SERIES EXPANSIONS FOR RANDOM DISC-POLYGONS IN SMOOTH PLANE CONVEX BODIES

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Abstract. We establish power series expansions for the asymptotic expectations of the vertex number and missed area of random disc-polygons in planar convex bodies with C^{k+1}_+ smooth boundaries. These results extend asymptotic formulas proved in [11].

1. Introduction and results

Reconstructing a possibly unknown set, or some of its characteristic quantities, from a random sample of points is a classical and a much investigated problem that arises naturally in various fields, like stereology (see, for example, Baddeley and Jensen [1]), computational geometry (see Goodman, O'Rourke and Tóth [13], statistical quality control (see Devroye and Wise [8]), etc. Estimating the shape, volume, surface area, and other characteristic quantities of sets is of interest both in geometry and statistics, although the investigated aspects are in many cases different in the respective fields. For an overview of set estimation see, for example, Cuevas and Rodríguez-Casal [7]. The set may be quite arbitrary but often various restrictions are imposed on it. One common such restriction that received much attention is when the set is required to be convex. In such a setting polytopes spanned by random samples of points from the set form a natural estimator. The theory of random polytopes is a rich and lively field with numerous applications. For a recent review and further references see, for example, Schneider [31]. The convex hull is an optimal estimator if no other restrictions are imposed on the set other than convexity. However, in this paper we study another estimator under further assumptions on K, namely, that the degree of smoothness of the boundary of K is prescribed to be C^{k+1} and it also assumed that the curvature is positive everywhere. Under these circumstances, using congruent circles to form the hull of the sample yields better performance than the classical convex hull.

Since the case when the number of random points is fixed is notoriously difficult, it has become common to investigate the asymptotic behaviour of functionals associated with random polytopes as the number of points in the sample tends to infinity. The investigations of the asymptotic behaviour of random polytopes started with the classical papers by Rényi and Sulanke [26, 27] in the 1960s. They studied the following particular model in the plane. Let K be a convex body (a compact convex set with nonempty interior) in d dimensional Euclidean space $\mathbb{R}^d$ and let $x_1, \dots, x_n$ be independent random points from K selected according to the uniform probability distribution.

The convex hull $K_n = [x_1, \dots, x_n]$ of $x_1, \dots, x_n$ is called a (uniform) random polytope in K. Rényi and Sulanke [26, 27] proved asymptotic formulas in the plane for the expected number $f_0(K_n)$ of vertices of K_n and the expectation of the missed area $A(K \setminus K_n)$ under the assumption that the boundary ∂K of K is sufficiently smooth, and also in the case when K itself is a convex polygon. Wieacker [35] extended this to the d-dimensional ball B^d, and Bárány [2] for d-dimensional convex bodies with at least C^3_+ smooth boundary (three times continuously differentiable with everywhere positive Gauss-Kronecker curvature). Schütt [33] removed all smoothness conditions, and Böröczky, Fodor and Hug [6] extended the results for nonuniform distributions and weighted volume difference.

Let $V_i(\cdot)$, $i = 1, \ldots, d$ denote the i-th intrinsic volume of a convex body. Reitzner [24] established a power series expansion of the quantity $\mathbb{E}(V_i(K) - V_i(K_n))$ for all $i = 1, \ldots, d$ as $n \to \infty$ under stronger smoothness conditions on the boundary of K.

Theorem 1 ([24]). *Let K be a convex body in $\mathbb{R}^d$ with $V_d(K) = 1$ whose boundary ∂K is C^{k+1}_+ for some integer $k \geq 2$. Then*

$$\begin{aligned}&\mathbb{E}(V_i(K) - V_i(K_n))\\ &\quad = c_2^{(i,d)}(K)n^{-\frac{2}{d+1}} + c_3^{(i,d)}(K)n^{-\frac{3}{d+1}} + \ldots + c_k^{(i,d)}(K)n^{-\frac{k}{d+1}} + O(n^{-\frac{k+1}{d+1}}) \qquad (1)\end{aligned}$$

as $n \to \infty$. Moreover, $c_{2m+1}^{(i,d)} = 0$ for all $m \leq d/2$ if d is even, and $c_{2m+1}^{(i,d)} = 0$ for all m if d is odd.

Under the same conditions as in Theorem 1, one can obtain from (1) a series expansion for the number of vertices $\mathbb{E}(f_0(K_n))$ via Efron's identity [10]

$$\mathbb{E}(f_0(K_n)) = d_2(K)n^{\frac{d-1}{d+1}} + d_3 n^{\frac{d-2}{d+1}} + \ldots + d_k(K)n^{\frac{d-k+1}{d+1}} + O(n^{\frac{d-k+2}{d+1}})$$

as $n \to \infty$.

Gruber [14] proved the case of Theorem 1 when $i = 1$. Using properties of the convex floating body, Reitzner established the planar case for the area ($d = 2$, $i = 2$) of Theorem 1 in [23]. In particular, Reitzner proved that

$$d_4(K) = c_4^{(2,2)}(K) = -\Gamma\left(\frac{7}{3}\right)\frac{1}{5}\sqrt[3]{\frac{3}{2}}\int_{\partial K} k(x)\kappa^{\frac{1}{3}}(x)\,\mathrm{d}x, \qquad (2)$$

where $\Gamma(\cdot)$ is Euler's gamma function, $k(x)$ is the affine curvature (for information about the affine curvature see, for example [5, pp. 12–15] or [15, Section 7.3].) and $\kappa(x)$ is the curvature of ∂K at x, and integration on the boundary ∂K of K is with respect to arc-length.

For more information about approximations of convex bodies by classical random polytopes we refer to the book by Schneider and Weil [32], and the survey articles by Bárány [3], Reitzner [25], and by Schneider [31], and by Weil and Wieacker [34].

When estimating a planar convex body under curvature restrictions, naturally, it may be more advantageous to use suitably curved arcs to form the boundary of the approximating set that fit K better than line segments. One of the simplest such constructions uses radius R circular arcs and the resulting (convex) hull is called, among other names, the R-spindle convex hull, for precise definitions see below. The radius should be chosen in such a way that the (generalised) random polygon is still contained in K. This imposes the condition on R that it should be at least as large as the maximum radius of curvature of ∂K. However, similarly to the classical convex case, difficulties arise when R is equal to the maximal radius of curvature, so this case usually needs a separate treatment using different methods.

