ON THE VARIANCE OF THE MEAN WIDTH OF RANDOM POLYTOPES CIRCUMSCRIBED AROUND A CONVEX BODY

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ABSTRACT. Let K be a convex body in $\mathbb{R}^d$ in which a ball rolls freely and which slides freely in a ball at the same time. Let $K^{(n)}$ be the intersection of n i.i.d. random half-spaces containing K chosen according to a certain prescribed probability distribution. We prove an asymptotic upper bound on the variance of the mean width of $K^{(n)}$ as $n \to \infty$. We achieve this result by first proving an asymptotic upper bound on the variance of the weighted volume of random polytopes generated by n i.i.d. random points selected according to certain probability distributions, then, using polarity, we transfer this to the circumscribed model. Our work combines arguments from Reitzner [17] and Böröczky, Fodor, Hug [6].

1. Introduction and results

In this paper we study both random polytopes contained in a convex body and random polyhedral sets that contain a convex body. In the literature, the overwhelming majority of results are about the former types of models. Our results are asymptotic upper bounds on variances and laws of large numbers. The first order asymptotic properties of random polytopes have been investigated extensively since the ground breaking works of Rényi and Sulanke [18–20] in the 1960s, and their literature has grown enormous since then. Results on variances, higher moments and limit theorems are, however, much more scarce in the literature. For an overview of these extensive topics we refer to the surveys by Bárány [2], Hug [14], Reitzner [16], Schneider [22, 23], and Weil, Wieacker [25], and the references therein. In this paper we only mention those results that most directly related to our investigations.

Our first main result (Theorem 1.2) is an asymptotic upper bound for the variance of the volume of random polytopes in a model where the i.i.d. random points that generate the polytope are not necessarily uniform in distribution and the volume is measured according to a weight function. Also, the convex bodies we consider satisfy only weak but still meaningful smoothness conditions that have already been assumed in similar cases. This upper bound is an extension of a result of Reitzner [17]. Our main motivation for such an extended bound is that we can transfer it, via polarity, to a circumscribed model. Based on this extended asymptotic upper bound on the weighted volume, our second main result (Theorem 1.7) is an asymptotic upper bound for the variance of the mean width of random polyhedral sets that are circumscribed about the convex body in a model considered recently, for example, by Böröczky, Schneider [8], Böröczky, Fodor, Hug [6] and Fodor, Hug, Ziebarth [12].

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We work in d-dimensional Euclidean space $\mathbb{R}^d$, in which points (vectors) are denoted by lowercase letters and sets of points by capitals. We use the symbol $\langle \cdot, \cdot \rangle$ for the usual Euclidean scalar product, and $\| \cdot \|$ for the induced norm. Let $B^d = \{x \in \mathbb{R}^d : \|x\| \leq 1\}$ denote the unit ball of $\mathbb{R}^d$ centred at the origin o, and let $S^{d-1} = \{x \in \mathbb{R}^d : \|x\| = 1\}$ be the unit sphere, which is the boundary ∂B^d of B^d. We denote by $V(X)$ the volume (Lebesgue measure) of a measurable set $X \subset \mathbb{R}^d$, and by $\sigma(Y)$ the spherical Lebesgue measure of a measurable set $Y \subset S^{d-1}$. We use the notations $\kappa_d = V(B^d)$ and $\omega_d = \sigma(S^{d-1})$. Throughout the paper K denotes a convex body (compact convex set with non-empty interior) in $\mathbb{R}^d$. We say that K is C^k_+ for $k \geq 2$ if its boundary is a regular hypersurface in $\mathbb{R}^d$ that is k times continuously differentiable and has positive Gauss-Kronecker curvature, which is denoted by $\kappa(x)$ for $x \in \partial K$. If ∂K is not C^2, then it is still possible to define a notion of generalised second order derivative such that ∂K is differentiable in this generalised sense at almost all boundary points with respect to the $(d-1)$-dimensional Hausdorff measure $\mathcal{H}^{d-1}$ on ∂K, cf. Alexandrov's theorem. For more information and precise definition of generalised second order differentiability, see [21, Sections 1.5, 2.5, 2.6]. In those points where ∂K is differentiable twice in the generalised sense, a generalised Gauss-Kronecker curvature can naturally be defined, which coincides with the usual Gauss-Kronecker curvature if, in the particular point, ∂K is differentiable twice in the usual sense. Therefore, we use the symbol $\kappa(x)$ for the generalised Gauss-Kronecker curvature as well.

Let the functions f and g be defined on a space I. If there exists a constant $\gamma > 0$ such that $|f| < \gamma g$ on I, then we denote this fact with the symbol $f \ll g$, or the common Landau symbol $f = O(g)$.

In the first part of the paper we study the following probability model. Let $\tilde{\varrho} : K \to \mathbb{R}$ be a bounded non-negative measurable function on K which is positive on the boundary ∂K of K and continuous in a neighbourhood of ∂K (relative to K). Let $\varrho = (\int_K \tilde{\varrho}(x)) \, \mathrm{d}x)^{-1} \tilde{\varrho}$, where integration is with respect to the Lebesgue measure in $\mathbb{R}^d$. Then ϱ determines a probability measure on K as follows. For any measurable set $A \subset K$,

$$\mathbb{P}_{\varrho,K}(A) := \int_A \varrho(x) \, \mathrm{d}x. \tag{1}$$

Let $p_1, \ldots, p_n$ be i.i.d. random points from K distributed according $\mathbb{P}_\varrho$. The convex hull $K_{(n)} = [p_1, \ldots, p_n]$ is a random polytope in K. Expectation and variance with respect to $\mathbb{P}_{\varrho,K}$ will be denoted by $\mathbb{E}_{\varrho,K}$ and $\mathrm{Var}_{\varrho,K}$, respectively. If K is clear from the context, we may also use the simpler notations $\mathbb{P}_\varrho$, $\mathbb{E}_\varrho$ and Var_ϱ. In the special case when $\varrho \equiv 1$, one obtains the uniform model (in that case we use the even simpler notations K_n for the random polytope, $\mathbb{E}$ for the expectation and Var for variance). The majority of results in the literature concern the uniform model.

Let $\lambda : K \to \mathbb{R}$ be a bounded, non-negative measurable function on K which is positive on ∂K and continuous in a neighbourhood of ∂K. For a (Lebesgue) measurable set $A \subset K$, we define the λ-weighted volume of A as

$$V_\lambda(A) = \int_A \lambda(x) \, \mathrm{d}x. \tag{2}$$

If $\lambda \equiv 1$, then $V_\lambda(A) = V(A)$, which is the volume of A.

Of the various functionals on $K_{(n)}$, in this paper, we concentrate on the weighted volume $V_\lambda(K \setminus K_{(n)})$ and the number of vertices $f_0(K_{(n)})$. For results on other interesting functionals, we refer to the survey papers listed above.

The following asymptotic formula was proved in [6].

