* (H1). *If $|m|>1$, then the zero fixed point of* (2.1) *is* not *GAS.*

Proof. Indeed, if $|m|>1$ then we readily get that $\min\{|x_0|,|y_0|\} > \frac{M_\varphi|\alpha|}{|m|-1}$ implies $\min\{|x_1|,|y_1|\} > \frac{M_\varphi|\alpha|}{|m|-1}$, excluding the global stability of the fixed point $(0,0)$ of F in this case. □

Lemma 3.2. *The $(0,0)$ fixed point of* (1.3) *is globally asymptotically stable if*

(a) $(\alpha,m)\in\mathscr{R}_0(m)$ with $\alpha<1-|m|$ or
(b) φ satisfies (H2) and $(\alpha,m)\in\mathscr{R}(m)$ with $\alpha<1-|m|$.

See Figure 2 for a visualization of these parameter regions.

Proof. The idea of the proof has been used for more general difference equations via Halanay-type results; see, for example in [19]. We prove statement *(a)* first. For any point $(x_0,y_0)\in\mathbb{R}^2$, we have $|y_1|\le|m||y_0|+|\alpha||\varphi(x_0)|$, thus the inequalities $\max\{|x_1|,|y_1|\}\le\max\{|x_0|,|y_0|\}$ and $\max\{|x_2|,|y_2|\}\le(|m|+|\alpha|)\max\{|x_0|,|y_0|\}$ are satisfied. Since $|m|+|\alpha|<1$, by induction we obtain that $\max\{|x_{2k+1}|,|y_{2k+1}|\}\le\max\{|x_{2k}|,|y_{2k}|\}$ for $k\in\mathbb{N}$ and $\lim_{k\to\infty}\max\{|x_{2k}|,|y_{2k}|\}=0$, thus global asymptotic stability holds in this case.

Consider now statement *(b)*. For $\alpha \in (|m|-1, 1-|m|)$ the same argument works, thus let $\alpha = |m|-1 \in [-1,0)$ and $m \in (-1,1)$. In the same way, for a point $(x_0,y_0) \in \mathbb{R}^2$, we obtain that $\max\{|x_{2k}|,|y_{2k}|\}$ is decreasing, thus $\max\{|x_{2k}|,|y_{2k}|\} \to c \geq 0$. In order to show that $c=0$, due to the continuity of φ, it is enough to establish that, for any (x_0,y_0) such that $\max\{|x_0|,|y_0|\} = c > 0$, the orbit satisfies $\limsup_{k\to\infty} \max\{|x_{2k}|,|y_{2k}|\} < c$. This easily follows from the condition on φ and $m \in (-1,1)$, since for any point $(x,y) \neq (0,0)$, we have $|my - \alpha\varphi(x)| < \max\{|x|,|y|\}$. □

Even though we will supply a stronger condition, note that using an analogous argument, GAS is easily shown for $(\alpha,m) \in \{1-|m|\} \times (-1,1)$ if we assume that φ fulfills (H2).

Let us define the following sets for $a,b \in (0,\infty]$

$$\begin{aligned}
H_1(a,b) &= \{(x,y) : 0 \leq x \leq a;\ 0 < y \leq b\},\\
H_2(a,b) &= \{(x,y) : 0 < x \leq a;\ -b \leq y \leq 0\},\\
H_3(a,b) &= \{(x,y) : -a \leq x \leq 0;\ -b \leq y < 0\},\\
H_4(a,b) &= \{(x,y) : -a \leq x < 0;\ 0 \leq y \leq b\}
\end{aligned}$$

and $H_i = H_i(\infty,\infty)$ for $i \in \{1,2,3,4\}$. Figure 3 shows these four sets for a pair of values (a,b).

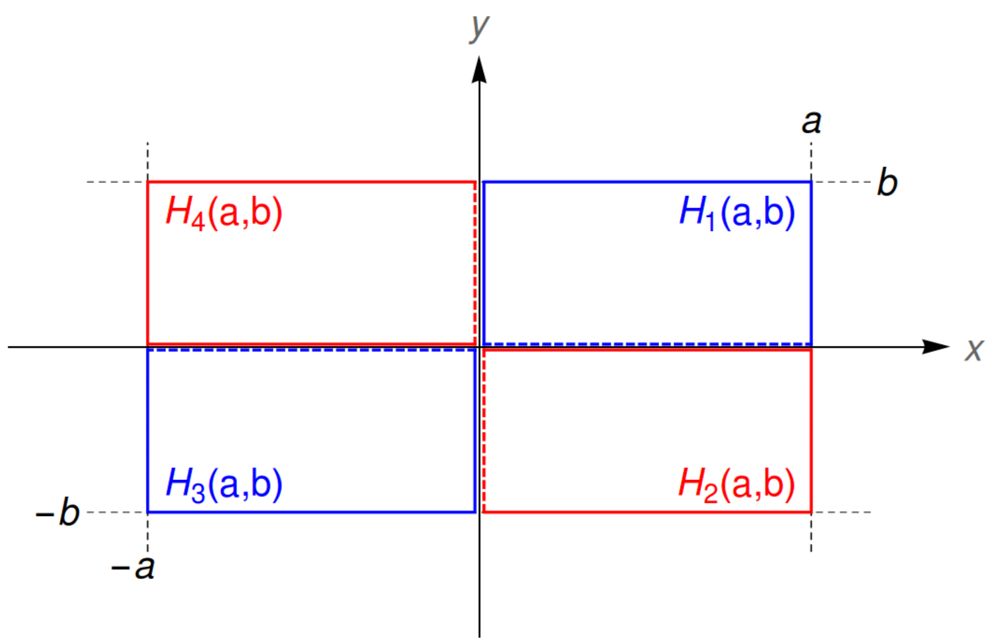

FIGURE 3. The sets $H_1(a,b), H_2(a,b), H_3(a,b)$ and $H_4(a,b)$.

Proposition 3.3. *For $m \in [0,1]$ and $\alpha \in [0,1]$ the following statements hold.*

(i) *If $(x_0,y_0) \in H_1(a,b)$, then $(x_1,y_1) \in H_1(b,mb) \cup H_2(b,\alpha a)$.*
(ii) *If $(x_0,y_0) \in H_2(a,b)$, then $(x_1,y_1) \in H_3(b,mb+\alpha a)$.*
(iii) *If $(x_0,y_0) \in H_3(a,b)$, then $(x_1,y_1) \in H_3(b,mb) \cup H_4(b,\alpha a)$.*
(iv) *If $(x_0,y_0) \in H_4(a,b)$, then $(x_1,y_1) \in H_1(b,mb+\alpha a)$.*

(v) Having $m \in [0,1)$ and $(x_k,y_k) \in H_1$ or $(x_k,y_k) \in H_3$ for all $k \in \mathbb{N}_0$ implies $\lim_{k\to\infty}(x_k,y_k) = (0,0)$. In addition, if (H2) is satisfied, then the claim holds for $m = 1$ as well.