In this paper, we study the R-spindle convex variant of the above probability model in the Euclidean plane $\mathbb{R}^2$. Let $R > 0$ be fixed, and let $x, y \in \mathbb{R}^2$ be such that their distance is at most $2R$. We call the intersection $[x, y]_R$ of all (closed circular) discs of radius R that contain both x and y the R-spindle of x and y. A set $X \subseteq \mathbb{R}^2$ is called *R-spindle convex* if from $x, y \in X$ it follows that $[x, y]_R \subseteq X$. Spindle convex sets are also convex in the usual linear sense. In this paper we restrict our attention to compact spindle convex sets. One can show (cf. Corollary 3.4 on page 205 in [4]) that a convex body in $\mathbb{R}^2$ is R-spindle convex if it is the intersection of (not necessarily finitely many) closed discs of radius R. The intersection of finitely many closed discs of radius R is called a convex R-disc-polygon. Let X be a compact set which is contained in a closed disc of radius R. The intersection of all planar R-spindle convex bodies containing X is called the R-spindle convex hull of X, and it is denoted by $[X]_R$. Perhaps it is easier to grasp this notion if we point out the similarity with the classical convex hull. In the R spindle convex case the radius R discs play a similar role to what closed half-spaces do for classical convex hulls. Thus, in a heuristic way, one can consider the classical convex hull as a limiting case as $R \to \infty$. If $X \subset K$ for an R-spindle convex body K in $\mathbb{R}^2$, then $[X]_R \subset K$. A prominent class of R-spindle convex sets in $\mathbb{R}^2$ that are directly relevant in this paper is provided by convex bodies whose boundary is C^2_+ smooth with curvature $\kappa(x) \geq 1/R$ for all boundary points $x \in \partial K$ (see [30, §2.5 and 3.2]). For more detailed information about spindle convexity we refer to Bezdek et al. [4] and Martini, Montejano and Oliveros [19].

We note that there exist further generalisations of spindle convexity, most notably, the concept of L-convexity in which the translates of a fixed convex body L play the role of the radius R closed disc, for more information see, for example, Lángi, Naszódi and Talata [17]. Another further generalisation is H-convexity introduced by Kabluchko, Marynych and Molchanov [16], where the hull of a set is generated by intersections of transformed copies of a fixed convex set C by a set H of affine transformations. A similar concept (see, for example, Mani-Levitska [18]) to R-spindle convexity, called α-convexity, also exists, where the α-convex hull of a set is defined as the complement of the union of all radius r open balls disjoint from the set. The α-convex hull of a finite sample is different from its R-spindle convex hull as it is nonconvex while the R-convex hull is always convex. We note that the α-convex hull can be used to estimate not necessarily convex sets as well, see, for example, Paterio-Lopez and Rodríguez-Casal [22], Rodríguez-Casal [28] and Pateiro-López [21], where several such results are proved about random samples chosen from the set according to an absolute continuous probability distribution.

A convex R-disc-polygon is clearly R-spindle convex. We consider a single radius R disc and a single point also R-disc-polygons, albeit trivial ones. The non-smooth points of the boundary of a nontrivial convex R-disc-polygon are called vertices. The vertices divide the boundary into a union of radius R circular arcs of positive arc-length, we call edges. Thus, a nontrivial convex R-disc-polygon has an equal number of edges and vertices, just like a classical convex polygon, except the sides are radius R circular arcs. The radius R disc has one edge and no side, and a single point has one vertex and no side.

Our probability model is the following. Let K be convex body in $\mathbb{R}^2$ with at least C^2_+ smooth boundary and let R be such that $\kappa(x) > 1/R$ for all $x \in \partial K$. Let $x_1, \ldots, x_n$ be independent random points in K chosen according to the uniform probability distribution. The R-spindle convex hull $K_n^R = [x_1, \ldots, x_n]_R$ is called a *uniform random R-disc-polygon* in K, which is a convex R-disc-polygon. It is clear that K_n^R has an equal number of vertices and sides with probability one, and its vertex set is formed by some of the random points $x_1, \ldots, x_n$. Let $f_0(K_n^R)$ denote the number of vertices of K_n^R. We note that in [21] the radius r_n of the discs used in the estimation of an α-convex set tends to zero as $n \to \infty$. In our model, we use suitable fixed radius discs in order to guarantee that the R-spindle convex hull of the random sample is contained in K. However, after the statements of our main results, we briefly discuss what happens to the quality of the approximation when the radius R tends to the limits of its possible range.

Fodor, Kevei and Vígh proved [11, Thm 1.1 on p. 901] that under the above conditions the following hold.

$$\mathbb{E}(f_0(K_n^R)) = z_1(K) n^{\frac{1}{3}} + o\left(n^{\frac{1}{3}}\right), \tag{3}$$

$$\mathbb{E}(A(K \setminus K_n^R)) = A(K) z_1(K) n^{-\frac{2}{3}} + o\left(n^{-\frac{2}{3}}\right), \tag{4}$$

as $n \to \infty$, where

$$z_1(K) = \sqrt[3]{\frac{2}{3A(K)}} \cdot \Gamma\left(\frac{5}{3}\right) \int_{\partial K} \left(\kappa(x) - \frac{1}{R}\right)^{1/3} \mathrm{d}x.$$

In the above formula $A(K)$ denotes the area of K.

We note that (3) and (4) are connected by an Efron-type [10] identity (see [11, (5.10.) on p. 910]), which states that

$$\mathbb{E}(f_0(K_n^R)) = n \frac{\mathbb{E}(A(K \setminus K_{n-1}^R))}{A(K)}.$$

In this paper we prove the following theorems that provide a power series expansion of $\mathbb{E}(f_0(K_n^R))$ and $\mathbb{E}(A(K \setminus K_n^R))$ in the case when ∂K satisfies stronger differentiability conditions.

Theorem 2. *Let $k \geq 2$ be an integer, and let K be a convex body in $\mathbb{R}^2$ with C_+^{k+1} smooth boundary. Then for all $R > \max_{x \in \partial K} 1/\kappa(x)$ it holds that*

$$\mathbb{E}(f_0(K_n^R)) = z_1(K) n^{\frac{1}{3}} + \ldots + z_{k-1}(K) n^{-\frac{k-3}{3}} + O(n^{-\frac{k-2}{3}})$$

as $n \to \infty$. All coefficients $z_1, \ldots, z_k$ can be determined explicitly. In particular,

$$\begin{aligned}
z_1(K) &= \sqrt[3]{\frac{2}{3A(K)}} \Gamma\left(\frac{5}{3}\right) \int_{\partial K} \left(\kappa(x) - \frac{1}{R}\right)^{\frac{1}{3}} \mathrm{d}x, \\
z_2(K) &= 0, \\
z_3(K) &= -\Gamma\left(\frac{7}{3}\right) \frac{1}{5} \sqrt[3]{\frac{3A(K)}{2}} \int_{\partial K} \frac{\kappa''(x)}{3(\kappa(x) - \frac{1}{R})^{\frac{4}{3}}} \\
&\qquad + \frac{2R^2\kappa^2(x) + 7R\kappa(x) - 1}{2R^2(\kappa(x) - \frac{1}{R})^{\frac{1}{3}}} - \frac{5(\kappa'(x))^2}{9(\kappa(x) - \frac{1}{R})^{\frac{7}{3}}} \mathrm{d}x.
\end{aligned}$$

By the spindle convex version of Efron's identity we obtain the following corollary.

Theorem 3. *Let $k \geq 2$ be an integer, and let K be a convex body in $\mathbb{R}^2$ with C_+^{k+1} smooth boundary. Then for all $R > \max_{x\in\partial K} 1/\kappa(x)$ it holds that*

$$\mathbb{E}(A(K \setminus K_n^R)) = z_1'(K)n^{-\frac{2}{3}} + \ldots + z_{k-1}'(K)n^{-\frac{k}{3}} + O(n^{-\frac{k+1}{3}})$$

as $n \to \infty$, where $z_i'(K) = A(K)z_i(K)$ for $i = 1, \ldots, k$.