Theorem 1.1 ([6]). *For a convex body $K \subset \mathbb{R}^d$, a probability density function ϱ on K, and an integrable function $\lambda : K \to \mathbb{R}$ such that, on a neighbourhood of ∂K relative to K, λ and ϱ are continuous and ϱ is positive, we have*

$$\lim_{n\to\infty} n^{\frac{2}{d+1}} \mathbb{E}_\varrho \int_{K\setminus K_{(n)}} \lambda(x)\,\mathrm{d}x = c_d \int_{\partial K} \varrho(x)^{\frac{-2}{d+1}} \lambda(x) \kappa(x)^{\frac{1}{d+1}} \mathcal{H}^{d-1}(\mathrm{d}x). \tag{3}$$

The exact value of the constant c_d was determined by Wieacker [26]. The special case of (3) when $\varrho \equiv 1$ and $\lambda \equiv 1$ was proved for sufficiently smooth bodies in a series of papers by the following authors. Rényi and Sulanke [18] proved the case when $d = 2$ and K is C^3_+. Wieacker [26] proved it for $K = B^d$ and general d, Bárány [1] extended it to the case when K is C^3_+. Schütt [24] removed the smoothness condition on K. Finally, in Böröczky, Fodor, Hug [6], the probability density ϱ and weight function λ were added.

Recently, variance estimates, laws of large numbers and central limit theorems have been proved in different models in a sequence of articles. In particular, in the case $\varrho \equiv 1$ and $\lambda \equiv 1$, Küfer [15] proved that $\mathrm{Var}(V(B^d_n)) \ll n^{-(d+3)/(d+1)}$. Reitzner [17], using the Efron-Stein jackknife inequality [10], extended this upper bound $\mathrm{Var}(V(K_n)) \ll n^{-(d+3)/(d+1)}$ for C^2_+ bodies for general d. He also proved the strong law of large numbers for the volume in the form

$$\lim_{n\to\infty} n^{\frac{d+3}{d+1}} V(K \setminus K_n) = c_d \int_{\partial K} \kappa(x)^{\frac{1}{d+1}} \mathcal{H}^{d-1}(\mathrm{d}x),$$

where convergence is with probability 1. This asymptotic upper bound and the strong law of large numbers were extended to all intrinsic volumes of K_n by Bárány, Fodor, Vígh [3] in the case when K is C^2_+.

Our first main result, Theorem 1.2, is an extension of Reitzner's [17] asymptotic upper bound on the variance of the volume for non-constant ϱ and λ under milder smoothness conditions than C^2_+.

We say that a ball of radius $r > 0$ rolls freely in K if for any $x \in \partial K$ there exists a $v \in \mathbb{R}^d$ such that $x \in rB^d + v \subset K$. On the other hand, K slides freely in a ball of radius $R > 0$ if for each $x \in RS^{d-1}$ there exists $p \in \mathbb{R}^d$ with $x \in K + p \subset RB^d$. Requiring that K has a rolling ball or slides freely in ball are mild conditions on the smoothness of the boundary. If K has a rolling ball and slides freely in a ball at the same time, then ∂K is a C^1 submanifold of $\mathbb{R}^d$ and it is strictly convex. However, ∂K need not be C^2. From now on, we always assume that K has a rolling ball and slides freely in ball, and $o \in \operatorname{int} K$.

We note that the choice of these particular smoothness conditions is due, on the one hand, to the fact that the main idea of Reitzner's [17] proof can be adapted to fit this case, on the other hand, these conditions are preserved under polarity (see Section 4 for details), which make it applicable in the transfer to the circumscribed model.

Let $\sigma(K, \cdot) : \partial K \to S^{d-1}$ denote the spherical image map that assigns to a boundary point $x \in \partial K$, the outer unit normal $\sigma(K, x) \in S^{d-1}$. Furthermore, let $\tau(K, \cdot) : S^{d-1} \to \partial K$ be the reverse spherical image map that assigns to a unit vector $u \in S^{d-1}$ the boundary point $\tau(K, u) \in \partial K$ with the property that u is an outer normal to ∂K at $\tau(K,u)$. Under the assumption that K has a rolling ball and slides freely in a ball, both $\sigma(K,\cdot)$ and $\tau(K,\cdot)$ are well-defined and inverses to each other.

Our first main result is the following upper bound on the variance of $V_\lambda(K_{(n)})$.

Theorem 1.2. *For a convex body $K\subset\mathbb{R}^d$ that has a rolling ball and which slides freely in a ball, and a probability density function ϱ on K, and a non-negative integrable function $\lambda: K\to\mathbb{R}$ such that, on a neighbourhood of ∂K relative to K, λ and ϱ are continuous and positive, we have*

$$\operatorname{Var}_\varrho(V_\lambda(K_{(n)}))\ll n^{-\frac{d+3}{d+1}},$$

where the implied constant depends only on K, ϱ, λ and the dimension d.

Theorem 1.2 is a generalisation of Theorem 1 of Reitzner [17, p. 2138]. The need for this level of generality in ϱ and λ will be explained by its applicability in the circumscribed model in Theorem 1.7.

We note that even if $\varrho\equiv 1$ and $\lambda\equiv 1$, only in the planar case ($d=2$) is an asymptotic upper bound known for $\operatorname{Var}(V(K_n))$ for general convex bodies (discs) without smoothness condition on ∂K, see Bárány and Steiger [5]. Upper bounds were also proved for $\operatorname{Var}(V(K_n))$ by Bárány and Reitzner for polytopes in [4] in the uniform model. For further results on variances (upper and lower bounds, asymptotic formulas), deviation estimates and limit laws of other quantities associated with K_n we refer to the the surveys mentioned above.

From Theorem 1.2, one can derive the law of large numbers for $V_\lambda(K\setminus K_{(n)})$ by standard methods.

Theorem 1.3. *Under the same assumptions as in Theorem 1.2, it holds that*

$$\lim_{n\to\infty} V_\lambda(K\setminus K_{(n)})\, n^{\frac{2}{d+1}} = c_d\int_{\partial K}\varrho(x)^{-\frac{2}{d+1}}\lambda(x)\kappa(x)^{\frac{1}{d+1}}\,\mathrm{d}x$$

with probability 1.

The proof of Theorem 1.3 is very similar to that of Theorem 2 in Reitzner [17, pp. 2150–2151].

The following asymptotic formula was also obtained in [6] by virtue of an Efron-type argument (see [9]) that connects the expectation of the number of vertices $\mathbb{E}_\varrho(f_0(K_{(n)}))$ with $\mathbb{E}_\varrho(V_\lambda(K\setminus K_{(n)}))$.

Theorem 1.4. [6] *For a convex body $K\subset\mathbb{R}^d$, and for a probability density function ϱ on K which is continuous and positive on a neighbourhood of ∂K relative to K, it holds that*

$$\lim_{n\to\infty} n^{-\frac{d-1}{d+1}}\mathbb{E}_\varrho(f_0(K_{(n)})) = c_d\int_{\partial K}\varrho(x)^{\frac{d-1}{d+1}}\kappa(x)^{\frac{1}{d+1}}\,\mathcal{H}^{d-1}(\mathrm{d}x).$$

Reitzner [17] proved that if K is C^2_+, then

$$\operatorname{Var}(f_0(K_n))\ll n^{\frac{d-1}{d+1}}.$$

With a minor modification of the proof of Theorem 1.2, we obtain the following extension of Reitzner's upper bound.

Theorem 1.5. *For a convex body $K \subset \mathbb{R}^d$ that has a rolling ball and which slides freely in a ball, and a probability density function ϱ on K such that, on a neighbourhood of ∂K relative to K, ϱ is continuous and positive, we have*

$$\operatorname{Var}_{\varrho}(f_0(K_{(n)})) \ll n^{\frac{d-1}{d+1}},$$

where the implied constant depends only on K, ϱ and the dimension d.

The proof of theorem 1.5 is essentially the same as that of Theorem 1.2 with minor adjustments that we briefly discuss in Section 3. We note that Reitzner [17] also proved the strong law of large numbers for the number of vertices in the case when $d \geq 4$.