Proof. The proof is elementary. We denote the first point with (x_0,y_0). To see that statement *(i)* holds, first note that $0 < x_1 = y_0 \le b$. Since we have $y_1 = my_0 - \alpha\varphi(x_0)$ and $0 \le x\varphi(x) \le x^2$, therefore $(x_0,y_0) \in H_1(a,b)$ readily implies $-\alpha a \le y_1 \le mb$, resulting in $(x_1,y_1) \in H_1(b,mb) \cup H_2(b,\alpha a)$. Statements *(ii)–(iv)* can be proved in a similar manner.

To prove statement *(v)*, let us suppose that the point $(x_0,y_0) \in H_1$ is such that $(x_k,y_k) \in H_1$ holds for all $k \in N_0$. Using the notation $a = \max\{x_0,y_0\} > 0$ and statement *(i)*, we obtain by induction that

$$(x_{2k},y_{2k}) \in H_1(m^k a, m^k a),\ (x_{2k+1},y_{2k+1}) \in H_1(m^k a, m^{k+1}a) \text{ and } 0 < y_{2k+1} \le y_{2k}$$

hold for all $k \in \mathbb{N}_0$ implying that

$$\lim_{k\to\infty} \max\{x_k,y_k\} = \lim_{k\to\infty} x_k = \lim_{k\to\infty} y_k = c \ge 0,$$

which results in $c = 0$ if $m \in [0,1)$. We finish our argument by noting, that for $m = 1$, the continuity of φ implies that it is enough to show that (c,c) cannot be a fixed point for $c > 0$. This easily follows from $0 < \varphi(c)$.

The case of $(x_k,y_k) \in H_3$ is analogous. □

Remark 3.4. It is possible to formulate similar statements for $m \in [-1,0]$ as well, but as they will not be used in the paper, we omit the details.

The following theorem is a corollary of Lemma 3.2 and the results of Nenya and Nenya *et al.* [25, 26, 27] for more general, non-autonomous difference equations with Yorke and generalized Yorke condition.

Theorem 3.5. *The $(0,0)$ fixed point of (1.3) is globally asymptotically stable, if any of the following conditions is satisfied*

(a) $(\alpha,m) \in \mathscr{R}_0(m) \setminus \left[\frac{m^2+1}{|m|+1}, 1\right] \times (-1,1)$,

(b) φ *satisfies condition (H2) and* $(\alpha,m) \in \mathscr{R}(m) \setminus \left(\frac{m^2+1}{|m|+1}, 1\right] \times [-1,1]$.

See Figure 2 for a visualization of these parameter regions.

Proof. It is easy to verify that the fixed point is locally stable so we only have to prove that it is globally attracting.

Let $m > 0$. For $\alpha > 0$, the statement is a special case of Theorem 3 in [27] and Theorem 2 in [25]. If $m = 0$ or $\alpha \le 0$, then Lemma 3.2 applies (it also applies for $\alpha \le 1 - |m|$).

Assume now that $m < 0$. The substitution $y_n := (-1)^n x_n$ transforms (1.1) into $y_{n+1} = (-m)y_n - \alpha(-1)^{n-1}\varphi((-1)^{n-1}y_{n-1})$. As the referred results from [25, 27] apply to non-autonomous difference equations and the map $(-1)^{n-1}\varphi((-1)^{n-1}x)$ satisfies (1.2) for all $n \in \mathbb{N}$, the statement follows from GAS for $(\alpha,|m|)$. □

In the remaining part of the section we limit our analysis to the case when $m \in [0,1]$, and conditions (H1) and (H2) are satisfied. Let $M \ge 0$ and consider the sets $\mathbb{T}(M,m)$ and $\mathscr{S}(M,m)$ given by

$$\mathbb{T}(M,m) = \begin{cases} \mathbb{R}^2, & \text{for } m = 0, \\ H_1\left(\frac{2M}{m},\frac{2M}{m}\right) \cup H_2\left(\frac{M}{m},\frac{M}{m}\right) \cup H_3\left(\frac{2M}{m},\frac{2M}{m}\right) \cup H_4\left(\frac{M}{m},\frac{M}{m}\right) \cup \{(0,0)\}, & \text{for } m \in (0,1], \end{cases}$$

and

$$\mathscr{S}(M,m) = \begin{cases} [-\frac{2M}{1-m}, \frac{2M}{1-m}]^2, & \text{for } m \in [0,1), \\ \mathbb{R}^2, & \text{for } m = 1. \end{cases}$$

We sketched $\mathbb{T}(M,m)$ on Figure 4 for $m \neq 0$.

The following proposition has an essential role in the proof of our main result.

Proposition 3.6. *Assume that* $(\alpha,m) \in [0,1]^2$, φ *satisfies conditions* (H1) *and* (H2) *and let* $M = M_\varphi$. *Then the following statements hold.*

(i) $(x_0,y_0) \in \mathbb{T}(M,m)$ *implies* $(x_1,y_1) \in \mathbb{T}(M,m)$*, moreover, for* $(x_0,y_0) \in \mathbb{R}^2$*, there exists* $k \in \mathbb{N}_0$ *such that* $(x_k,y_k) \in \mathbb{T}(M,m)$ *is satisfied.*

(ii) $(x_0,y_0) \in \mathscr{S}(M,m)$ *implies* $(x_1,y_1) \in \mathscr{S}(M,m)$*, moreover, for* $(x_0,y_0) \in \mathbb{R}^2$*, there exists* $k \in \mathbb{N}_0$ *such that* $(x_k,y_k) \in \mathscr{S}(M,m)$ *holds.*

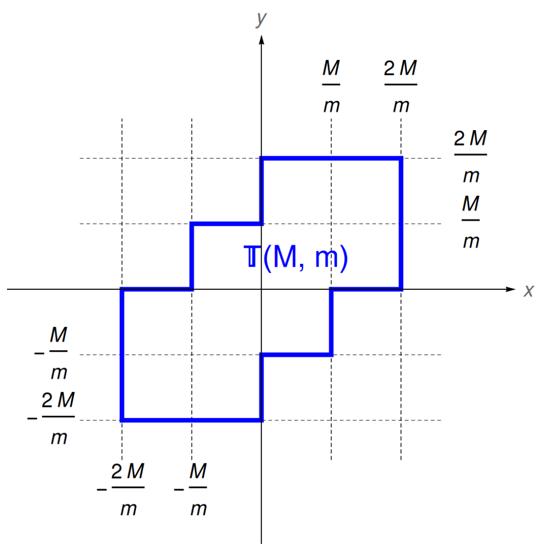

FIGURE 4. The set $\mathbb{T}(M,m)$ for $m > 0$.

Proof. Let m and M be fixed and let us use notations $\mathbb{T} = \mathbb{T}(M,m)$ and $\mathscr{S} = \mathscr{S}(M,m)$.