We note that we only evaluate $z_i(K)$, $i = 1, 2, 3$ explicitly in this paper because the calculation, although possible, becomes more complicated as i increases, even when K is a closed disc. The coefficients $z_i(K)$ depend only on R, the area of K, and on the power series expansion of the local representation of the boundary of K, see (9), in particular, the derivatives of κ up to order $i - 1$.

Although Theorems 2 and 3 are only valid for $R > R_M = \max_{x\in\partial K} 1/\kappa(x)$, it may also be interesting to look at the behaviour of the coefficients $z_i(K)$ at the limits of the range of R. When $R \to \infty$, the integral in $z_1(K)$ tends to the affine arc-length of ∂K, see [11]. For $z_3(K)$, direct calculation yields that

$$\lim_{R\to\infty} \frac{\kappa''(x)}{3(\kappa(x) - \frac{1}{R})^{\frac{4}{3}}} + \frac{2R^2\kappa^2(x) + 7R\kappa(x) - 1}{2R^2(\kappa(x) - \frac{1}{R})^{\frac{1}{3}}} - \frac{5(\kappa'(x))^2}{9(\kappa(x) - \frac{1}{R})^{\frac{7}{3}}} = k(x)\kappa^{\frac{1}{3}}(x),$$

where $k(x)$ is the affine curvature of ∂K at x, cf. also (1).

On the other hand, when $R \to R_M^+$, then

$$\lim_{R\to R_M^+} z_1(K) = \sqrt[3]{\frac{2}{3A(K)}}\Gamma\left(\frac{5}{3}\right) \int_{\partial K} \left(\kappa(x) - \frac{1}{R_M}\right)^{\frac{1}{3}} \mathrm{d}x, \tag{5}$$

where the integrand is bounded, nonnegative, and zero in exactly those points where $\kappa(x) = 1/R_M$. We conjecture that the right-hand-side of (5) is equal to $\lim_{n\to\infty} \mathbb{E}f_0(K_n^R))n^{-1/3}$ when $R = R_M$ and K is not a closed disc. However, this asymptotic expectation is not known.We also note, that $z_1(K)$ is a monotonically decreasing function of R, which shows that it is indeed more advantageous to use circular arcs to form the hull of the random sample of n point in order to approximate K better. Although the order of magnitude in n of the approximation is the same as in the linearly convex case, the main coefficient is smaller.

Furthermore, we note that in the particular case when $K = B^2$ and $R > 1$, then

$$z_1(B) = \sqrt[3]{\frac{2}{3\pi}}\Gamma\left(\frac{5}{3}\right) 2\pi \left(1 - \frac{1}{R}\right)^{\frac{1}{3}}, \quad z_2(B) = 0,$$
$$z_3(B) = -\Gamma\left(\frac{7}{3}\right) \frac{1}{5} \sqrt[3]{\frac{3\pi}{2}} 2\pi \frac{2R^2 + 7R - 1}{2R^2(1 - \frac{1}{R})^{\frac{1}{3}}}.$$

If $R \to 1^+$, then $z_1(B) \to 0$, and $z_3(B) \to -\infty$, and both are monotonically increasing functions showing that the quality of approximation improves as R tends to 1. This behaviour comes as no surprise as the expected number of vertices behaves fundamentally differently from the previously discussed situation when $K \neq B$; the order of magnitude in n is different if $K = B$ as we will see below. Finally, we note that we also suspect that $z_3(K)$ behaves similarly as $z_3(B)$ when $R \to R_M^+$ but this is not clear from its current form.

It was proved in [11] that

$$\mathbb{E}(f_0(B(R)_n^R)) = \frac{\pi^2}{2} + o(1),$$
$$\mathbb{E}(A(B(R) \setminus B(R)_n^R)) = \frac{R^2\pi^3}{2}\frac{1}{n} + o\left(\frac{1}{n}\right)$$

as $n \to \infty$. The unusual behaviour of $\mathbb{E}(f_0(B(R)_n^R))$, i.e. that it tends to a finite constant, was explained by Marynych and Molchanov [20]. They proved, in the much wider context of L-convexity (see also Fodor, Papvári, Vígh [12]) that $\mathbb{E}(f_0(B(R)_n^R))$ tends to the expectation of the number of vertices of the polar of the zero cell of a Poisson line process whose intensity measure on $\mathbb{R}$ is the $A(B(R))^{-1} = 1/(R^2\pi)$ times the Lebesgue measure, and whose directional distribution is uniform on S^1, see [20, (6.1) on page 29]. In Section 4, we calculate the (the first three terms of) the power series expansion of $\mathbb{E}(f_0(B(R)_n^R))$ for the sake of completeness. This gives the speed of convergence of $\mathbb{E}(f_0(B(R)_n^R))$ to $\pi^2/2$. We note that here we only quoted the result of Marynych and Molchanov in the plane, however, they proved it in $\mathbb{R}^d$.

The rest of the paper is organised as follows. In Section 2, we briefly recall from [11] the necessary background and describe how $\mathbb{E}(f_0(B(R)_n^R))$ can be calculated. In Section 3, we provide the power series expansions of the involved geometric quantities. In Section 4, we quote a power series expansion of the incomplete beta function from Gruber [14]. We prove Theorem 2 in Section 5. Finally, in Section 6, we treat the case when $K = B(R)$.

2. Expectation of the number of vertices of K_n^R

Our arguments are based on the methods of Rényi and Sulanke [26] and Gruber [14]. We also note that, compared to those of [21], our methods essentially depend on the higher regularity and smoothness of the boundary of K and the explicit local power series expansion of ∂K. Notice that it is enough to prove the theorem for $R = 1$, from that the statement for general R follows by a scaling argument.

Due to the C_+^{k+1} condition, K is both smooth, i.e. has a unique supporting line at each boundary point, and strictly convex. Let $u_x \in S^1$ denote the unique outer unit normal vector to K at x, and for $u \in S^1$ let x_u be the (again) unique boundary point where the outer unit normal is equal to u.

We use B° to denote the interior of B. A subset D of K is a *disc-cap of K* if $D = K \setminus (B^\circ + p)$ for some point $p \in \mathbb{R}^2$. It was proved in [11] that for a disc-cap of K $D = K \setminus (B^\circ + p)$ there exists a unique point $x_0 \in \partial K \cap D$ and $t \geq 0$ such that $B + p = B + x_0 - (1+t)u_{x_0}$. We call x_0 the vertex and t the height of D.