Next comes the main application of Theorems 1.2 and 1.5 where we apply them in the following circumscribed model, which was recently studied, for example, in Böröczky, Schneider [8], Böröczky, Fodor, Hug [6] and Fodor, Hug, Ziebarth [12].

The width of a convex body in the direction $u \in S^{d-1}$ is the distance between its two parallel supporting hyperplanes orthogonal to u. The mean width $W(K)$ of K is the average of its width over all directions, see precise definition in Section 4.

Let $K_1 = K + B^d$ the radius 1 parallel domain of K. By $A(d, d-1)$ we denote the space of hyperplanes in $\mathbb{R}^d$ with its usual topology, and by $\mathcal{H}_K$ the subspace of $A(d, d-1)$ with the property that for any $H \in \mathcal{H}_K$, $H \cap K_1 \neq \emptyset$ and $H \cap \operatorname{int} K = \emptyset$. For $H \in \mathcal{H}_K$, let H^- denote the closed half-space bounded by H that contains K. Let the motion invariant Borel measure μ_d on $A(d, d-1)$ be normalised in such a way that $\mu_d(\{H \in A(d, d-1) : H \cap M \neq \emptyset\})$ is the mean width $W(M)$ of M, for every convex body $M \subset \mathbb{R}^d$. Let $2\mu_K$ be the restriction of μ_d to $\mathcal{H}_K$. Thus μ_K is a probability measure on $\mathcal{H}_K$. Let $H_1, \ldots, H_n$ be independent random hyperplanes in $\mathbb{R}^d$, each distributed according to μ_K. The intersection $K^{(n)} = \bigcap_{i=1}^n H_i^-$ is a (possibly unbounded) random polyhedron containing K. Since $K^{(n)}$ is unbounded with positive probability, we consider $K^{(n)} \cap K_1$ instead (which is no longer a polyhedron). As already noted in [6], the choice of K_1 does not affect the asymptotic behaviour of $W(K^{(n)} \cap K_1)$ only some normalisation constants. In fact, one could replace K_1 by any other convex body M with $\operatorname{int} K \subset M$, or we can consider $W(K^{(n)})$ under the condition that $K^{(n)} \subset K_1$. In fact, we will use the latter in our argument in Section 4.

It was proved in [6] that the following holds for $W(K^{(n)} \cap K_1)$.

Theorem 1.6. [6] *If K is a convex body in $\mathbb{R}^d$, then*

$$\lim_{n \to \infty} n^{\frac{2}{d+1}} \mathbb{E}_{\mu_K}(W(K^{(n)} \cap K_1) - W(K)) = 2c_d \omega_d^{-\frac{d-1}{d+1}} \int_{\partial K} \kappa(x)^{\frac{d}{d+1}} \mathcal{H}^{d-1}(\mathrm{d}x).$$

Our main statement regarding this circumscribed model is the following theorem.

Theorem 1.7. *For a convex body $K \subset \mathbb{R}^d$ that has a rolling ball and which slides freely in a ball, it holds that*

$$\operatorname{Var}_{\mu_K}(W(K^{(n)} \cap K_1)) \ll n^{-\frac{d+3}{d+1}},$$

where the implied constant depends only on K and d.

In fact, we prove a more general statement in Theorem 4.1.

From Theorem 1.7, we can also obtain the strong law of large numbers for $W_{\mu_K}(K^{(n)} \cap K_1)$ by standard methods.

Theorem 1.8. *For a convex body $K \subset \mathbb{R}^d$ that has a rolling ball and which slides freely in a ball, it holds that*

$$\lim_{n\to\infty} n^{\frac{2}{d+1}} \left(W(K^{(n)} \cap K_1) - W(K)\right) = 2\, c_d\, \omega_d^{-\frac{d-1}{d+1}} \int_{\partial K} \kappa(x)^{\frac{d}{d+1}}\, \mathcal{H}^{d-1}(\mathrm{d}x)$$

with probability 1.

Using Theorem 1.5 we also prove the following upper bound for the number of facets $f_{d-1}(K^{(n)})$ of $K^{(n)}$.

Theorem 1.9. *For a convex body $K \subset \mathbb{R}^d$ that has a rolling ball and which slides freely in a ball, we have that*

$$\mathrm{Var}_{\mu_K}(f_{d-1}(K^{(n)})) \ll n^{\frac{d-1}{d+1}},$$

where the implied constant depends only on K and d.

Again, in Section 4 we prove a more general statement, see Theorem 4.5.

2. Proof of Theorem 1.2

Our proof is essentially based on the argument of Reitzner [17]. The main idea is to use the Efron-Stein jacknife inequality [10] to bound the variance from above by the second moment of the increment of the weighted volume of $K_{(n)}$ when adding a new random point. Then, one obtains a geometric integral that involves cap volumes, which can be estimated based on the geometric assumptions on K. This is where the existence of the rolling ball and sliding ball are important.

For $u \in S^{d-1}$ and $t \geq 0$, let $H(t,u) \in A(d, d-1)$ be the hyperplane $H(t,u) = \{x \in \mathbb{R}^d : \langle x, u\rangle = t\}$. Let $H^+(t,u)$ and $H^-(t,u)$ be the closed half-spaces bounded by $H(t,u)$, in particular, $H^+(t,u) = \{x \in \mathbb{R}^d : \langle x,u\rangle \geq t\}$ and $H^-(t,u) = \{x \in \mathbb{R}^d : \langle x,u\rangle \leq t\}$.

The intersection of K with a closed half-space is called a cap. In particular, let $C(t,u) = K \cap H^+(t,u)$, and let $V(t,u) = V(C(t,u))$. The (unique) boundary point $\tau(K,u)$ is called the vertex, and $h = h(K,u) - t$ the height of the cap $C(t,u)$. We will also use the notation $\overline{C}(h,u)$ ($\overline{V}(h,u) = V(\overline{C}(h,u))$) when we describe the cap $C(t,u)$ using its height h.

Assume that the radius of the rolling ball is r and K slides freely in a ball of radius R. Then for all $h \leq r$ and $u \in S^{d-1}$ it holds that

$$\gamma_1 h^{\frac{d+1}{2}} = \frac{2\kappa_{d-1} r^{\frac{d-1}{2}} h^{\frac{d+1}{2}}}{d+1} \leq V(\overline{C}(h,u)) \leq \gamma_2 h^{\frac{d+1}{2}} \tag{4}$$

for some positive constant γ_2 that depends on R.

