# NOTE ON THE VARIANCE OF GENERALIZED RANDOM POLYGONS

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Abstract. We consider a probability model in which the hull of a sample of i.i.d. uniform random points from a convex disc $K$ is formed by the intersection of all translates of another suitable fixed convex disc $L$ that contain the sample. Such an object is called a random $L$-polygon in $K$. We assume that both $K$ and $L$ have $C^2_+$ smooth boundaries, and we prove upper bounds on the variance of the number of vertices and missed area of random $L$-polygons assuming different curvature conditions. We also transfer some of our result to a circumscribed variant of this model.

## 1. Introduction and results

Approximation of convex bodies by random polytopes that arise as the convex hull of i.i.d. random points is an actively investigated topic in contemporary geometry. Since the ground-braking papers of Rényi and Sulanke [RS63, RS64, RS68] one of the major directions is the study of asymptotic properties of random polytopes as the number of generating points tends to infinity. We refer to the surveys by Bárány [Bár08], Reitzner [Rei10], and Schneider [Sch18], and Weil, Wieacker [WW93] for further information on the extensive literature of this rich field. In this paper, rather than adhering to the classical notion of convexity, we use a modified definition, the so-called $L$-convex hull. This alternative approach allows for a more nuanced perspective on the approximation properties of convex bodies depending on the curvature of the boundary.

The classical convex hull of a closed set is the intersection of all closed half-spaces containing the set. When reconstructing a convex domain from a random sample of points, the most natural estimator is the convex hull of the sample. However, if the boundary of the set is curved, then it may be more advantageous to replace the half-spaces by, for example, translates of a suitable convex domain, with possibly non-constant curvature. This idea naturally leads to the notion of $L$-convexity.

The geometric properties of $L$-convexity are treated in detail in the articles, for example, by Balashov and Polovinkin [BP00], Polovinkin [Pol00], Lángi, Naszódi and Talata [LNT13], and in a more general setting, by Kabluchko, Marynych and Molchanov [KMM22], and Marynych and Molchanov [MM22]. Here we only summarize the most important information on $L$-convex sets that is directly used in our arguments.

Let $K$ and $L$ two convex discs (compact convex sets with nonempty interior) in the Euclidean plane. We say that $K$ is *$L$-convex* if it is equal to the intersection of

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all translates of $L$ that contain $K$. Let $X \subset \mathbb{R}^2$ be a compact set that is contained in a translate of $L$. We call the intersection of all translates of $L$ containing $X$ the $L$-*convex hull* of $X$, and denote it by $[X]_L$. It is known that $K$ is $L$-convex if and only if $L$ is a Minkowski summand of $K$, that is, there exists a convex set $M$ such that $L = K + M$, see Schneider [Sch14, Section 3.2].

The classical notion of convexity is usually defined using segments. In analogy to this, we define the $L$-segment or $L$-spindle of the points $x$ and $y$ as the $L$-convex hull of $\{x, y\}$. We say, see [LNT13, Def. (1.2)], that a convex disc $K$ is $L$-*spindle convex* if it is contained in a translate of $L$ and for any $x, y \in K$ it holds that $[x, y]_L \subset K$. These two definitions are equivalent in the plane, but both can be naturally extended to higher dimensions, where the set of $L$-spindle convex bodies is a proper subset of $L$-convex ones, cf. [LNT13], [BP00] or [Pol00]. The $L$-spindle convex hull of finitely many points is called an $L$-*polygon*. If the boundary $\partial L$ of $L$ is smooth, an $L$-polygon $P$ is said to have a vertex at $x \in \partial P$ if $x$ is not smooth. We note that in the special case when $L = rB^2$, the radius $r$ circular disc in $\mathbb{R}^2$, the term $r$-spindle convexity is used. Properties of the random $r$-spindle convex model have been investigated recently in, for example, [FKV14, FV18].

We consider the following probability model. Let $K$ and $L$ be two convex discs with $C^2_+$ smooth boundaries, and assume that $K$ is $L$-convex. Let $x_1, \ldots, x_n$ be i.i.d. random points in $K$ chosen according to the uniform probability distribution (normalized Lebesgue measure). Let $K_n^L = [x_1, \ldots, x_n]_L$ be the $L$-convex hull of $x_1, \ldots, x_n$, a random $L$-polygon in $K$. We are interested in the distribution of the following two random variables: the missed area $A(K \setminus K_n^L)$ and the number of vertices $f_0(K_n^L)$. Our main results are asymptotic upper bounds for the variance of $A(K \setminus K_n^L)$ and $f_0(K_n^L)$ as $n \to \infty$.