We may assume that $o \in \operatorname{int} K$. Let $A = A(K) = V_2(K)$. Let $X_n = \{x_1, \ldots, x_n\}$ be a sample of i.i.d. uniform random points from K. For $x_i, x_j \in X_n$, we denote by x_ix_j the shorter unit circular arc connecting x_i and x_j with the property that x_i and x_j are in counterclockwise order on the arc. Let

$$\mathcal{E}(K_n^1) = \{x_ix_j : x_i, x_j \in X_n \text{ and } x_ix_j \text{ is an edge of } K_n^1\}$$

the set of directed edges of K_n^1. For $x_i, x_j \in X_n$, let C_{ij} be the disc-cap of K determined by the disc of x_ix_j, and $A_{ij} = A(C_{ij})$. Note that $x_ix_j \in \mathcal{E}(K_n^1)$ exactly when all the other $n-2$ random points of X_n are in $K \setminus C_{ij}$. Thus, due to the independence of the random points,

$$\begin{aligned}\mathbb{E}(f_0(K_n^1)) &= \sum \frac{1}{A^n}\int_K \cdots \int_K \mathbf{1}\{x_i x_j \in \mathcal{E}(K_n^1)\}\,\mathrm{d}x_1 \ldots \mathrm{d}x_n \\ &= \binom{n}{2}\frac{1}{A^2}\int_K\int_K \left(1-\frac{A_{12}}{A}\right)^{n-2} + \left(1-\frac{A_{21}}{A}\right)^{n-2} \mathrm{d}x_1\mathrm{d}x_2, \qquad (6)\end{aligned}$$

where in the first line summation extends over all ordered pairs of distinct points from X_n. Now, we use the same re-parametrization for the pair (x_1, x_2) as in [11]. Let

$$(x_1, x_2) = \Phi(u, t, u_1, u_2),$$

where $u, u_1, u_2 \in S^1$ and $0 \le t \le t_0(u)$ are chosen such that

$$C(u, t) = C_{12},$$

where $C(u,t)$ is the unique disc-cap of K with vertex x_u and height t, and

$$(x_1, x_2) = (x_u - (1+t)u + u_1, x_u - (1+t)u + u_2).$$

The vectors u_1 and u_2 are the unique outer unit normals of $\partial B + x_u - (1+t)u$ at x_1 and x_2, respectively. For fixed u and t, both u_1 and u_2 are contained in the same arc $L(u,t)$ of S^1 whose length is denoted by $\ell(u,t)$. The uniqueness of the vertex and height of disc-caps guarantees that the map Φ is well-defined, bijective, and differentiable on a suitable domain of (u, t, u_1, u_2). The Jacobian of Φ is

$$|J\Phi| = \left(1 + t - \frac{1}{\kappa(x_u)}\right)|u_1 \times u_2|.$$

Let $A(u,t)$ denote the area of the disc-cap with vertex x_u and height t. For each $u \in S^1$, let $t_0(u)$ be maximal such that $A(u, t_0(u)) \ge 0$. Then, after the change of variables we get from (6) that

$$\begin{aligned}\mathbb{E}(f_0(K_n^1)) &= \binom{n}{2}\frac{1}{A^2}\int_{S^1}\int_0^{t_0(u)}\int_{L(u,t)}\int_{L(u,t)}\left(1-\frac{A(u,t)}{A}\right)^{n-2} \\ &\qquad \times \left(1+t-\frac{1}{\kappa(x_u)}\right)|u_1\times u_2|\mathrm{d}u_1\mathrm{d}u_2\mathrm{d}t\mathrm{d}u \\ &= \binom{n}{2}\frac{1}{A^2}\int_{S^1}\int_0^{t_0(u)}\left(1-\frac{A(u,t)}{A}\right)^{n-2} J(u,t)\mathrm{d}t\mathrm{d}u,\end{aligned}$$

where

$$\begin{aligned}J(u,t) &= \left(1+t-\frac{1}{\kappa(x_u)}\right)\int_{L(u,t)}\int_{L(u,t)}|u_1\times u_2|\mathrm{d}u_1\mathrm{d}u_2 \\ &= 2\left(1+t-\frac{1}{\kappa(x_u)}\right)(\ell(u,t) - \sin\ell(u,t)).\end{aligned}$$

We note that due to the C^2_+ property of ∂K, $J(u,t) \le C$ for some $0 < C \le 6(2\pi+1)$ that depends only on K.

Let $0 < \delta < A$ be an arbitrary but fixed small number. Let $0 < t_1$ be such that for arbitrary $t \in [t_1, t_0(u)]$ and $u \in S^1$ it holds that $A(u,t) \ge \delta$.

Then

$$\begin{aligned}
&\int_{S^1}\int_{t_1}^{t_0(u)}\left(1-\frac{A(u,t)}{A}\right)^{n-2} J(u,t)\mathrm{d}t\mathrm{d}u\\
&\leq C\int_{S^1}\int_{t_1}^{t_0(u)}\left(1-\frac{A(u,t)}{A}\right)^{n-2}\mathrm{d}t\mathrm{d}u\\
&\leq 2\pi C\int_{t_1}^{2}\left(1-\frac{\delta}{A}\right)^{n-2}\mathrm{d}t\\
&\leq 4\pi C\left(1-\frac{\delta}{A}\right)^{n-2},
\end{aligned}$$

thus, in particular, with the choice of a suitably small δ,

$$\mathbb{E}(f_0(K_n^1)) = \binom{n}{2}\frac{1}{A^2}\int_{S^1}\int_0^{t_1}\left(1-\frac{A(u,t)}{A}\right)^{n-2} J(u,t)\,\mathrm{d}t\mathrm{d}u + O(n^{-k}). \tag{7}$$

In the following sections we evaluate the integral (7) under different smoothness assumptions on ∂K.

3. Power series expansions

Let $k \geq 2$ be an integer and $K \subset \mathbb{R}^2$ a convex body with a C_+^{k+1} boundary ($(k+1)$ times continuously differentiable with everywhere positive curvature). We will use the following statement from Gruber [14] (see also Schneider [29]). We state it in the form used by Reitzner [24], but only for $d = 2$.

Lemma 1. *Let K be a convex body in $\mathbb{R}^2$ with C_+^{k+1} boundary for some integer $k \geq 2$. Then there exist constants $\alpha, \beta > 0$ depending only on K such that the following holds for every boundary point x of K. If $x = 0$ and the (unique) tangent line of K at x is $\mathbb{R}$, then there is an α neighbourhood of x in which the boundary of K can be represented by a convex function $f(\sigma)$ of differentiability class C^{k+1} in $\mathbb{R}$. Moreover, all derivatives of f up to order $k+1$ are uniformly bounded by β.*

Let $u \in S^1$ and let $x = x_u \in \partial K$. Assume that K is in the position described in Lemma 1. Let f be the function that represents the boundary of K in an α neighbourhood of x. Then f is of the form

$$f(\sigma) = b_2(u)\sigma^2 + \ldots + b_k(u)\sigma^k + O(\sigma^{k+1}),$$

where the coefficients $b_i = b_i(u)$, $i = 2, \ldots, k$ depend on u. In the foregoing we will suppress the dependence of coefficients on u (and thus on x) when we work with a fixed u. We will only indicate dependence when u is used in the argument.