(i) The case $m = 0$ is trivial, therefore we may assume $m \in (0,1]$. First, let us show the second part of the statement. Let $(x_0,y_0) \in \mathbb{R}^2$ be an arbitrary point. According to Proposition 3.3 either there exists $k_0 \in \mathbb{N}_0$ such that $(x_{k_0},y_{k_0}) \in H_1$ or we have $(x_k,y_k) \to (0,0)$ as $k \to \infty$, which implies $(x_k,y_k) \in \mathbb{T}$ for large enough values of k. Thus we may assume $(x_0,y_0) \in H_1$.

a) If $0 < y_0 \leq \frac{M}{m}$, then we readily get that $(x_1,y_1) \in H_1(\frac{M}{m},\frac{M}{m}) \cup H_2(\frac{M}{m},\frac{M}{m}) \subset \mathbb{T}$.

b) $y_0 > \frac{M}{m}$ leads to $0 < x_1 = y_0$ and $0 < y_1 = my_0 - \alpha\varphi(x_0) \leq my_0 \leq y_0$. Now if $y_1 \leq \frac{M}{m}$, then we are in case a). Otherwise $y_1 > \frac{M}{m}$ and $(x_1,y_1) \in H_1$. We obtain by induction, that either there exists $k_0 \in \mathbb{N}$ such that $0 < y_{k_0} \leq \frac{M}{m}$ and $x_{k_0} > \frac{M}{m}$ and the claim follows from case a), or $(x_k,y_k) \in H_1 \setminus H_1(\frac{M}{m},\frac{M}{m})$ for all $k \in \mathbb{N}$. In the latter case, Proposition 3.3 leads to $\lim_{k\to\infty}(x_k,y_k) = (0,0)$, implying a contradiction.

Now, we may prove the first part of statement *(i)*. The argument above also shows that for $(x_0,y_0) \in H_1(\frac{2M}{m},\frac{2M}{m})$, $(x_1,y_1) \in \mathbb{T}$ is guaranteed. For $(x_0,y_0) \in H_2(\frac{M}{m},\frac{M}{m})$, statement *(ii)* of Proposition 3.3 with $a = b = \frac{M}{m}$ yields $(x_1,y_1) \in \mathbb{T}$. A similar argument can be applied to show that for $(x_0,y_0) \in H_3(\frac{2M}{m},\frac{2M}{m}) \cup H_4(\frac{M}{m},\frac{M}{m})$, $(x_1,y_1) \in \mathbb{T}$ holds, which completes the proof of *(i)*.

(ii) The statement is trivial for $m = 1$, thus we may assume that $m \in [0,1)$. To prove the first part, let us suppose that $(x_0,y_0) \in \mathscr{S}$. Then $|x_1| = |y_0| \le \frac{2M}{1-m}$ together with $|y_1| \le m|y_0| + \alpha M \le m\frac{2M}{1-m} + M < \frac{2M}{1-m}$ yields $(x_1,y_1) \in \mathscr{S}$.

To prove the second part of the statement let us assume that $(x_0,y_0) \notin \mathscr{S}$.

a) If $|y_0| \ge \frac{2M}{1-m}$, then $|x_1| = |y_0| \ge \frac{2M}{1-m}$ and $|y_1| \le m|y_0| + M \le \frac{m+1}{2}|y_0| < |y_0|$. By induction we get a geometrically decreasing series y_k, thus there exists $k_0 \in \mathbb{N}$ such that $|x_{k_0}| \ge \frac{2M}{1-m}$ and $|y_{k_0}| < \frac{2M}{1-m}$. Now $|x_{k_0+1}| = |y_{k_0}| < \frac{2M}{1-m}$ and $|y_{k_0+1}| \le m|y_{k_0}| + M < m\frac{2M}{1-m} + M < \frac{2M}{1-m}$, thus $(x_{k_0+1}, y_{k_0+1}) \in \mathscr{S}$.

b) If $|y_0| < \frac{2M}{1-m}$, then $(x_0,y_0) \notin \mathscr{S}$ implies $|x_0| > \frac{2M}{1-m}$ which reduces to case a) and makes our proof complete.

□

Corollary 3.7. *Let us assume that* $(\alpha,m) \in [0,1]^2$ *and* φ *satisfies conditions* (H1) *and* (H2). *Given* $M = M_\varphi$, *the sets* $\mathbb{T}(M,m)$ *and* $\mathscr{S}(M,m)$ *are well defined. Their intersection* $S = \mathbb{T}(M,m) \cap \mathscr{S}(M,m)$ *is compact. Moreover,* S *is positive invariant and globally attracting for* (1.3). *In addition, the following inclusion holds*

$$S = \mathbb{T}(M,m) \cap \mathscr{S}(M,m) \subseteq \left[-\frac{2M}{\max\{m,1-m\}}, \frac{2M}{\max\{m,1-m\}}\right]^2 \subseteq [-4M,4M]^2.$$

4. **Main result: necessary and sufficient condition for global stability.** In this section we restrict our attention to equation (1.4), namely

$$x_{n+1} = mx_n - \alpha \tanh(x_{n-1}),$$

where $(\alpha,m) \in \mathbb{R}^2$, that is (1.1) with $\varphi(x) \equiv \tanh(x)$. Keeping the notations of the previous section we get that

$$F\colon \mathbb{R}^2 \to \mathbb{R}^2,\ F(x,y) = F_{\alpha,m}(x,y) = (y, my - \alpha \tanh(x)). \tag{4.1}$$

Remark 4.1. Note that, the function $\varphi \equiv \tanh$ satisfies conditions *(H1)* and *(H2)*. In addition $M_{\tanh} = 1$ and $\tanh'(0) = 1$. These observations imply, in accordance with the results of Section 3, that GAS of the zero solution may only hold when

$$(\alpha,m) \in \mathscr{R}(m) = [|m|-1,1] \times [-1,1] \setminus \{(0,-1),(0,1)\}.$$

Notice that the zero solution is LAS for $(\alpha,m) \in (|m|-1,1) \times (-2,2)$ but it is not GAS for $|m| > 1$. This shows the importance of restricting the parameter range to $m \in (0,1)$ for the Clark type equations. Moreover, as m has a biological meaning in both the neural network model and in the Clark type models – it represents an internal decay or a mortality rate – this restriction is natural.

Theorem 3.5 establishes GAS for $(\alpha,m) \in \mathscr{R}(m) \setminus (\frac{m^2+1}{|m|+1},1] \times [-1,1]$. As tanh is odd, the same substitution may be used as in the proof of the formerly mentioned theorem in order to show that GAS holds for $(\alpha,m) \in \mathscr{R}(m)$ if and only if it holds for $(\alpha,-m)$. Thus it is sufficient to concentrate on the parameter range $(\alpha,m) \in [0,1]^2$.

Figure 5 summarizes the most important results on global stability of (4.1). Global stability is elementary to prove on the triangle marked with A (see Lemma 3.2) and in this case it is also a consequence of e.g. [10]. Theorem 2.1.1 in the monograph of Kocić and

Ladas [13] establishes that the fixed point is GAS in the triangle labeled with B. The results of Nenya *et al.* [25, 27] cover the areas marked with C. Finally, we deal with D and E using computational tools in the proof of Theorem 4.2 presented in this section.