Let $\varepsilon > 0$ and denote by $\partial K + \varepsilon B^d$ the radius ε parallel domain of ∂K. Let $K(\varepsilon) = K \cap (\partial K + \varepsilon B^d)$. Let ε be sufficiently small that both λ and ϱ are positive and continuous on $K(\varepsilon)$. For such ε, let

$$\varrho_m(\varepsilon) = \min_{x\in K(\varepsilon)} \varrho(x), \quad \varrho_M(\varepsilon) = \max_{x\in K(\varepsilon)} \varrho(x),$$

and

$$\lambda_m(\varepsilon) = \min_{x\in K(\varepsilon)} \lambda(x), \quad \lambda_M(\varepsilon) = \max_{x\in K(\varepsilon)} \lambda(x)$$

Then for any measurable set $A \subset K(\varepsilon)$,

$$\varrho_m(\varepsilon) V(A) \leq \mathbb{P}_\varrho(A) \leq \varrho_M(\varepsilon) V(A),$$

$$\lambda_m(\varepsilon)V(A) \leq V_\lambda(A) \leq \lambda_M(\varepsilon)V(A). \tag{5}$$

In order to prove the upper bound in Theorem 1.2, we use the Efron-Stein jackknife inequality [10], which, when applied to $V_\lambda(K_{(n)})$, yields that

$$\mathrm{Var}_\varrho V_\lambda(K_{(n)}) \leq (n+1)\mathbb{E}_\varrho V_\lambda^2(K_{(n+1)} \setminus K_{(n)}). \tag{6}$$

Let $c_1 = 18R/r$, and let $\varepsilon_0 > 0$ be sufficiently small that the following conditions are all satisfied:

(i) $c_1\varepsilon_0 < r/2$.
(ii) Both λ and ϱ are positive and continuous on $K(c_1\varepsilon_0)$.
(iii) $\varrho_0\gamma_1\varepsilon_0^{\frac{d+1}{2}} < 1$, where $\varrho_0 = \varrho_m(c_1\varepsilon_0)$.

Let $\delta(\cdot,\cdot)$ denote the Hausdorff distance of compact sets in $\mathbb{R}^d$. Let D denote the event $\delta(K_{(n)}, K) < \varepsilon_0$ and let D^c be its complement. Assume that D^c happens. Then $K_{(n)}$ has a facet whose affine hull cuts off a cap of height more than ε_0 from K that contains none of the other $n-d$ random points. Then, taking into account the bounds in (5), it follows (see also the argument in [17, pp. [2146–2147]) that

$$\mathbb{P}_\varrho(D^c) \leq O(n^d(1-c_0)^n)$$

for some suitable constant c_0 depending on ε_0. Therefore,

$$\begin{aligned}\mathrm{Var}_\varrho V_\lambda(K_{(n)}) \leq (n+1)\int_K \cdots \int_K \mathbb{1}(D)V_\lambda^2(K_{(n+1)} \setminus K_{(n)})\,\mathrm{d}p_1 \ldots \mathrm{d}p_{n+1}& \\ + O(n^{d+1}(1-c_0)^n)&.\end{aligned}$$

If $p_{n+1} \in K_{(n)}$, then the set $K_{(n+1)} \setminus K_{(n)}$ is empty so $V_\lambda(K_{(n+1)} \setminus K_{(n)}) = 0$. If $p_{n+1} \notin K_{(n)}$, then $K_{(n+1)} \setminus K_{(n)}$ is the union of simplices (with pairwise disjoint interiors) that are the convex hull of p_{n+1} and a facet of $K_{(n)}$ that is visible from p_{n+1}.

Let $x_1, \ldots, x_n, x_{n+1}$ be arbitrary points in K. Let $I = \{i_1, \ldots, i_d\} \subset \{1, \ldots, n\}$. We use the notation $F_I = [x_{i_1}, \ldots, x_{i_d}]$ for the convex hull of $x_{i_1}, \ldots, x_{i_d}$, and H_I for the affine hull of $x_{i_1}, \ldots, x_{i_d}$. Then F_I is almost always a $(d-1)$-dimensional simplex and H_I is a hyperplane. If H_I is a supporting hyperplane of the polytope $[x_1, \ldots, x_n]$, then we denote the half-space of H_I containing $[x_1, \ldots, x_n]$ by H_I^-, and the other half-space by H_I^+.

Let $\mathcal{F} = \mathcal{F}(x_{n+1})$ denote the set of facets of $[x_1, \ldots, x_n]$ that are visible from x_{n+1}, that is,

$$\mathcal{F} = \{F_I : F_I \text{ is a facet of } [x_1, \ldots, x_n], x_{n+1} \in H_I^+, I = \{i_1, \ldots, i_d\} \subset \{1, \ldots, n\}\}. \tag{7}$$

If $x_{n+1} \in [x_1, \ldots, x_n]$, then $\mathcal{F}(x_{n+1}) = \emptyset$. We obtain from (6) that

$$\begin{aligned}(n+1)&\int_K \cdots \int_K \mathbb{1}(D)V_\lambda^2(K_{(n+1)} \setminus K_{(n)})\,\mathrm{d}p_1 \ldots \mathrm{d}p_{n+1} \\ &\ll n\int_K \cdots \int_K \mathbb{1}(D)\left(\sum_{F\in\mathcal{F}} V_\lambda([F, x_{n+1}])\right)^2 \prod_{i=1}^{n+1}\varrho(x_i)\,\mathrm{d}x_1 \ldots \mathrm{d}x_{n+1},\end{aligned} \tag{8}$$

where $\int_K \ldots \mathrm{d}x_i$, $i = 1, \ldots, n+1$ denotes integration with respect to the Lebesgue measure on K.

Let $F_I^+ = K \cap H_I^+$ be the cap corresponding to F_I, and $V_I^+ = V(F_I^+)$. Taking into account the second set of inequalities in (5), we obtain

$$(8) \ll n \int_K \cdots \int_K \left(\sum_I \mathbb{1}(D) \mathbb{1}\left(F_I \in \mathcal{F}\right) V_\lambda(F_I^+) \right)^2 \prod_{i=1}^{n+1} \varrho(x_i)\,\mathrm{d}x_1 \ldots \mathrm{d}x_{n+1}$$

$$\ll n \int_K \cdots \int_K \left(\sum_I \mathbb{1}(D) \mathbb{1}\left(F_I \in \mathcal{F}\right) V_I^+ \right)^2 \prod_{i=1}^{n+1} \varrho(x_i)\,\mathrm{d}x_1 \ldots \mathrm{d}x_{n+1}. \qquad (9)$$

Expanding the square in the integral yields that

$$(9) = n \sum_I \sum_J \int_K \cdots \int_K \mathbb{1}(D) \mathbb{1}\left(F_I \in \mathcal{F}\right) V_I^+ \mathbb{1}\left(F_J \in \mathcal{F}\right) V_J^+$$
$$\times \prod_{i=1}^{n+1} \varrho(x_i)\,\mathrm{d}x_1 \ldots \mathrm{d}x_{n+1}, \quad (10)$$

where the double summation extends to all subsets $I = \{i_1, \ldots, i_d\}$ and $J = \{j_1, \ldots, j_d\}$ of $\{1, \ldots, n\}$. Let the number of common elements of I and J be $|I \cap J| = k$, and let $F_1 = [x_1, \ldots, x_d]$ and $F_2 = [x_{d-k+1}, \ldots, x_{2d-k}]$. Let $V_1^+ = V(F_1^+)$ and $V_2^+ = V(F_2^+)$. By the independence of the random points

$$(10) = n \sum_{k=0}^{d} \binom{n}{d} \binom{d}{k} \binom{n-d}{d-k} \int_K \cdots \int_K \mathbb{1}(D) \mathbb{1}\left(F_1 \in \mathcal{F}\right) V_1^+$$
$$\times \mathbb{1}\left(F_2 \in \mathcal{F}\right) V_2^+ \prod_{i=1}^{n+1} \varrho(x_i)\,\mathrm{d}x_1 \ldots \mathrm{d}x_{n+1}. \quad (11)$$