We recall the following facts from the differential geometry of plane curves. Let $r(s)$ be the arc-length parametrization of ∂K with $r(0) = x$ in the neighbourhood of x such that the following hold. With the above assumptions on K, let the vector $r'(0)$, and the unit normal vector $r''(0)/\kappa(0) = -u$ form the basis of a Cartesian coordinate system, in which we denote the coordinate along the r'-axis by σ, and the r''-axis by η. Then

$$\sigma = \sigma(s) = s - \frac{\kappa^2(0)}{3!}s^3 - 3\kappa(0)\kappa'(0)\frac{s^4}{4!} + O(s^5), \tag{8}$$

$$\eta = \eta(s) = \kappa(0)\frac{s^2}{2} + \kappa'(0)\frac{s^3}{3!} + (\kappa''(0) - \kappa^3(0))\frac{s^4}{4!} + O(s^5), \tag{9}$$

see, for example, [9, Section 1.6]. From the equality $f(\sigma(s)) = \eta(s)$ we can identify the coefficients $b_2, \dots, b_k$. In particular,

$$b_2 = \frac{\kappa(0)}{2}, \quad b_3 = \frac{\kappa'(0)}{6}, \quad b_4 = \frac{\kappa''(0) + 3\kappa^3(0)}{24}.$$

With a slight abuse of notation, in the above formulas we use κ to denote the curvature as a function of s, which is different from previous usage. Later, we will also use the same letter when the curvature is a function of the outer unit normal u. Moreover, when u (s or x) is fixed, we suppress the dependence of κ on u (s or x, respectively). It will always be clear from the context which function we consider.

We will also use the following statement due to Gruber [14], see also Reitzner [24] (we state it again only for $d = 2$, so this is a simpler version of the original theorem):

Lemma 2. *Let*

$$\eta = \eta(\sigma) = b_m\sigma^m + \dots + b_k\sigma^k + O(\sigma^{k+1})$$

for $0 \le \sigma \le \alpha$, $2 \le m \le k$ be a strictly increasing function. Then there are coefficients $c_1, \dots, c_{k-m+1}$ and a constant $\gamma > 0$ such that the inverse function $\sigma = \sigma(\eta)$ has the following representation

$$\sigma = \sigma(\eta) = c_1\eta^{\frac{1}{m}} + \dots + c_{k-m+1}\eta^{\frac{k-m+1}{m}} + O(\eta^{\frac{k-m+2}{m}})$$

for $0 \le \eta \le \gamma$. The coefficients $c_1, \dots, c_{k-m+1}$ can be determined explicitly in terms of $b_m, \dots, b_k$. In particular,

i) $c_1 = \frac{1}{b_m^{\frac{1}{m}}}$,
ii) $c_2 = -\frac{b_{m+1}}{m b_m^{\frac{m+2}{m}}}$,
iii) $c_3 = -\frac{b_{m+2}}{m b_m^{\frac{m+3}{m}}} + \frac{(m+3)b_{m+1}^2}{2m^2 b_m^{\frac{2m+3}{m}}}$.

For $t \ge 0$, let the unit radius lower semicircle with centre $(0, 1+t)$ be represented by the function

$$\begin{aligned} g_t(\sigma) &= t + 1 - \sqrt{1-\sigma^2} = t + 1 - \sum_{i=0}^{\infty}(-1)^i\binom{\frac{1}{2}}{i}\sigma^{2i} \\ &= t + g_2\sigma^2 + \dots + g_{2i}\sigma^{2i} + \dots, \end{aligned}$$

for $\sigma \in [-1, 1]$, where

$$g_2 = \frac{1}{2}, \quad g_3 = 0, \quad g_4 = \frac{1}{8}.$$

Let $\sigma_+ = \sigma_+(t) > 0$ and $\sigma_- = \sigma_-(t) < 0$ such that

$$f(\sigma_+) = g_t(\sigma_+), \quad \text{and} \quad f(\sigma_-) = g_t(\sigma_-).$$

For sufficiently small $\sigma > 0$, it holds that

$$t = t(\sigma) = f(\sigma) - 1 + \sqrt{1-\sigma^2} = u_2\sigma^2 + \dots + u_k\sigma^k + O(\sigma^{k+1}),$$

where, in particular,

$$u_2 = b_2 - g_2, \quad u_3 = b_3, \quad u_4 = b_4 - g_4.$$

We note that, subsequently, we express coefficients in terms of the u_i's (as long as it does not become too complicated) as they carry all information about ∂K and the circle. We will only substitute their values when we determine our final answer.

Since $u_2 > 0$ by the conditions on ∂K, Lemma 2 yields

$$\sigma_+ = \sigma_+(t) = c_1 t^{\frac{1}{2}} + \ldots + c_{k-1} t^{\frac{k-1}{2}} + O(t^{\frac{k}{2}}), \tag{10}$$

where

$$c_1 = u_2^{-\frac{1}{2}}, \quad c_2 = -\frac{u_3}{2u_2^2}, \quad c_3 = \frac{5u_3^2 - 4u_2u_4}{8u_2^{\frac{7}{2}}}.$$

Similarly, we obtain that

$$\sigma_- = \sigma_-(t) = \tilde{c}_1 t^{\frac{1}{2}} + \ldots + \tilde{c}_{k-1} t^{\frac{k-1}{2}} + O(t^{\frac{k}{2}}), \tag{11}$$

where the coefficients $\tilde{c}_1, \ldots, \tilde{c}_{k-1}$ can be determined explicitly. In particular,

$$\tilde{c}_1 = -c_1, \quad \tilde{c}_2 = c_2, \quad \tilde{c}_3 = -c_3.$$

Thus, using (10) and (11), the area of the disc cap $C(u, t)$ is

$$\begin{aligned} A(u,t) &= \int_{\sigma_-}^{\sigma_+} g_t(\sigma) - f(\sigma)\, \mathrm{d}\sigma = \int_{\sigma_-}^{\sigma_+} t - u_2\sigma^2 - \ldots - u_k\sigma^k + O(\sigma^{k+1})\, \mathrm{d}\sigma \\ &= \left[t\sigma - \frac{u_2}{3}\sigma^3 - \ldots - \frac{u_k}{k+1}\sigma^{k+1} + O(\sigma^{k+2}) \right]_{\sigma_-}^{\sigma_+} \\ &= a_1 t^{\frac{3}{2}} + a_2 t^2 + \ldots + a_{k-1} t^{\frac{k+1}{2}} + O(t^{\frac{k+2}{2}}), \end{aligned} \tag{12}$$

where the coefficients $a_1, \ldots, a_{k-1}$ can be expressed explicitly. In particular,

$$a_1 = \frac{4}{3} u_2^{-\frac{1}{2}}, \quad a_2 = 0, \quad a_3 = \frac{5u_3^2 - 4u_2u_4}{10u_2^{\frac{7}{2}}}.$$

Now we turn to expressing the Jacobian $J(u, t)$ in the form of a series expansion in t. Using (10) and (11), we get

$$\begin{aligned} \ell(u,t) &= \int_{\sigma_-}^{\sigma^+} \sqrt{1 + (g_t'(\sigma))^2}\, \mathrm{d}\sigma = \int_{\sigma_-}^{\sigma^+} \sqrt{\frac{1}{1-\sigma^2}}\, \mathrm{d}\sigma = [\arcsin\sigma]_{\sigma_-}^{\sigma_+} \\ &= h_1 t^{\frac{1}{2}} + h_2 t + \ldots + h_{k-1} t^{\frac{k-1}{2}} + O(t^{\frac{k}{2}}), \end{aligned} \tag{13}$$

where the coefficients $h_1, \ldots, h_{k-1}$ can be expressed explicitly. In particular,

$$h_1 = 2u_2^{-\frac{1}{2}}, \quad h_2 = 0, \quad h_3 = \frac{15u_3^2 + 4u_2(u_2 - 3u_4)}{12u_2^{\frac{7}{2}}}.$$

We note that the coefficients c_1, c_2, c_3 (also $\tilde{c}_1, \tilde{c}_2, \tilde{c}_3$), a_1, a_2, a_3 and h_1, h_2, h_3 were calculated in [11, pp. 911–912] with a different notation.