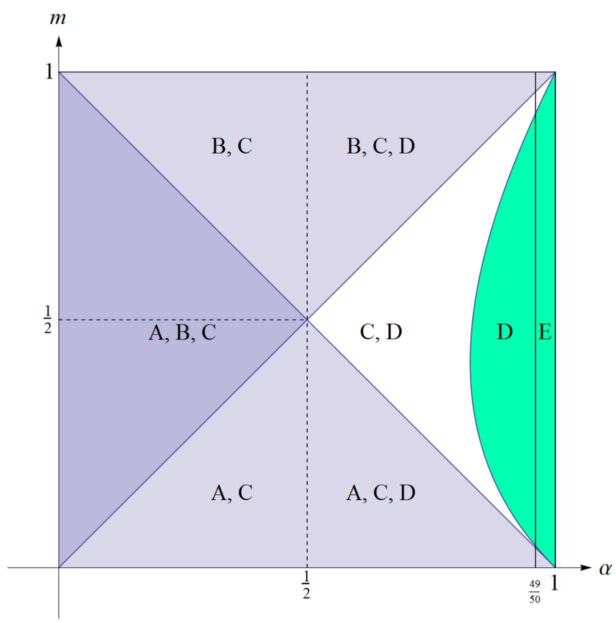

FIGURE 5. The parameter region $(\alpha,m) \in [0,1]^2$.

Notice that at $\alpha = 1$ a Neimark–Sacker bifurcation takes place with a 1:4 strong resonance at $(\alpha,m) = (1,0)$. Our main result is, as we show in the following theorem, that condition $(\alpha,m) \in \mathscr{R}(m)$ is not only necessary but also sufficient for global asymptotic stability of the origin.

Theorem 4.2. *The fixed point* $(0,0)$ *of the map* F *is globally asymptotically stable if and only if* $(\alpha,m) \in \mathscr{R}(m)$.

As already noted, it is sufficient to consider $(\alpha,m) \in [\frac{m^2+1}{m+1},1] \times [0,1]$. The proof of global stability in this region consists of two parts. In Part I, for every such pair (α,m), we obtain a compact neighborhood $U(\alpha)$ inside the basin of attraction of $(0,0)$, such that for any parameter interval $[\alpha] \times [m] \subseteq [(m^2+1)/(m+1),1] \times [0,1]$, the set $U([\alpha]) := \cap_{\alpha\in[\alpha]} U(\alpha)$ contains a closed disk around the origin. For this we shall study the linearized equation and the 1:4 resonant normal form of the Neimark–Sacker bifurcation. After we have derived this neighborhood and the compact set $S = \mathbb{T}(M_{\tanh},m) \cap \mathscr{S}(M_{\tanh},m)$ from Section 3, in Part II we analyze the equation using a rigorous computer program. The results prove that having $(\alpha,m) \in [\alpha] \times [m]$, every trajectory starting in S will enter $U([\alpha]) \subseteq U(\alpha)$.

Part I: Obtaining the compact neighborhood $U(\alpha)$. Linearizing F at the $(0,0)$ fixed point yields

$$(x,y) \mapsto F(x,y) = A(\alpha,m)(x,y)^T + f_{\alpha,m}(x,y) \tag{4.2}$$

where the linear part is

$$A(\alpha,m) = \begin{pmatrix} 0 & 1 \\ -\alpha & m \end{pmatrix},$$

and the remainder is given by

$$f_\alpha(x,y) = \begin{pmatrix} 0 \\ \alpha x - \alpha \tanh(x) \end{pmatrix}.$$

First recall that the eigenvalues of $A(\alpha,m)$ are $\mu_{1,2}(\alpha,m) = \frac{m\pm i\sqrt{4\alpha-m^2}}{2} \in \mathbb{C}$. Let $\mu = \mu_1(\alpha,m)$ and q denote the eigenvector $q = q(\alpha,m) = \left(\frac{m-i\sqrt{-m^2+4\alpha}}{2\alpha},1\right)^T \in \mathbb{C}^2$. Let also $p = p(\alpha,m) \in \mathbb{C}^2$ denote the eigenvector of $A(\alpha,m)^T$ corresponding to $\overline{\mu}$ such that $\langle p,q\rangle = 1$. This results in

$$p = \left(-\frac{i\alpha}{\sqrt{4\alpha-m^2}}, \frac{1}{2} + \frac{im}{2\sqrt{4\alpha-m^2}}\right). \tag{4.3}$$

We shall introduce the complex variable

$$z = z(x,y,\alpha,m) = \langle p,(x,y)\rangle = \frac{\alpha\left(mx-2y-ix\sqrt{4\alpha-m^2}\right)}{m^2-4\alpha-im\sqrt{4\alpha-m^2}}. \tag{4.4}$$

The inverse of the transformation may also be given by

$$(x,y) = zq+\overline{zq} = \left(\frac{1}{\alpha}\left(-iz\sqrt{4\alpha-m^2}+\left(m+i\sqrt{4\alpha-m^2}\right)\mathrm{Re}z\right), 2\mathrm{Re}z\right). \tag{4.5}$$

System (4.1) is now transformed into the complex system

$$\begin{aligned} z \mapsto G(z) = G(z,\bar{z},\alpha,m) &= \langle p,\, A(\alpha,m)(zq+\overline{zq}) + f_{\alpha,m}(zq+\overline{zq})\rangle \\ &= \mu z + g(z,\bar{z},\alpha,m), \end{aligned} \tag{4.6}$$

where g is a complex valued smooth function of $z,\bar{z},\alpha$ and m defined by

$$\begin{aligned} g(z,\bar{z},\alpha,m) = 2\alpha &\left(m\mathrm{Re}z + \sqrt{4\alpha-m^2}\mathrm{Im}z - \alpha\tanh\left(\frac{m\mathrm{Re}z+\sqrt{4\alpha-m^2}\mathrm{Im}z}{\alpha}\right)\right) \\ &\cdot \left(4\alpha-m^2+im\sqrt{4\alpha-m^2}\right)^{-1}. \end{aligned} \tag{4.7}$$

It is also clear that for fixed α and m, g is an analytic function of z and $\bar{z}$. Calculating the Taylor expansion of g around 0 with respect to z and $\bar{z}$ we get that it has only cubic and higher order terms (due to the fact that $\tanh''(0) = 0$). That is,

$$g(z,\bar{z},\alpha,m) = \sum_{k+l=3} \frac{g_{kl}}{k!l!} z^k \bar{z}^l + R_1(z), \quad \text{with } k,l \in \{0,1,2,3\}, \tag{4.8}$$

where $g_{kl} = g_{kl}(\alpha,m) = \frac{\partial^{k+l}}{\partial z^k \partial \bar{z}^l} g(z,\bar{z},\alpha,m)\Big|_{z=0}$ for $k+l=3,\ k,l \in \{0,1,2,3\}$ and $R_1(z) = R_1(z,\bar{z},\alpha,m) = O(|z|^4)$ for fixed (α,m).

Theorem 4.3. *Let* $\alpha \in [\frac{1}{2},1)$ *and* $m \in [0,1]$. *If* $(x_0,y_0) \in U(\alpha) = \mathrm{K}_{\varepsilon(\alpha)}$, *where*

$$\varepsilon(\alpha) = \sqrt[4]{\frac{27}{800}}\sqrt{1-\sqrt{\alpha}},$$

then $\lim_{k\to\infty}(x_k,y_k) = (0,0)$.