Let $\operatorname{diam}(\cdot)$ denote the diameter of a set. Let A denote the event that $\operatorname{diam}(F_2^+) < \operatorname{diam}(F_1^+)$. Then

$$(11) \ll n^{2d-k+1} \sum_{k=0}^{d} \int_K \cdots \int_K \mathbb{1}(D) \mathbb{1}\left(F_1 \in \mathcal{F}\right) V_1^+$$
$$\times \mathbb{1}\left(F_2 \in \mathcal{F}\right) \mathbb{1}(A) V_2^+ \prod_{i=1}^{n+1} \varrho(x_i)\,\mathrm{d}x_1 \ldots \mathrm{d}x_{n+1}. \quad (12)$$

Replacing $\mathbb{1}\left(F_2 \in \mathcal{F}\right)$ by $\mathbb{1}\left(F_1^+ \cap F_2^+ \neq \emptyset\right)$, we obtain

$$(12) \ll n^{2d-k+1} \sum_{k=0}^{d} \int_K \cdots \int_K \mathbb{1}(D) \mathbb{1}\left(F_1 \in \mathcal{F}\right) V_1^+$$
$$\times \mathbb{1}\left(F_1^+ \cap F_2^+ \neq \emptyset\right) \mathbb{1}(A) V_2^+ \prod_{i=1}^{n+1} \varrho(x_i)\,\mathrm{d}x_1 \ldots \mathrm{d}x_{n+1}. \quad (13)$$

The facet F_1 can be seen from x_{n+1} if and only if the random points $x_{2d-k+1}, \ldots, x_n$ are in H_1^- and x_{n+1} is in H_1^+. Therefore if we fix the points $x_1, \ldots, x_{2d-k}$, then integration with respect to $x_{2d-k+1}, \ldots, x_n$ yields

$$(13) \ll n^{2d-k+1} \sum_{k=0}^{d} \int_K \cdots \int_K \mathbb{1}(D) (1 - \mathbb{P}_\varrho(F_1^+))^{n-2d+k} \mathbb{P}_\varrho(F_1^+) V_1^+$$

$$\times\, \mathbb{1}\left(F_1^+ \cap F_2^+ \neq \emptyset\right) \mathbb{1}(A) V_2^+ \prod_{i=1}^{2d-k} \varrho(x_i)\, \mathrm{d}x_1 \dots \mathrm{d}x_{2d-k}. \tag{14}$$

Now, for a fixed $0 \leq k \leq d-1$ and $x_1, \dots, x_d$ we evaluate the following integral

$$\int_K \dots \int_K \mathbb{1}(D)\mathbb{1}\left(F_1^+ \cap F_2^+ \neq \emptyset\right) \mathbb{1}(A) V_2^+ \prod_{i=d+1}^{2d-k} \varrho(x_i)\, \mathrm{d}x_{d+1} \dots \mathrm{d}x_{2d-k}. \tag{15}$$

In order to do this, we need the following statement. Let y_i be the vertex and h_i the height of the cap F_i^+, $i = 1, 2$. We show that if $h_1 < \varepsilon_0$, then

$$F_2^+ = \overline{C}(y_2, h_2) \subset \overline{C}(y_1, c_1 h_1). \tag{16}$$

We note that a careful analysis of the argument in Reitzner [17] shows that, under the assumptions on K, this statement holds in each case when ∂K is twice differentiable in the generalised sense at both y_1 and y_2, from which it follows that it is true for almost all pairs y_1, y_2 with the prescribed conditions on F_1^+ and F_2^+. However, here we give a short and direct proof that verifies (16) for all possible combinations of y_1 and y_2.

Let H be the supporting hyperplane of K at y_1. Let B be the radius R ball (in which K slides freely) that supports K at y_1, that is, $y_1 \in \partial B$, $K \subset B$, and let B' be the radius r rolling ball containing y_1.

Then the intersection $H_1 \cap B$ is a $(d-1)$-dimensional ball of radius $\sqrt{2Rh_1 - h_1^2} < \sqrt{2Rh_1}$. From $F_1^+ \subset H_1^+ \cap B$ it follows that $\operatorname{diam}(F_1^+) < 2\sqrt{2Rh_1}$. Since $\operatorname{diam}(F_2^+) < \operatorname{diam}(F_1^+)$ and $F_1^+ \cap F_2^+ \neq \emptyset$, the orthogonal projection of F_2^+ to H is contained in the $(d-1)$-dimensional ball B'' of radius $3\sqrt{2Rh_1}$ centred at o. Let h' be chosen such that $\sqrt{rh'} = 3\sqrt{2Rh_1}$, that is, $h' = 18(R/r)h_1 = c_1 h_1 < r/2$ by the choice of ε_0. The hyperplane H' parallel to H at height $c_1 h_1$ intersects the rolling ball B' in a $(d-1)$-dimensional ball of radius at least $\sqrt{rc_1h_1} = 3\sqrt{2Rh_1}$, so the orthogonal projection of $H' \cap B'$ to H contains B'', therefore, $F_2^+ \subset \overline{C}(y_1, c_1 h_1)$.

Using (16), (4) and (5), we obtain that for fixed $x_1, \dots, x_d$,

$$\begin{aligned}
(15) &\ll \int_K \dots \int_K \mathbb{1}\left(x_{d+1}, \dots, x_{2d-k} \in \overline{C}(y_1, c_1 h_1)\right) \\
&\qquad\qquad \times V(\overline{C}(y_1, c_1 h_1)) \prod_{i=d+1}^{2d-k} \varrho(x_i)\, \mathrm{d}x_{d+1} \dots \mathrm{d}x_{2d-k} \\
&\ll \mathbb{P}_\varrho(\overline{C}(y_1, c_1 h_1))^{d-k} V(\overline{C}(y_1, c_1 h_1)) \\
&\leq (\varrho_M(c_1\varepsilon_0))^{d-k} (V(\overline{C}(y_1, c_1 h_1)))^{d-k+1} \\
&\ll (V_1^+)^{d-k+1},
\end{aligned}$$

which yields that

$$(14) \ll n^{2d-k+1} \sum_{k=0}^{d} \int_K \dots \int_K (1 - \mathbb{P}_\varrho(F_1^+))^{n-2d+k} (V_1^+)^{d-k+3} \times \prod_{i=1}^{d} \varrho(x_i)\, \mathrm{d}x_1 \dots \mathrm{d}x_d. \tag{17}$$

Note that in (17) the range of integration is extended to the whole of K. This does not contribute to the order of magnitude of the variance, as we will see later. We will show that the order of magnitude of (17) is less than $n^{-\frac{d+3}{d+1}}$.

We use the following special case of the affine Blaschke-Petkantschin formula (see, for example, [23, Theorem 7.2.7 on p. 278]). Let $\Delta_{d-1} = \Delta_{d-1}(x_1, \dots, x_d)$ be the $(d-1)$-dimensional volume of the simplex whose vertices are the points $x_1, \dots, x_d$.

Theorem 2.1. *Let $f : (\mathbb{R}^d)^d \to \mathbb{R}$ be a non-negative (Lebesgue) measurable function. Then*

$$\int_{(\mathbb{R}^d)^d} f \,\mathrm{d}x_1 \dots \mathrm{d}x_d = \frac{\omega_d}{\omega_1}(d-1)! \int_{A(d,d-1)} \int_{H^d} f \Delta_{d-1} \,\mathrm{d}x_1 \dots \mathrm{d}x_d \mathrm{d}\mu_d(H). \tag{18}$$

The measure $\mathrm{d}\mu_d = \mathrm{d}u\mathrm{d}t$ assuming that $\mathrm{d}u$ is the surface area element of the unique rotation invariant probability measure (normalised spherical Lebesgue measure) on S^{d-1} and $\mathrm{d}t$ is the volume element of the one-dimensional Lebesgue measure.