Now, using (13), we get that

$$\ell(u,t) - \sin \ell(u,t) = \sum_{i=0}^{\infty} (-1)^i \frac{\ell^{2i+1}(u,t)}{(2i+1)!} = l_1 t^{\frac{3}{2}} + \ldots + l_{k-1} t^{\frac{k+1}{2}} + O(t^{\frac{k+2}{2}}),$$

where the coefficients $l_1, \ldots, l_{k-1}$ can be calculated explicitly. In particular,

$$l_1 = \frac{4}{3} u_2^{-\frac{3}{2}}, \quad l_2 = 0, \quad l_3 = \frac{25u_3^2 + 4u_2(u_2 - 5u_4)}{10u_2^{\frac{9}{2}}}. \tag{14}$$

Then

$$\begin{aligned} J(u,t) &= 2\left(1+t-\frac{1}{\kappa(x_u)}\right)(\ell(u,t)-\sin\ell(u,t)) \\ &= j_1 t^{\frac{3}{2}} + \ldots + j_{k-1} t^{\frac{k+1}{2}} + O(t^{\frac{k+2}{2}}), \end{aligned} \tag{15}$$

where the coefficients $j_1, \ldots, j_{k-1}$ can be calculated explicitly. In particular,

$$j_1 = \frac{8u_2^{-\frac{3}{2}}(\kappa-1)}{3\kappa}, \quad j_2 = 0, \quad j_3 = \frac{8u_2^{-\frac{3}{2}}}{3} + \frac{25u_3^2 + 4u_2(u_2 - 5u_4)}{5u_2^{\frac{9}{2}}}\frac{(\kappa-1)}{\kappa}.$$

For a fixed n, let $y = y(u,t)$ be defined by

$$\frac{y}{n-2} = \frac{A(u,t)}{A}.$$

Then, by (12) and using Lemma 2 for $\sqrt{t}$ and then squaring, we obtain that

$$t = p_1\left(\frac{y}{n-2}\right)^{\frac{2}{3}} + \ldots + p_{k-1}\left(\frac{y}{n-2}\right)^{\frac{k}{3}} + O\left(\left(\frac{y}{n-2}\right)^{\frac{k+1}{3}}\right), \tag{16}$$

where the coefficients $p_1, \ldots, p_{k-1}$ can be calculated explicitly. In particular,

$$p_1 = \left(\frac{3A}{4}\right)^{\frac{2}{3}} u_2^{\frac{1}{3}}, \quad p_2 = 0, \quad p_3 = \frac{9A(-5u_3^2 + 4u_2u_4)}{320u_2^2}.$$

Then, substituting (16) into (15), we obtain

$$J\left(u, \frac{y}{n-2}\right) = q_1\left(\frac{y}{n-2}\right) + \ldots + q_{k-1}\left(\frac{y}{n-2}\right)^{\frac{k+1}{3}} + O\left(\left(\frac{y}{n-2}\right)^{\frac{k+2}{3}}\right), \tag{17}$$

where the coefficients $q_1, \ldots, q_{k-1}$ can be calculated explicitly. In particular,

$$q_1 = j_1 p_1^{\frac{3}{2}}, \quad q_2 = 0, \quad q_3 = j_3 p_1^{\frac{5}{2}} + \frac{3j_1p_3p_1^{\frac{1}{2}}}{2}.$$

In the coefficients q_1, q_3 we used j_1, j_3 and p_1, p_3 instead of the u_i's in order to simplify notation.

4. The incomplete beta function

In evaluating the integral (7), we use the following expansion of the incomplete beta-function from Gruber [14].

Lemma 3 (Gruber [14])**.** *Let $\beta \in \mathbb{R}$. There are coefficients $\gamma_1, \gamma_2, \ldots \in \mathbb{R}$ depending on β which can be determined explicitly such that for a fixed $l = 1, 2, \ldots$ and $0 < \alpha \le 1$*

$$\int_0^{\alpha n}\left(1-\frac{t}{n}\right)^n t^\beta\, dt = \Gamma(\beta+1) + \frac{\gamma_1}{n} + \ldots + \frac{\gamma_l}{n^l} + O\left(\frac{1}{n^{l+1}}\right), \text{ as } n \to \infty.$$

In particular,

$$\gamma_1 = -\frac{\Gamma(\beta+3)}{2}, \quad \gamma_2 = -\frac{\Gamma(\beta+4)}{3} + -\frac{\Gamma(\beta+5)}{8}.$$

If α is chosen from a closed subinterval of $(0,1]$, then the constant in $O(\cdot)$ can be chosen independent of α.

In our calculations, we need the following corollary of Lemma 3.

Lemma 4. *Under the same assumptions as in Lemma 3, there are coefficients* $\gamma_1', \gamma_2', \ldots \in \mathbb{R}$ *such that*

$$\int_0^{\alpha(n-2)} \left(1 - \frac{t}{n-2}\right)^{n-2} t^\beta \, dt = \Gamma(\beta+1) + \frac{\gamma_1'}{n} + \ldots + \frac{\gamma_l'}{n^l} + O\left(\frac{1}{n^{l+1}}\right), \text{ as } n \to \infty.$$

In particular,

$$\gamma_1' = -\frac{\Gamma(\beta+3)}{2}, \quad \gamma_2' = -\frac{\Gamma(\beta+4)}{3} - 2\Gamma(\beta+3).$$

If α *is chosen from a closed subinterval of* $(0,1]$*, then the constant in* $O(\cdot)$ *can be chosen independent of* α*.*

Proof. Using (3) and

$$\frac{n}{n-2} = \frac{1}{1-\frac{2}{n}} = 1 + \frac{2}{n} + \frac{4}{n^2} + \ldots$$

we obtain

$$\begin{aligned}
&\int_0^{\alpha(n-2)} \left(1 - \frac{t}{n-2}\right)^{n-2} t^\beta \, dt \\
&= \Gamma(\beta+1) + \frac{\gamma_1}{n}\frac{n}{n-2} + \ldots + \frac{\gamma_l}{n^l}\frac{n^l}{(n-2)^l} + O\left(\frac{1}{n^{l+1}}\frac{n^{l+1}}{(n-2)^{l+1}}\right) \\
&= \Gamma(\beta+1) + \frac{\gamma_1'}{n} + \ldots + \frac{\gamma_l'}{n^l} + O\left(\frac{1}{n^{l+1}}\right),
\end{aligned}$$

from which we can get the coefficients $\gamma_1', \ldots, \gamma_l'$ by simple calculation. □