Proof. Let us study our map in the form (4.6). Let also $(x,y) \in \mathrm{K}_{\varepsilon(\alpha)} \setminus \{(0,0)\}$ be an arbitrary point and $z = z(x,y,\alpha,m)$ be defined by (4.4). We are going to show, that $|G(z,\bar{z},\alpha,m)| < |z|$ if $z \neq 0$. Using equations (4.4) and (4.5) it can be easily shown that for all $\alpha \in [\frac{1}{2},1]$ and $m \in [0,1]$

$$\begin{aligned} \max\{|x|,|y|\} \le \tfrac{2|z|}{\sqrt{\alpha}} \le 2\sqrt{2}|z|, \quad \text{and} \\ |z| \le \sqrt{\tfrac{\alpha(1+\alpha+m)}{4\alpha-m^2}}\max\{|x|,|y|\} \le \tfrac{\sqrt{5}}{2}\max\{|x|,|y|\} \end{aligned} \tag{4.9}$$

hold with $z = z(x,y,\alpha,m)$. Using the Taylor expansion of the tanh function, inequality $\varepsilon(\alpha) < 1$ and that $\max_{|x|\le 1}\left\{\left|\frac{\mathrm{d}^3}{\mathrm{d}x^3}\tanh(x)\right|\right\} = 2$, we get that

$$\begin{aligned}|g(z,\bar{z},\alpha,m)| &= \left|\left\langle p(\alpha,m), f_{\alpha,m}\left(zq(\alpha,m)+\overline{zq(\alpha,m)}\right)\right\rangle\right| \\ &= \sqrt{\tfrac{\alpha}{4\alpha-m^2}}\,\alpha^2|x-\tanh(x)| \\ &\le \sqrt{\tfrac{\alpha}{4\alpha-m^2}}\,\tfrac{\alpha^2}{6}\max_{|x|\le\varepsilon(\alpha)}\left\{\left|\tfrac{\mathrm{d}^3}{\mathrm{d}x^3}\tanh(x)\right|\right\}|x|^3 \\ &= \sqrt{\tfrac{\alpha}{4\alpha-m^2}}\,\tfrac{\alpha^2}{3}|x|^3.\end{aligned}$$

Now, by the first inequality in (4.9) and equation $|\mu| = \sqrt{\alpha}$, we obtain

$$\begin{aligned}|G(z,\bar{z},\alpha,m)| &\le \sqrt{\alpha}|z| + \sqrt{\tfrac{\alpha}{4\alpha-m^2}}\,\tfrac{\alpha^2}{3}(2\sqrt{2})^3|z|^3 \\ &= |z|\cdot\left(\sqrt{\alpha}+\sqrt{\tfrac{\alpha}{4\alpha-m^2}}\,\tfrac{\alpha^2}{3}16\sqrt{2}|z|^2\right).\end{aligned}$$

As $\sqrt{\frac{\alpha}{4\alpha-m^2}}\frac{\alpha^2}{3} \le \frac{1}{3\sqrt{3}}$ holds for all $\alpha\in[\frac{1}{2},1]$ and $m\in[0,1]$, thus $0\ne|z|<\varepsilon_0(\alpha)=\sqrt[4]{\frac{27}{512}}\sqrt{1-\sqrt{\alpha}}$ guarantees $|G(z)|<|z|$. Using the second inequality of (4.9) yields that for $(x,y)\in\mathrm{K}_{\varepsilon(\alpha)}$, inequality $|z| = |z(x,y,\alpha,m)| < \varepsilon_0(\alpha)$ is satisfied. Now, we have $|G(z)|<|z|$ if $z\ne 0$. This implies $G^k(z)\to 0$ as $k\to\infty$. Thus the $(0,0)\in\mathbb{R}^2$ solution is asymptotically stable in $U(\alpha)=\mathrm{K}_{\varepsilon(\alpha)}$ for our original system $(x,y)\mapsto F(x,y)$. ☐

Theorem 4.4. *Let $\alpha\in[0.98,1]$ and $m\in[0,1]$. If $(x_0,y_0)\in U(\alpha)=\mathrm{K}_{\varepsilon(\alpha)}$, where*

$$\varepsilon(\alpha)=\tfrac{1}{6},$$

then $\lim_{k\to\infty}(x_k,y_k)=(0,0)$.

The proof is based on the argument applied in our previous work [2]. As already noted, at $\alpha=1$, the dynamical system defined by $F_{\alpha,m}$ undergoes a Neimark–Sacker bifurcation. However, at $(\alpha,m)=(1,0)$, a strong $1:4$ resonance occurs. We shall transform our system into its $1:4$ resonant normal-form (according to Kuznetsov [18]) to prove the claim of the theorem, as the non-resonant normal form of the Neimark–Sacker bifurcation would not be not applicable near the parameter values $(\alpha,m)=(1,0)$. The reason for that is that we shall need, among others, uniform estimates on the transformation, which is impossible as the parameters tend to the critical pair $(1,0)$. However, the resonant normal form is applicable over the whole region $(\alpha,m)\in[0.98,1]\times[0,1]$. In the following proof we used the assistance of the symbolic toolbox of *Wolfram Mathematica*.

Proof of Theorem 4.4. In this proof, we shall present several estimations. The given bounds shall always be uniform, that is, they hold for all parameter values $\alpha\in[0.98,1]$ and $m\in[0,1]$.

Step 1: Transformation into the 1:4 resonant normal form. Let us consider our system in the form (4.6). We are looking for a smooth complex function $h=h_{\alpha,m}\colon\mathbb{C}\to\mathbb{C}$, which is defined and is invertible on a neighborhood of $0\in\mathbb{C}$ and which transforms our system (4.6) into the following normal form $w\mapsto G_{1:4}(w)=G_{1:4}(w,\overline{w},\alpha,m)$, where

$$G_{1:4}(w)=h^{-1}(G(h(w),\overline{h(w)},\alpha,m))=\mu w+c(\alpha,m)w^2\overline{w}+d(\alpha,m)\overline{w}^3+R_2(w), \quad (4.10)$$

and $R_2(w) = R_2(w,\overline{w},\alpha,m) = O(|w|^4)$ for (α,m) fixed. One can find such a function h by assuming it to be a polynomial of w and $\overline{w}$ with at most cubic terms. This results in

$$h(w) = h(w,\overline{w},\alpha,m) = w + \tfrac{h_{30}}{6}w^3 + \tfrac{h_{12}}{2}w\overline{w}^2, \tag{4.11}$$

and

$$h^{-1}(z) = h^{-1}(z,\bar{z},\alpha,m) = z - \tfrac{h_{30}}{6}z^3 - \tfrac{h_{12}}{2}z\bar{z}^2 + R_3(z), \tag{4.12}$$

where

$$h_{30} = h_{30}(\alpha,m) = \tfrac{g_{30}}{\mu(\mu^2-1)}, \quad h_{12} = h_{12}(\alpha,m) = \tfrac{g_{12}}{2\mu(\overline{\mu}^2-1)}$$

and $R_3(z) = R_3(z,\bar{z},\alpha,m) = O(|z|^4)$ for (α,m) fixed. The domains of h and h^{-1} are to be defined later.