Let $0 \le k \le d$ be fixed. Using (18) we obtain

$$\begin{aligned}
&\int_K \dots \int_K (1 - \mathbb{P}_\varrho(F_1^+))^{n-2d+k} (V_1^+)^{d-k+3} \,\mathrm{d}x_1 \dots \mathrm{d}x_d \\
&\ll \int_{A(d,d-1)} \int_{H\cap K} \dots \int_{H\cap K} (1 - \mathbb{P}_\varrho(F_1^+))^{n-2d+k} (V_1^+)^{d-k+3} \\
&\qquad\qquad \times \Delta_{d-1} \,\mathrm{d}x_1 \dots \mathrm{d}x_d \,\mathrm{d}\mu_d(H) \\
&= \int_{S^{d-1}} \int_0^{h(K,u)} (1 - \mathbb{P}_\varrho(C(t,u)))^{n-2d+k} V(t,u)^{d-k+3} \\
&\qquad\qquad \times \left(\int_{H(t,u)\cap K} \dots \int_{H(t,u)\cap K} \Delta_{d-1} \,\mathrm{d}x_1 \dots \mathrm{d}x_d \right) \mathrm{d}t\mathrm{d}u,
\end{aligned} \tag{19}$$

Due to the existence of the rolling ball and the conditions on ϱ, there exists a $\delta > 0$ such that for any $u \in S^{d-1}$ and $0 \le t \le h(K,u) - \varepsilon_0$, it holds that $\mathbb{P}_\varrho(C(t,u)) > \delta$. Since the innermost d-fold integral in (19) is bounded above by a constant, it follows that

$$\begin{aligned}
&\int_{S^{d-1}} \int_0^{h(K,u)-\varepsilon_0} (1 - \mathbb{P}_\varrho(C(t,u)))^{n-2d+k} V(t,u)^{d-k+3} \\
&\qquad\qquad \times \left(\int_{H(t,u)\cap K} \dots \int_{H(t,u)\cap K} \Delta_{d-1} \,\mathrm{d}x_1 \dots \mathrm{d}x_d \right) \mathrm{d}t\mathrm{d}u \\
&\ll (1-\delta)^{n-2d+k}
\end{aligned}$$

For a fixed $u \in S^{d-1}$, let B be the supporting ball (of radius R) of K at $\tau(K,u)$. Then $H \cap B$ is a $(d-1)$-dimensional ball with $H \cap K \subset H \cap B$. Its radius is $r(t) = \sqrt{2R(h(K,u)-t) - (h(K,u)-t)^2} \ll h^{1/2}$, where $h = h(K,u) - t$.

The quantity

$$\int_{H(t,u)\cap K} \dots \int_{H(t,u)\cap K} \Delta_{d-1} \,\mathrm{d}x_1 \dots \mathrm{d}x_d \tag{20}$$

is clearly monotone with respect to the integration domain, therefore we obtain that

$$\begin{aligned}
&\int_{S^{d-1}} \int_{h(K,u)-\varepsilon_0}^{h(K,u)} (1-\mathbb{P}_\varrho(C(t,u)))^{n-2d+k} V(t,u)^{d-k+3} \\
&\qquad\qquad \times \left(\int_{H(t,u)\cap K} \cdots \int_{H(t,u)\cap K} \Delta_{d-1}\, \mathrm{d}x_1 \ldots \mathrm{d}x_d \right) \mathrm{d}t \mathrm{d}u \\
&\ll \int_{S^{d-1}} \int_{h(K,u)-\varepsilon_0}^{h(K,u)} (1-\mathbb{P}_\varrho(C(t,u)))^{n-2d+k} V(t,u)^{d-k+3} \\
&\qquad\qquad \times \left(\int_{r(t)B^{d-1}} \cdots \int_{r(t)B^{d-1}} \Delta_{d-1}\, \mathrm{d}x_1 \ldots \mathrm{d}x_d \right) \mathrm{d}t \mathrm{d}u. \qquad (21)
\end{aligned}$$

Let us substitute $h = h(K,u) - t$ in (21). By the choice of ε_0, if $h(K,u) - \varepsilon_0 \leq t \leq h(K,u)$, then $\mathbb{P}_\varrho(C(t,u)) = \mathbb{P}_\varrho(\overline{C}(h,u)) > \varrho_0\gamma_1 h^{(d+1)/2}$, and $\varrho_0\gamma_1\varepsilon_0^{(d+1)/2} < 1$. Using that the degree of homogeneity of (20) is d^2-1, we obtain

$$\begin{aligned}
(21) &\ll \int_{S^{d-1}} \int_0^{\varepsilon_0} \left(1-\varrho_0\gamma_1 h^{\frac{d+1}{2}}\right)^{n-2d+k} h^{\frac{d+1}{2}(d-k+3)} h^{\frac{d^2-1}{2}}\, \mathrm{d}h \mathrm{d}u \\
&\ll \int_0^{\varepsilon_0} \left(1-\varrho_0\gamma_1 h^{\frac{d+1}{2}}\right)^{n-2d+k} h^{\frac{d+1}{2}(d-k+3)} h^{\frac{d^2-1}{2}}\, \mathrm{d}h. \qquad (22)
\end{aligned}$$

We evaluate (22) using the following asymptotic formula (see [11, formula (5.6)] and also [7, formula (11)]). For any $\beta \geq 0$, $\omega > 0$ and $\alpha > 0$, it holds that

$$\int_0^{g(n)} h^\beta (1-\omega h^\alpha)^n\, \mathrm{d}h \sim \frac{1}{\alpha\omega^{\frac{\beta+1}{\alpha}}} \Gamma\left(\frac{\beta+1}{\alpha}\right) n^{-\frac{\beta+1}{\alpha}}$$

as $n \to \infty$, assuming that

$$\left(\frac{(\beta+\alpha+1)\ln n}{\alpha\omega n}\right)^{\frac{1}{\alpha}} \leq g(n) \leq \omega^{-\frac{1}{\alpha}}$$

for sufficiently large n. The symbol $\sim$ denotes the asymptotic equality of sequences.

By the choice of ε_0, it holds that $\varepsilon_0 < (\varrho_0\gamma_1)^{-2/(d+1)}$. Let $g(n) = \varepsilon_0$ and

$$\alpha = \frac{d+1}{2}, \quad \beta = \left(\frac{d+1}{2}\right)(d-k+3) + \frac{d^2-1}{2}, \quad \omega = \varrho_0\gamma_1.$$

Simple calculation yields that

$$\frac{\beta+1}{\alpha} = \frac{2d^2-dk+4d-k+4}{d+1} = \frac{d+3}{d+1} + 2d - k + 1.$$

Since $0 \leq k \leq d$ was arbitrary, this finishes the proof of the theorem.