5. Proof of Theorem 2

Substituting (17) in the integral (7) and using (16), we obtain that

$$\begin{aligned}
\mathbb{E}(f_0(K_n^1)) &= \binom{n}{2}\frac{1}{A^2}\int_{S^1}\int_0^{t_1}\left(1 - \frac{A(u,t)}{A}\right)^{n-2} J(u,t)\,\mathrm{d}t\mathrm{d}u + O(n^{-k}) \\
&= \binom{n}{2}\frac{1}{A^2}\frac{1}{n-2}\int_{S^1}\int_0^{\tau(n-2)}\left(1 - \frac{y}{n-2}\right)^{n-2} \\
&\qquad\qquad \times J\left(u, \frac{y}{n-2}\right) t'\left(\frac{y}{n-2}\right)\mathrm{d}y\mathrm{d}u + O(n^{-k}).
\end{aligned}$$

We evaluate the inner integral as follows. Collecting the terms according to the exponent of $y/(n-2)$ and also the error term yield

$$\begin{aligned}
&\binom{n}{2}\frac{1}{A^2}\frac{1}{n-2}\int_0^{\tau(n-2)}\left(1 - \frac{y}{n-2}\right)^{n-2} J\left(u, \frac{y}{n-2}\right) t'\left(\frac{y}{n-2}\right)\mathrm{d}y \\
&= v_1\binom{n}{2}\frac{1}{A^2}\frac{1}{(n-2)^{\frac{5}{3}}}\int_0^{\tau(n-2)}\left(1 - \frac{y}{n-2}\right)^{n-2} y^{\frac{2}{3}}\,\mathrm{d}y + \ldots + \qquad (18) \\
&\quad + v_{k-1}\binom{n}{2}\frac{1}{A^2}\frac{1}{(n-2)^{\frac{k+3}{3}}}\int_0^{\tau(n-2)}\left(1 - \frac{y}{n-2}\right)^{n-2} y^{\frac{k}{3}}\,\mathrm{d}y \\
&\quad + O\left(\frac{1}{(n-2)^{\frac{k-2}{3}}}\int_0^{\tau(n-2)}\left(1 - \frac{y}{n-2}\right)^{n-2} y^{\frac{k+1}{3}}\,\mathrm{d}y\right)
\end{aligned}$$

as $n \to \infty$. The coefficients $v_1, \dots, v_{k-1}$ can be determined explicitly. In particular,

$$v_1 = \frac{2}{3} p_1 q_1, \quad v_2 = 0, \quad v_3 = \frac{4}{3} q_1 p_3 + \frac{2}{3} p_1 q_3.$$

Here we use p_1, p_3 and q_1, q_3 to express v_1, v_3 for the sake of brevity. Of course, they can also be expressed explicitly in terms of the u_i's.

We evaluate the above integrals one-by-one using Lemma 4. In particular, the first integral is as follows:

$$\begin{aligned}
& v_1 \binom{n}{2} \frac{1}{A^2} \frac{1}{(n-2)^{\frac{5}{3}}} \int_0^{\tau(n-2)} \left(1 - \frac{y}{n-2}\right)^{n-2} y^{\frac{2}{3}} \, \mathrm{d}y \\
&= \sqrt[3]{\frac{2}{3A}} \frac{(\kappa-1)^{\frac{1}{3}}}{\kappa} \frac{n(n-1)}{(n-2)^{\frac{5}{3}}} \left(\Gamma\left(\frac{5}{3}\right) - \frac{\Gamma\left(\frac{10}{3}\right)}{2} \frac{1}{n} + \dots \right) \\
&= \sqrt[3]{\frac{2}{3A}} \frac{(\kappa-1)^{\frac{1}{3}}}{\kappa} \left(\Gamma\left(\frac{5}{3}\right) n^{\frac{1}{3}} + \left(\frac{7}{3} \Gamma\left(\frac{5}{3}\right) - \frac{\Gamma\left(\frac{10}{3}\right)}{2} \right) \frac{1}{n^{\frac{2}{3}}} + \dots \right),
\end{aligned}$$

where in the last line we used the binomial series expansion

$$\frac{n(n-1)}{(n-2)^{5/3}} = n^{\frac{1}{3}} + \frac{7}{3} n^{-\frac{2}{3}} + \dots.$$

The second (nonzero) integral is the following:

$$\begin{aligned}
& v_3 \binom{n}{2} \frac{1}{A^2} \frac{1}{(n-2)^{\frac{7}{3}}} \int_0^{\tau(n-2)} \left(1 - \frac{y}{n-2}\right)^{n-2} y^{\frac{4}{3}} \, \mathrm{d}y \\
&= \frac{v_3}{2A^2} \frac{n(n-1)}{(n-2)^{\frac{7}{3}}} \left(\Gamma\left(\frac{7}{3}\right) - \frac{\Gamma\left(\frac{13}{3}\right)}{2} \frac{1}{n} + \dots \right) \\
&= \frac{v_3}{2A^2} \left(\Gamma\left(\frac{7}{3}\right) n^{-\frac{1}{3}} + \left(\frac{11\Gamma\left(\frac{7}{3}\right)}{3} - \frac{\Gamma\left(\frac{13}{3}\right)}{2} \right) n^{-\frac{4}{3}} + \dots \right),
\end{aligned}$$

where we used the binomial series expansion

$$\frac{n(n-1)}{(n-2)^{\frac{7}{3}}} = n^{-\frac{1}{3}} + \frac{11}{3} n^{-\frac{4}{3}} + \dots.$$

Evaluating the $k-1$ integrals in (18) and collecting the terms, including the error term, we obtain that

$$\begin{aligned}
& \binom{n}{2} \frac{1}{A^2} \int_0^{t_1} \left(1 - \frac{A(u,t)}{A}\right)^{n-2} J(u,t) \mathrm{d}t \\
& \qquad = w_1 n^{\frac{1}{3}} + w_2 n^0 + \dots + w_{k-1} n^{-\frac{k-3}{3}} + O(n^{-\frac{k-2}{3}}),
\end{aligned}$$

where, in principle, all coefficients $w_1, \dots, w_{k-1}$ can be calculated explicitly. In particular,

$$\begin{aligned}
w_1(u) &= \sqrt[3]{\frac{2}{3A}} \Gamma\left(\frac{5}{3}\right) \frac{(\kappa(u)-1)^{\frac{1}{3}}}{\kappa(u)}, \\
w_2(u) &= 0, \\
w_3(u) &= -\Gamma\left(\frac{7}{3}\right) \frac{1}{5} \sqrt[3]{\frac{3A}{2}} \left(\frac{\kappa''(u)}{3(\kappa(u)-1)^{\frac{4}{3}} \kappa(u)} + \frac{2\kappa^2(u) + 7\kappa(u) - 1}{2(\kappa(u)-1)^{\frac{1}{3}} \kappa(u)} - \frac{5(\kappa'(u))^2}{9(\kappa(u)-1)^{\frac{7}{3}} \kappa(u)} \right),
\end{aligned}$$

where we recall that κ is a function of u.

We note here that, when calculating further coefficients, one must also take into account some of the lower order terms from previous integrals. This does not yet affect the evaluation of w_3, as the second largest term in the first integral is $n^{-2/3}$. However, this would have to be added when calculating w_4, and so on.