Our aim is now to find $\varepsilon_0 > 0$ such that for all $(x,y) \in \left[-\frac{1}{6},\frac{1}{6}\right]^2$, $|w| < \varepsilon_0$ is satisfied and for all $0 \neq |w| < \varepsilon_0$ the following inequality holds

$$|G_{1:4}(w)| = |\mu w + c(\alpha,m)w^2\overline{w} + d(\alpha,m)\overline{w}^3 + R_2(w,\overline{w},\alpha,m)| < |w|. \tag{4.13}$$

To find ε_0, we need several uniform estimations on the higher order (error) terms R_1, R_2 and R_3, on the transformations h and $(x,y) \mapsto z$ and their inverses and on the functions g, c and d, as well.

Step 2: Estimations. ***Estimation of g and R_1***

First of all, it can be easily shown from equations (4.4) and (4.5) that the following inequalities hold

$$\begin{gathered} \max\{|x|,|y|\} \leq \tfrac{2|z|}{\sqrt{\alpha}} \leq 2.03|z| \quad \text{and} \\ |z| \leq \sqrt{\tfrac{\alpha(1+\alpha+m)}{4\alpha-m^2}}\max\{|x|,|y|\} \leq 1.01\cdot\max\{|x|,|y|\}, \end{gathered} \tag{4.14}$$

for all $\alpha \in [0.98,1]$, $m \in [0,1]$. Now, it is clear from the Taylor expansion of the tanh function and from equations (4.2), (4.3) and (4.5) that

$$|R_1(z)| \leq \left|\frac{1}{2} + \frac{im}{2\sqrt{4\alpha-m^2}}\right| \cdot \frac{\alpha}{120}\cdot \max_{|x|\leq\frac{1}{6}}\left\{\left|\frac{\mathrm{d}^5}{\mathrm{d}x^5}\tanh(x)\right|\right\}\cdot|x|^5 = \sqrt{\frac{\alpha}{4\alpha-m^2}}\frac{2\alpha^2}{15}\cdot|x|^5$$

is satisfied if $z = z(x,y,\alpha,m)$. Now, using rigorous estimations, from the first inequality of (4.14) and from $|x| \leq \frac{1}{6}$ it can be readily shown that

$$|R_1(z)| \leq 0.22|z|^4. \tag{4.15}$$

We have the following explicit formulas for the third order terms of g

$$\begin{aligned} g_{30} &= \frac{2i\alpha - m\left(im+\sqrt{4\alpha-m^2}\right)}{\alpha\sqrt{4\alpha-m^2}}, & g_{21} &= -\frac{2i}{\sqrt{4\alpha-m^2}}, \\ g_{12} &= \frac{2\left(m+i\sqrt{4\alpha-m^2}\right)}{4\alpha-m^2+im\sqrt{4\alpha-m^2}}, & g_{03} &= \frac{\left(m+i\sqrt{4\alpha-m^2}\right)^3}{2\alpha\left(4\alpha-m^2+im\sqrt{4\alpha-m^2}\right)}. \end{aligned} \tag{4.16}$$

By symbolic calculations, we obtain that

$$\sum_{k+l=3}\frac{|g_{kl}|}{k!l!} = \frac{8}{3\sqrt{4\alpha-m^2}} < 1.57, \text{ with } k,l \in \{0,1,2,3\}. \tag{4.17}$$

Inequalities (4.15) and (4.17) together with equations (4.6) and (4.8) yield in particular that

$$|G(z)| \leq |z| + 1.57|z|^3 + 0.22|z|^4. \tag{4.18}$$

A region where transformation h is valid, and estimation of h

We are going to show that the transformation h, defined by equation (4.11), is injective on $\overline{\mathrm{B}_{1/2}} \subset \mathbb{C}$ and that its inverse h^{-1} is defined on $\overline{\mathrm{B}_{1/3}}$ and has the form (4.12).

The following equations and upper bound can be easily obtained

$$|h_{30}| = |h_{12}| = \frac{2}{\sqrt{\alpha\left(4\alpha-m^2\right)\left((1+\alpha)^2-m^2\right)}} < 0.7. \tag{4.19}$$

Let

$$H_z = H_{\alpha,m,z} \colon \mathbb{C} \ni w \mapsto w + z - h(w) \in \mathbb{C}.$$

By this notation, $H_z(w) = w$ holds if and only if $h(w) = z$. Let us make the following observation.

$$\begin{aligned}|H_z(w_1) - H_z(w_2)| &= |w_1 - h(w_1) - w_2 + h(w_2)| \\ &\leq \tfrac{|h_{30}|}{6}\left|w_1^3 - w_2^3\right| + \tfrac{|h_{12})|}{2}\left|\overline{w}_1|w_1|^2 - \overline{w}_2|w_2|^2\right|.\end{aligned}$$

Note also that

$$\begin{aligned}\left|\overline{w}_1|w_1|^2 - \overline{w}_2|w_2|^2\right| &\leq \left|\overline{w}_1|w_1|^2 - \overline{w}_1|w_2|^2\right| + \left|\overline{w}_1|w_2|^2 - \overline{w}_2|w_2|^2\right| \\ &= |w_1|\left(|w_1|^2 - |w_2|^2\right) + |w_2|^2|\overline{w}_1 - \overline{w}_2| \\ &\leq |w_1|(|w_1| - |w_2|)(|w_1| + |w_2|) + |w_2|^2|w_1 - w_2| \\ &\leq |w_1 - w_2|\left(|w_1|^2 + |w_1||w_2| + |w_2|^2\right).\end{aligned}$$

Now, if $w_1, w_2 \in \mathrm{B}_{1/2}$ are arbitrary and $z \in \mathrm{B}_{1/3}$ is fixed, then we have the following estimations

$$\begin{aligned}|H_z(w_1) - H_z(w_2)| &\leq |w_1 - w_2| \cdot \left(\tfrac{|h_{30}|}{6} + \tfrac{|h_{12}|}{2}\right)\left(|w_1|^2 + |w_1||w_2| + |w_2|^2\right) \\ &\leq 0.47|w_1 - w_2| \cdot 3 \cdot \tfrac{1}{4} \leq |w_1 - w_2|,\end{aligned}$$

and

$$|H_z(w)| \leq |z| + |w - h(w)| \leq |z| + \left(\tfrac{|h_{30}|}{6} + \tfrac{|h_{12}|}{2}\right)|w|^3 \leq \tfrac{1}{3} + 0.47 \cdot 2 \cdot \tfrac{1}{8} < \tfrac{1}{2}.$$

We obtained that $H_{\alpha,m,z} \colon \overline{\mathrm{B}_{1/2}} \to \overline{\mathrm{B}_{1/2}}$ is a contraction. Hence for all fixed $z \in \overline{\mathrm{B}_{1/3}}$ there exists exactly one $w = w(z) \in \overline{\mathrm{B}_{1/2}}$ such that $H_z(w(z)) = w(z)$, that is $h(w(z)) = z$. This means that h^{-1} can be defined on $\overline{\mathrm{B}_{1/3}}$.