3. Sketch of the proof of Theorem 1.5

Note that the probability measure $\mathbb{P}_\varrho$ is absolutely continuous with respect to the Lebesgue measure. Therefore, $K_{(n)}$ is a simplicial polytope with probability one. Applying the Efron-Stein inequality [10] to the number of vertices $f_0(K_{(n)})$, one obtains that

$$\mathrm{Var}_\varrho\left(f_0(K_{(n)})\right) \leq (n+1)\mathbb{E}_\varrho(f_0(K_{(n+1)}) - f_0(K_{(n)}))^2.$$

If $p_{n+1} \in K_{(n)}$, then $f_0(K_{(n+1)}) - f_0(K_{(n)}) = 0$. If $p_{n+1} \notin K_{(n)}$, then, using the notation (7),

$$|f_0(K_{(n+1)}) - f_0(K_{(n)})| \leq (d+1)|\mathcal{F}(p_{n+1})|,$$

as $K_{(n)}$ is simplicial with probability 1. Thus,

$$\operatorname{Var}_\varrho (f_0(K_{(n)})) \ll n\mathbb{E}_\varrho|\mathcal{F}(p_{n+1})|^2.$$

On the other hand,

$$\mathbb{E}_\varrho|\mathcal{F}(p_{n+1})|^2 \leq \int_K \cdots \int_K \left(\sum_I \mathbb{1}(D)\mathbb{1}\left(F_I \in \mathcal{F}\right)\right)^2 \prod_{i=1}^{n+1} \varrho(x_i)\,\mathrm{d}x_1 \ldots \mathrm{d}x_{n+1} + O(n^{d+1}(1-c_0^n).$$

Repeating (essentially) the same argument as in Section 2, we obtain that

$$\mathbb{E}_\varrho|\mathcal{F}(p_{n+1})|^2 \ll n^{-\frac{2}{d+1}},$$

from which Theorem 1.5 follows.

4. The variance of the mean width of circumscribed polyhedral sets

We recall some of the notations and arguments from [6]. Let $K \subset \mathbb{R}^d$ be a convex body with $o \in \operatorname{int} K$. Let

$$K^* = \{z \in \mathbb{R}^d : \langle x, z\rangle \leq 1 \;\; \forall x \in K\}$$

be the polar body of K. It was proved by Hug [13] (see Proposition 1.40 on page 40, and Proposition 1.45 on page 42) that if K has a rolling ball and it slides freely in a ball, then K^* also has a rolling ball and slides freely in a ball (of different radii). Thus, polarity preserves the smoothness conditions we impose.

Let $K_1 = K + B^d$ be parallel body of radius 1 of K. Recall that $\mathcal{H}_K$ denotes set of hyperplanes in $\mathbb{R}^d$ that intersect K_1 but not $\operatorname{int} K$. The circumscribed model is based on random hyperplanes with the following (quite general) distribution (see [6])

$$\mu_q = 2\int_{S^{d-1}} \int_0^\infty \mathbb{1}(H(t,u) \in \cdot) q(t,u)\,\mathrm{d}t\mathrm{d}u,$$

where $q : [0,\infty) \times S^{d-1} \to [0,\infty)$ is a measurable function which has the following properties: It is

(1) concentrated on $D_K = \{(t,u) \in [0,\infty) \times S^{d-1} : h(K,u) \leq t \leq h(K_1,u)\}$,
(2) positive and continuous in the neighbourhood of $\{(t,u) \in [0,\infty) \times S^{d-1} : t = h(K,u)\}$ relative to D_K,
(3) a probability measure, i.e. $\mu_K(\mathcal{H}_K) = 1$.

Probabilities, expectations and variances with respect to μ_q are denoted by $\mathbb{P}_{\mu_q}$, $\mathbb{E}_{\mu_q}$ and $\operatorname{Var}_{\mu_q}$, respectively.

Let $H_1, \ldots, H_n$ be i.i.d. random hyperplanes in $\mathbb{R}^d$ with distribution μ_q. For each H_i, let H_i^- be the closed half-space that contains the origin. Let $K^{(n)} = \cap_{i=1}^n H_i^-$, a random polyhedron containing K. We note that $K^{(n)}$ may be unbounded with positive probability, so we consider $K^{(n)} \cap K_1$ instead, or the conditional event that $K^{(n)} \subset K_1$, which has the same asymptotics as $n \to \infty$.

The polar body of $K^{(n)}$ is the convex hull of the points $x_i = t_i^{-1}u_i$, where t_i is the distance between o and H_i, and $u_i \in S^{d-1}$ is the (outer) unit normal vector of H_i, namely

$$(K^{(n)})^* = \left[t_1^{-1}u_1, \dots, t_n^{-1}u_n\right].$$

We will use the radial function of K, which is defined as

$$\rho(K, x) = \sup\{\lambda \geq 0 : \lambda x \in K\} \quad x \in \mathbb{R}^d \setminus \{o\}.$$

Furthermore, we need the following extension of q:

$$\tilde{q}(x) = q\left(\frac{1}{\|x\|}, \frac{x}{\|x\|}\right), \quad x \in K^* \setminus \{o\}.$$

It was proved in [6] (see p. 516) that the probability density function of the points $t_1^{-1}u_1, \dots, t_n^{-1}u_n$ in the polar model is

$$\varrho(x) = \begin{cases} \omega_d^{-1}\tilde{q}(x)\|x\|^{-(d+1)}, & x \in K^* \setminus K_1^* \,, \\ 0, & x \in K_1^*. \end{cases}$$

Note that $\varrho(x)$ is a bounded probability density function on K^* that is positive and continuous in a neighbourhood of ∂K^* with respect to K^*, so it satisfies the conditions of Theorem 1.2. Following the notation conventions in [6], we denote $(K^*)_{(n)}$ by the simpler symbol $K^*_{(n)}$, that is, $K^*_{(n)}$ is a random polytope generated as the convex hull of n i.i.d. random points from K^* each distributed according to ϱ. The important thing for us here is that $K^{(n)}$ and $(K^*_{(n)})^*$ are equal in distributions, see Proposition 5.1 in [6].

We prove the following theorem.

Theorem 4.1. *Let $K \subset \mathbb{R}^d$ be a convex body with $o \in \operatorname{int} K$ which has a rolling ball and which slides freely in a ball. Assume that the function $q : [0,\infty) \times S^{d-1} \to [0,\infty)$ satisfies properties (1)-(3) as described above. Then*

$$\operatorname{Var}_{\mu_K}\left(W(K^{(n)} \cap K_1)\right) \ll n^{-\frac{d+3}{d+1}},$$

where the implied constant depends only on K, q and d.

Proof. It was proved in [8] that $\mathbb{P}_{\mu_q}(K^{(n)} \not\subset K_1) \ll \alpha^n$ for a suitable $\alpha \in (0,1)$ depending on K and μ_q. Furthermore, it was proved in [6] that $K^{(n)}$ and $(K^*_{(n)})^*$ are equal in distributions. Using these observations, we can calculate the variance of the mean width with the help of the Efron-Stein inequality in the following way

$$\begin{aligned}
&\operatorname{Var}_{\mu_q}\left(W(K^{(n)} \cap K_1)\right) \\
&\leq (n+1)\mathbb{E}_{\mu_q}\left(W(K^{(n)} \cap K_1) - W(K^{(n+1)} \cap K_1)\right)^2 \\
&\ll n\mathbb{E}_{\mu_q}\left(W(K^{(n)} \cap K_1) - W(K^{(n+1)} \cap K_1)\right)^2 \\
&\ll n\left(\mathbb{E}_{\mu_q}\left(\mathbb{1}(K^{(n)} \subset K_1)(W(K^{(n)}) - W(K^{(n+1)}))^2\right) + O(\alpha^n)\right) \\
&= n\left(\mathbb{E}_{\varrho,K^*}\left(\mathbb{1}((K^*_{(n)})^* \subset K_1)(W((K^*_{(n)})^*) - W((K^*_{(n+1)})^*))^2\right) + O(\alpha^n)\right),
\end{aligned}$$