Finally, integration with respect to u yields that

$$\begin{aligned}\mathbb{E}(f_0(K_n^1)) &= \int_{S^1} w_1(u)n^{\frac{1}{3}} + w_2(u)n^0 + \ldots + w_{k-1}(u)n^{-\frac{k-3}{3}} + O(n^{-\frac{k-2}{3}})\,\mathrm{d}u \\ &= z_1(K)n^{\frac{1}{3}} + z_2(K)n^0 + \ldots + z_{k-1}(K)n^{-\frac{k-3}{3}} + O(n^{-\frac{k-2}{3}}),\end{aligned}$$

where, again, all coefficient can be found explicitly. In particular,

$$\begin{aligned}z_1(K) &= \int_{S^1} w_1(u)\,\mathrm{d}u = \sqrt[3]{\frac{2}{3A}}\Gamma\left(\frac{5}{3}\right)\int_{\partial K}(\kappa(x)-1)^{\frac{1}{3}}\,\mathrm{d}x, \\ z_2(K) &= 0, \\ z_3(K) &= \int_{S^1} w_3(u)\,du = -\Gamma\left(\frac{7}{3}\right)\frac{1}{5}\sqrt[3]{\frac{3A}{2}}\int_{\partial K}\frac{\kappa''(x)}{3(\kappa(x)-1)^{\frac{4}{3}}} \\ &\qquad + \frac{2\kappa(x)^2+7\kappa(x)-1}{2(\kappa(x)-1)^{\frac{1}{3}}} - \frac{5(\kappa'(x))^2}{9(\kappa(x)-1)^{\frac{7}{3}}}\,dx\end{aligned}$$

where we use that if ∂K is C_+^2 smooth and $f(u)$ is a measurable function on S^1, then $\int_{S^1} f(u)\mathrm{d}u = \int_{\partial K} f(u_x)\kappa(x)\mathrm{d}x$, (cf. formula (2.62) in [30]). This finishes the proof of Theorem 2.

6. The case of the unit circle

For the sake of completeness, we consider the case when $K = B(R)$. Since $\mathbb{E}(f_0(B(R)_n^R))$ is independent of R, we may assume that $R = 1$. We will use the simpler notation $B_n^1 = B(1)_n^1$. In [11, p. 916] it was proved that

$$\mathbb{E}(f_0(B_n^1)) = \binom{n}{2}4\int_0^\pi \sin(\sigma)\left(1 - \frac{\sin(\sigma)+\sigma}{\pi}\right)^{n-1}\mathrm{d}\sigma.$$

Let

$$\frac{y}{n-1} = \frac{\sin(\sigma)+\sigma}{\pi}.$$

Since $\sin(\sigma)+\sigma$ is a strictly monotonically increasing analytic function on $[0,\pi]$, its inverse is also a strictly monotonically increasing analytic function by the Lagrange inversion theorem. Then σ has a power series expansion in terms of $y/(n-1)$ around $y = 0$ as follows

$$\sigma = c_1\left(\frac{y}{n-1}\right) + c_3\left(\frac{y}{n-1}\right)^3 + \ldots + c_{2k+1}\left(\frac{y}{n-1}\right)^{2k+1} + \ldots,$$

where all coefficients can be calculated explicitly. In particular,

$$c_1 = \frac{\pi}{2}, \quad c_3 = \frac{\pi^3}{96}, \quad c_5 = \frac{\pi^5}{1920}.$$

Thus,

$$\sin(\sigma) = e_1\left(\frac{y}{n-1}\right) + e_3\left(\frac{y}{n-1}\right)^3 + \ldots + e_{2k+1}\left(\frac{y}{n-1}\right)^{2k+1} + \ldots,$$

where the coefficients can be calculated explicitly. In particular,

$$e_1 = \frac{\pi}{2}, \quad e_3 = -\frac{\pi^3}{96}, \quad e_5 = -\frac{\pi^5}{1920}.$$

Therefore

$$\begin{aligned}
\mathbb{E}(f_0(B_n^1)) &= \binom{n}{2}\frac{4}{n-1}\int_0^{n-1}\left(1-\frac{y}{n-1}\right)^{n-1}\sin\left(\sigma\left(\frac{y}{n-1}\right)\right)\sigma'\left(\frac{y}{n-1}\right)\,\mathrm{d}y\\
&= f_1\binom{n}{2}\frac{4}{(n-1)^2}\int_0^{n-1}\left(1-\frac{y}{n-1}\right)^{n-1} y\,dy\\
&+ f_3\binom{n}{2}\frac{4}{(n-1)^4}\int_0^{n-1}\left(1-\frac{y}{n-1}\right)^{n-1} y^3\,dy\\
&+ \ldots + f_{2k+1}\binom{n}{2}\frac{4}{(n-1)^{2k+2}}\int_0^{n-1}\left(1-\frac{y}{n-1}\right)^{n-1} y^{2k+1}\,dy + \ldots,
\end{aligned}$$

where all coefficients $f_1, \ldots, f_{2k+1}, \ldots$ can be evaluated explicitly using Lemma 3 and the binomial series expansion of $n/(n-1)^{2k+1}$. In particular,

$$f_1 = \frac{\pi^2}{4}, \quad f_3 = \frac{\pi^4}{96}, \quad f_5 = \frac{11\pi^6}{15360}.$$

Thus, by Lemma 4 the first integral yields

$$\begin{aligned}
&\binom{n}{2}\frac{\pi^2}{(n-1)^2}\int_0^{n-1}\left(1-\frac{y}{n-1}\right)^{n-1} y\,dy\\
&= \frac{\pi^2}{2}\frac{n}{n-1}\left(\Gamma(2) - \frac{\Gamma(4)}{2}\frac{1}{n-1} + \left(\frac{-\Gamma(5)}{3}+\frac{\Gamma(6)}{8}\right)\frac{1}{(n-1)^2} + \ldots\right)\\
&= \frac{\pi^2}{2}\left(1-\frac{2}{n}+\frac{2}{n^2}+\ldots\right)
\end{aligned}$$

The second integral yields

$$\begin{aligned}
&f_3\binom{n}{2}\frac{4}{(n-1)^4}\int_0^{n-1}\left(1-\frac{y}{n-1}\right)^{n-1} y^3\,dy\\
&= \frac{\pi^4}{48}\frac{n}{(n-1)^3}\left(\Gamma(4) - \frac{\Gamma(6)}{3}\frac{1}{n-1} + \left(-\frac{\Gamma(7)}{3}+\frac{\Gamma(8)}{8}\right)\frac{1}{(n-1)^2} + \ldots\right)
\end{aligned}$$

Thus,

$$\mathbb{E}(f_0(B_n^1)) = w_0 n^0 + w_1 n^{-1} + w_2 n^{-2} + \ldots + w_k n^{-k} + \ldots,$$

where all coefficient $w_1, \ldots$ can be calculated explicitly. In particular,

$$w_0 = \frac{\pi^2}{2}, \quad w_1 = -\pi^2, \quad w_2 = \frac{\pi^4+8\pi^2}{8}, \quad w_3 = \frac{13\pi^2}{3} - \frac{11\pi^4}{24}.$$

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