It is also clear from equation (4.11) and inequality (4.19) that

$$|w| - 0.47|w|^3 \leq |h(w)| \leq |w| + 0.47|w|^3. \tag{4.20}$$

Estimation of h^{-1}

Using inequalities (4.20) and assuming $w \in \mathrm{B}_{1/5}$, $z = h(w)$ yield the following inequality

$$|w| \leq 1.02|h^{-1}(z)|. \tag{4.21}$$

In order to have a similar upper estimation on its inverse h^{-1}, as well, we need to estimate the remainder term R_3. Let us assume that $z \in \mathrm{B}_{1/3}$. Since h^{-1} is defined on $\mathrm{B}_{1/3}$, hence there exists exactly one number w in $\mathrm{B}_{1/2}$ such that $z = h(w)$. Now, we have

$$R_3(z) = R_3(h(w)) = h^{-1}(h(w)) - h(w) + \tfrac{h_{30}}{6}(h(w))^3 + \tfrac{h_{12}}{2}h(w)\left(\overline{h(w)}\right)^2,$$

a polynomial of w and $\overline{w}$ having only fourth to ninth order terms. Assuming now $w \in \mathrm{B}_{1/5}$ and using inequalities (4.19) and (4.21) we obtain that

$$R_3(z) \leq 0.14|w|^4 < 0.16|z|^4$$

is satisfied for $z = h(w)$. This inequality combined with equation (4.12) and inequality (4.19) yields that if $w \in \mathrm{B}_{1/5}$ and $z = h(w)$, then

$$\left|h^{-1}(z)\right| \leq |z| + 0.47|z|^3 + 0.16|z|^4 \tag{4.22}$$

holds.

Estimation of R_2

Now, we are ready to estimate R_2. Let us define the following three polynomials

$$\begin{aligned} h^{-1;\max}(s) &= s+0.47s^3+0.16s^4, \\ G^{\max}(s) &= s+1.57s^3+0.22s^4, \\ h^{\max}(s) &= s+0.47s^3. \end{aligned} \tag{4.23}$$

Let also $Q(s)=\sum_{k=1}^{48} q_k s^k = h^{-1;\max}\circ G^{\max}\circ h^{\max}(s)$. It is obvious from our previous estimations that for $0\neq w\in \mathrm{B}_{1/5}$, we have $|R_2(w)|<\sum_{k=4}^{48} q_k|w|^4\left(\frac{1}{5}\right)^{k-4}$, which leads to

$$|R_2(w)|<1.59|w|^4. \tag{4.24}$$

Step 3: A region of attraction for the fixed point 0 *of system* (4.10). From equations (4.10),(4.11), (4.12) and (4.16) one can readily derive the formulas

$$c=c(\alpha,m)=-\frac{i}{\sqrt{4\alpha-m^2}},\quad d=d(\alpha,m)=\frac{\left(m+i\sqrt{4\alpha-m^2}\right)^3}{12\alpha\left(4\alpha-m^2+im\sqrt{4\alpha-m^2}\right)}. \tag{4.25}$$

Let

$$\beta=\beta(\alpha,m)=\frac{|\mu(\alpha,m)|}{\mu(\alpha,m)}c(\alpha,m)=\frac{-im-\sqrt{4\alpha-m^2}}{2\sqrt{\alpha}\sqrt{4\alpha-m^2}}$$

and let $\gamma=\gamma(\alpha)$ denote the real part of β, which is $\gamma=-\frac{1}{2\sqrt{\alpha}}$. Using these notations and inequality (4.24) we obtain that for all $0\neq w\in \mathrm{B}_{1/5}$ we have

$$\begin{aligned} |G_{1:4}(w)| &= \left|\mu w+c(\alpha,m)w^2\overline{w}+d(\alpha,m)\overline{w}^3+R_2(w)\right| \\ &\le |w|\left(\left|\mu+c|w|^2\right|+|d||w|^2\right)+|R_2(w)| \\ &= |w|\left(\left||\mu|+\beta|w|^2\right|+|d||w|^2\right)+|R_2(w,\overline{w},\alpha)| \\ &< |w|\left(\left|\sqrt{\alpha}+\beta|w|^2\right|+|d||w|^2+1.59|w|^3\right) \\ &\le |w|\left(\left|\sqrt{\alpha}+\gamma|w|^2\right|\right) \\ &\quad +|w|\left(\left|\left|\sqrt{\alpha}+\beta|w|^2\right|-(\sqrt{\alpha}+\gamma|w|^2)\right|+|d||w|^2+1.59|w|^3\right). \end{aligned} \tag{4.26}$$

Note that $-1<-\frac{5}{7\sqrt{2}}\le\gamma\le-\frac{1}{2}$. Now supposing $0\neq w\in \mathrm{B}_{1/5}$ yields the following

$$\begin{aligned} \left|\left|\sqrt{\alpha}+\beta|w|^2\right|-(\sqrt{\alpha}+\gamma|w|^2)\right| &= \left|\sqrt{\alpha+2\sqrt{\alpha}\gamma|w|^2+|\beta|^2|w|^4}-(\sqrt{\alpha}+\gamma|w|^2)\right| \\ &= \left|\frac{(|\beta|^2-\gamma^2)|w|^4}{\sqrt{\alpha+2\sqrt{\alpha}\gamma|w|^2+|\beta|^2|w|^4}+\sqrt{\alpha}+\gamma|w|^2}\right| \\ &\le \frac{(|\beta|^2-\gamma^2)|w|^4}{\sqrt{25\alpha|w|^2+2\sqrt{\alpha}\gamma|w|^2}+5\sqrt{\alpha}|w|+\gamma|w|} \\ &\le \frac{(|\beta|^2-\gamma^2)}{\sqrt{25\alpha-2}+5\alpha-1}|w|^3. \end{aligned}$$

Using the formulas for γ,β and d, one readily get that the following inequalities hold

$$\left|\left|\sqrt{\alpha}+\beta|w|^2\right|-(\sqrt{\alpha}+\gamma|w|^2)\right|<0.02|w|^3, \tag{4.27}$$

and

$$|d|\le\frac{5}{3\sqrt{73}}<\frac{1}{5}. \tag{4.28}$$

Combining inequalities (4.26), (4.27) and (4.28) we obtain that for $0 \neq w \in \mathrm{B}_{1/5}$ we have

$$\begin{aligned} |G_{1:4}(w)| &< |w| \left(1 - 0.5|w|^2 + 0.2|w|^2 + 1.61|w|^3\right) \\ &= |w| \left(1 - |w|^2(0.3 - 1.61|w|)\right) < |w|, \end{aligned}$$

provided that $|w| < \varepsilon_0 = \frac{0.3}{1.61}$. This proves the asymptotic stability of the 0 fixed point of system (4.10) in the region $\mathrm{B}_{\varepsilon_0}$.