It was proved in [6] that

$$\mathbb{1}(K^{(n)} \subset K_1)(W(K^{(n)} \cap K_1) - W(K))$$

$$
\begin{aligned}
&= \mathbb{1}((K^*_{(n)})^* \subset K_1) \int_{K^* \setminus K^*_{(n)}} \lambda(x)\,\mathrm{d}x \\
&= \mathbb{1}((K^*_{(n)})^* \subset K_1)(V_\lambda(K^*) - V_\lambda(K^*_{(n)})), \qquad (23)
\end{aligned}
$$

where

$$
\lambda(x) = \begin{cases} \omega_d^{-1}\|x\|^{-(d+1)}, & x \in K^* \setminus K_1^* \ , \\ 0, & x \in K_1^*. \end{cases}
$$

Note that $\lambda(x)$ is a bounded, integrable function on K^* and it is positive and continuous on a neighbourhood of ∂K^* with respect to K^*, thus, it satisfies the conditions of Theorem 1.2. Therefore, it follows that

$$
\begin{aligned}
&\mathrm{Var}_{\mu_q}\left(W(K^{(n)} \cap K_1)\right) \\
&\ll n(\mathbb{E}_{\varrho,K^*}(\mathbb{1}((K^*_{(n)})^* \subset K_1)(V_\lambda(K^*_{(n+1)}) - V_\lambda(K^*_{(n)}))^2) + O(\alpha^n)) \\
&= n\left(\mathbb{E}_{\varrho,K^*}\left(V_\lambda(K^*_{(n+1)}) - V_\lambda(K^*_{(n)})\right)^2 + O(\alpha^n)\right) \\
&\ll n^{-\frac{d+3}{d+1}},
\end{aligned}
$$

where the last inequality follows from Theorem 1.2 as K^* also satisfies the imposed smoothness conditions, that is, it has a rolling ball and it slides freely in a ball. □

The following asymptotic formula was also proved in [6].

Theorem 4.2. [6] *Let $K \subset \mathbb{R}^d$ be a convex body with $o \in \operatorname{int} K$ and let $q : [0,\infty) \times S^{d-1} \to [0,\infty)$ be a measurable function satisfying properties (1)-(3). Then*

$$
\begin{aligned}
&\lim_{n\to\infty} n^{\frac{2}{d+1}} \mathbb{E}_{\mu_K}(W(K^{(n)} \cap K_1) - W(K)) \\
&\qquad = 2c_d\omega^{-\frac{d-1}{d+1}} \int_{\partial K} q(h(K,\sigma(K,x)),\sigma(K,x))^{-\frac{2}{d+1}} \kappa^{\frac{d}{d+1}}(x)\,\mathcal{H}^{d-1}(\mathrm{d}x).
\end{aligned}
$$

Using the asymptotic upper bound of Theorem 1.2 and taking into account the monotone decreasing property of $W(K^{(n)} \cap K_1)$, essentially the same argument as in [7] yields the following statement.

Theorem 4.3. *Under the same hypotheses as in Theorem 4.2*

$$
\begin{aligned}
&\lim_{n\to\infty} (W(K^{(n)} \cap K_1) - W(K)) n^{\frac{2}{d+1}} \\
&\qquad = 2c_d\omega^{-\frac{d-1}{d+1}} \int_{\partial K} q(h(K,\sigma(K,x)),\sigma(K,x))^{-\frac{2}{d+1}} \kappa^{\frac{d}{d+1}}(x)\,\mathcal{H}^{d-1}(\mathrm{d}x)
\end{aligned}
$$

with probability 1.

Finally, we turn to the number of facets $f_{d-1}(K^{(n)})$ of $K^{(n)}$. The following asymptotic formula was proved in [6].

Theorem 4.4. [6] *Let $K \subset \mathbb{R}^d$ be a convex body with $o \in \operatorname{int} K$, and let $q : [0,\infty) \times S^{d-1} \to [0,\infty)$ be a measurable function satisfying properties (1)-(3). Then*

$$
\begin{aligned}
&\lim_{n\to\infty} n^{-\frac{d-1}{d+1}} \mathbb{E}_{\mu_q}(f_{d-1}(K^{(n)})) \\
&\qquad = c_d\omega^{-\frac{d-1}{d+1}} \int_{\partial K} q(h(K,\sigma(K,x)),\sigma(K,x))^{\frac{d-1}{d+1}} \kappa^{\frac{d}{d+1}}(x)\,\mathcal{H}^{d-1}(\mathrm{d}x).
\end{aligned}
$$

Since for any polyhedral set $P \subset \mathbb{R}^d$ with $o \in \operatorname{int} P$, $f_0(P) = f_{d-1}(P^*)$, and the random polyhedral sets $K^{(n)}$ and $(K^*_{(n)})^*$ are equal in distribution, we obtain by the Efron-Stein inequality that

$$\begin{aligned}
\operatorname{Var}_{\mu_q}(f_{d-1}(K^{(n)})) &\le (n+1)\mathbb{E}_{\mu_q}\left(f_{d-1}(K^{(n+1)}) - f_{d-1}(K^{(n)})\right)^2 \\
&\ll n\mathbb{E}_{\mu_q}\left(f_{d-1}(K^{(n+1)}) - f_{d-1}(K^{(n)})\right)^2 \\
&= n\mathbb{E}_{\mu_q}\left(\mathbb{1}(K^{(n)} \subset K_1)\left(f_{d-1}(K^{(n+1)}) - f_{d-1}(K^{(n)})\right)^2\right) \\
&\quad + n\mathbb{E}_{\mu_q}\left(\mathbb{1}(K^{(n)} \not\subset K_1)\left(f_{d-1}(K^{(n+1)}) - f_{d-1}(K^{(n)})\right)^2\right) \\
&\ll n\left(\mathbb{E}_{\mu_q}\left(\mathbb{1}(K^{(n)} \subset K_1)\left(f_{d-1}(K^{(n+1)}) - f_{d-1}(K^{(n)})\right)^2\right) + O(n^2 \cdot \alpha^n)\right) \\
&= n\mathbb{E}_{\varrho,K^*}\left(\mathbb{1}((K^*_{(n)})^* \subset K_1)\left(f_{d-1}((K^*_{(n+1)})^*) - f_{d-1}((K^*_{(n)})^*)\right)^2\right) + O(n^3 \cdot \alpha^n) \\
&= n\mathbb{E}_{\varrho,K^*}\left(\mathbb{1}((K^*_{(n)})^* \subset K_1)\left(f_0(K^*_{(n+1)}) - f_0(K^*_{(n)})\right)^2\right) + O(n^3 \cdot \alpha^n) \\
&= n\mathbb{E}_{\varrho,K^*}\left(f_0(K^*_{(n+1)}) - f_0(K^*_{(n)})\right)^2 + O(n^3 \cdot \alpha^n) \\
&\ll n^{\frac{d-1}{d+1}}
\end{aligned}$$

by Theorem 1.5 as K^* also has a rolling ball and slides freely in a ball. Thus, we have proved the following statement.

Theorem 4.5. *Let $K \subset \mathbb{R}^d$ be a convex body with $o \in \operatorname{int} K$, and let $q : [0,\infty) \times S^{d-1} \to [0,\infty)$ be a measurable function satisfying properties (1)-(3). Then*

$$\operatorname{Var}_{\mu_q}(f_{d-1}(K^{(n)})) \ll n^{\frac{d-1}{d+1}},$$

where the implied constant depends only on K, q and d.

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