Step 4: The 0 *fixed point of system* (4.1) *is asymptotically stable in the region* $[-\frac{1}{6}, \frac{1}{6}]^2$. Inequalities (4.14) and (4.21) imply that for all $(x, y) \in [-\frac{1}{6}, \frac{1}{6}]^2$, $w \in \mathrm{B}_{\varepsilon_0}$ is satisfied. This guarantees that given $(x_0, y_0) \in U(\alpha) = [-\frac{1}{6}, \frac{1}{6}]^2$, we have $\lim_{k\to\infty}(x_k, y_k) = (0, 0)$ and completes our proof.

□

Figure 6 illustrates how $U(\alpha)$ changes with the parameter α.

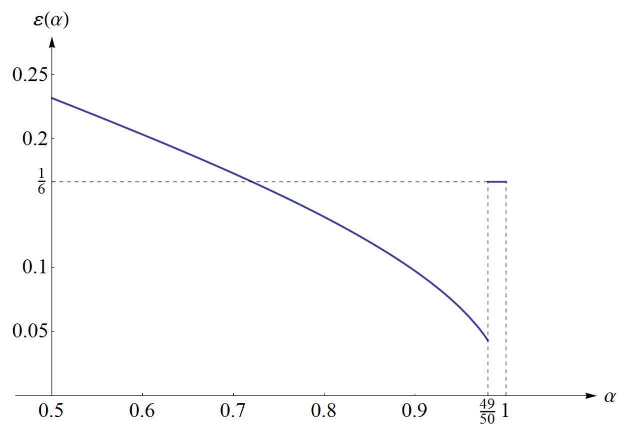

FIGURE 6. $U(\alpha)$ is the square with sides $2\varepsilon(\alpha)$, centered at $(0,0)$. $\varepsilon(\alpha)$ is obtained from Theorem 4.3 for $\alpha \in [\frac{1}{2}, \frac{49}{50}]$ and from Theorem 4.4 for $\alpha \in [\frac{49}{50}, 1]$.

Part II: Rigorous computations. Consider now a pair of parameter values $(\alpha, m) \in [\frac{m^2+1}{|m|+1}, 1] \times [0, 1]$. Given any starting point (x_0, y_0), the accumulation points of its orbit $((x_k, y_k))_{k=0}^{\infty}$ are non-wandering points of $F_{\alpha,m}$. In order to prove that the fixed point $(0,0)$ is globally attracting, it is enough to show that it is the only non-wandering point of $F_{\alpha,m}$. We know from Corollary 3.7, that all the non-wandering points are inside $S = [-4, 4]^2$. We shall show that S lies entirely in the basin of attraction of $(0,0)$, or equivalently, S contains exactly one non-wandering point, and that is $(0,0)$.

In the remaining part of the paper we emphasize, that $[\alpha], [m], [S]$ and $[U]$ are quantities that are represented in the computer as intervals or interval boxes, while $F_{[\alpha],[m]}$ is an interval valued function. Even though the sets are handled numerically, they provide rigorous enclosures of the number or set between the brackets. For any $(\alpha, m) \in [\alpha] \times [m]$ and for any $(x, y) \in [S]$, we have $F_{\alpha,m}(x, y) \in F_{[\alpha],[m]}(x, y)$. This is achieved by using the CAPD Library [38] for validated computations.

To proceed with the proof, first we divide the parameter range into small interval boxes $[\alpha] \times [m]$. Given one small box and a starting resolution δ, we shall run the procedure

`Global_Stability`, that appeared as Algorithm 3 together with a proof of its correctness in [2]. The algorithm uses partitions and graph representations. For a detailed introduction the reader is referred to [2].

1: **procedure** GLOBAL_STABILITY($[\alpha],[m],\delta$)
2: $[S] \leftarrow [-4,4]^2$
3: $[U] \leftarrow \cap_{\alpha\in[\alpha]} U(\alpha)$ ▷ from Theorems 4.3 and 4.4
4: $\mathscr{V} \leftarrow$ Partition($[S],\delta$) ▷ $\mathscr{V}$ is a partition of $[S]$, diam$(\mathscr{V}) \leq \delta$
5: **repeat**
6: $\mathscr{E} \leftarrow$ Transitions($\mathscr{V}$, $F_{[\alpha],[m]}$)
7: $\mathscr{G} \leftarrow$ GRAPH($\mathscr{V},\mathscr{E}$) ▷ $\mathscr{G}$ is a graph representation of $F_{[\alpha],[m]}$
8: $T \leftarrow \{v : v$ is in a directed cycle $\}$
9: **for all** $v \in \mathscr{V}$ **do**
10: **if** $v \notin T$ **or** $v \subseteq [U]$ **or** $F_{[\alpha],[m]}(v) \subseteq [U]$ **then**
11: **remove** v from $\mathscr{G}$
12: **end if**
13: **end for**
14: $\delta \leftarrow \delta/2$
15: $\mathscr{V} \leftarrow$ Partition($|\mathscr{V}|,\delta$)
16: **until** $|\mathscr{V}| = \emptyset$
17: **end procedure**

To obtain a simple picture of what the algorithm does, notice that it utilizes graph representations of the function $F_{[\alpha],[m]}$ over nested compact sets and with respect to partitions of decreasing diameter. The next (smaller) compact set is obtained by removing certain partition elements in line 11. A vertex v is removed from the graph representation only when we manage to establish that either it does not contain any non-wandering point or it lies inside the basin of attraction of the origin.

If the procedure ends in finite time, that is, at one point $|\mathscr{V}| = \emptyset$ is satisfied, it implies that the origin is the only non-wandering point in $[S]$, thus it is globally attracting for all parameter pairs inside the given box $[\alpha]\times[m]$.

The code is implemented in C++. The CAPD Library [38] and the Boost Graph Library [28] were used for obtaining rigorous computations and handling directed graphs respectively. We used Tarjan's algorithm [29] in order to find the directed cycles. We used different sizes for the parameter intervals and ran the computations on a cluster of the NIIF HPC centre at the University of Szeged parallelizing it with OpenMP. We covered the region $(\alpha,m) \in [\frac{m^2+1}{|m|+1},1]\times[0,1]$ using 6964 parameter intervals $[\alpha]\times[m]$ of size between 0.01×0.01 and 0.001×0.001. The iteration count (that is one cycle in program `Global_Stability`) varied from 10 to 25. The computation took 67 minutes and 54 seconds, while the total run time, summing for all the simultaneous processes was 11 hours 47 minutes and 3 seconds.

Proof of Theorem 4.2. The program `Global_Stability` ran successfully for every parameter box. Combining this with Theorem 4.3 and Theorem 4.4, proves that $(0,0)$ is globally attracting for $(\alpha,m) \in [\frac{m^2+1}{|m|+1},1]\times[0,1]$. The output of these computations can be found at [36]. These results, together with Theorem 3.5, Corollary 3.7 and Remark 4.1, prove the global attractivity of $(0,0)$ for $(\alpha,m) \in \mathscr{R}(m)$, and thus complete the proof of Theorem 4.2. □

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