Two special cases of this probability model were studied in [FPV20] which overlap with the cases discussed for circles in [FKV14]. In the first case, the curvatures of boundaries of $K$ and $L$ are strictly separated, that is, we assume that

$$\max_{x \in \partial L} \kappa_L(x) < 1 < \min_{y \in \partial K} \kappa_K(y), \tag{1}$$

where $\kappa_L(x)$ is the curvature at $x \in \partial L$, and $\kappa_K(y)$ is the curvature at $y \in \partial K$. In this case, it is known (cf. [Sch14, Sec. 3.2]) that at every point $x \in \partial K$ there exists a translate $L + p$ such that $K \subset L + p$ and $x \in \partial L + p$. We say that $L + p$ supports $K$ in $x$.

Fodor, Papvári and Vígh [FPV20] proved that if $K$ and $L$ are two $C^2_+$ convex discs that satisfy (1), then

$$\lim_{n \to \infty} \mathbb{E}(f_0(K_n^L)) \cdot n^{-\frac{1}{3}} = c \cdot A(K)^{-1/3} \int_{S^1} \frac{(\kappa_K(u) - \kappa_L(u))^{\frac{1}{3}}}{\kappa_K(u)} \,\mathrm{d}u, \tag{2}$$

$$\lim_{n \to \infty} \mathbb{E}(A(K \setminus K_n^L)) \cdot n^{\frac{2}{3}} = c \cdot A(K)^{2/3} \int_{S^1} \frac{(\kappa_K(u) - \kappa_L(u))^{\frac{1}{3}}}{\kappa_K(u)} \,\mathrm{d}u,$$

where $c = (2/3)^{1/3}\Gamma(5/3)$, $S^1$ is the unit circle and $\Gamma$ is Euler's gamma function. Integration on $S^1$ is with respect to arc-length. The special case when $L$ is a circular disc of radius $r > 1$ was proved in [FKV14, Theorem 1.1].

The second setup we study is when $K = L$ with $C^2_+$ boundary. It was observed in [FKV14] (see Theorem 1.2 (1.7)) that in the special case when $L = B^2$, the expected number of the vertices approaches a constant as $n \to \infty$. It was proved by Fodor, Papvári and Vígh [FPV20] that a similar phenomenon occurs for general $L$. This result was extended by Marynich and Molchanov for $d$-dimensions in [MM22] with no smoothness condition on $L$.

Fodor and Vígh [FV18] proved upper bounds for the variance of the area and vertex number in the case when $K = B^2$. Fodor, Grünfelder and Vígh [FGV22] proved matching lower bounds when the curvature of $\partial K$ is larger than 1 at all boundary points.

We use the following common notation for asymptotic inequalities. Let $f$ and $g$ be two real sequences. We write $f \ll g$ if there is a positive constant $\gamma_0$ such that $|f(n)| \leq \gamma_0 g(n)$ for every $n \in \mathbb{N}$. If both $f \ll g$ and $g \ll f$ hold we write $f \approx g$.

Our main results are the following.

**Theorem 1.** *If the convex discs $K$ and $L$ satisfy* (1)*, then*

$$\operatorname{Var}(f_0(K_n^L)) \ll n^{\frac{1}{3}}, \tag{3}$$

$$\operatorname{Var}(A(K_n^L)) \ll n^{-\frac{5}{3}}, \tag{4}$$

*where the implied constants depend only on $K$ and $L$.*

**Theorem 2.** *Let $L$ be a convex disc with $C^2_+$ smooth boundary. Then*

$$\operatorname{Var}(f_0(L_n^L)) \ll 1, \tag{5}$$

$$\operatorname{Var}(A(L_n^L))) \ll n^{-2}, \tag{6}$$

*where the implied constants depend only on $L$.*

The upper bound on the variance yields the strong law of large numbers for the missed area.

**Theorem 3.** *Let $K$ and $L$ be convex that satisfy* (1)*. Then it holds with probability* 1 *that*

$$\lim_{n\to\infty} A(K \setminus K_n^L) \cdot n^{\frac{2}{3}} = \sqrt[3]{\frac{2A(K)^2}{3}}\,\Gamma\left(\frac{5}{3}\right) \int_{S^1} \frac{(\kappa_K(u) - \kappa_L(u))^{\frac{1}{3}}}{\kappa_K(u)}\,\mathrm{d}u.$$

In proving the upper bounds in Theorems 1 and 2, we follow the methods of [Rei03] and [FV18], but the transition from circles to the $L$-convex case is non-trivial. Besides borrowing methods from [FPV20] new ideas are also needed in several places.

Theorem 3 can be proved by standard methods, see, for instance, [BFRV09], [FV18] and [Rei03].

## 2. Geometric preparations

In this section we recall some definitions and statements from [FPV20]. Let $K$ and $L$ be convex discs such that $K$ is $L$-convex. A subset $D$ of $K$ is called an $L$-cap if there exists a $p \in \mathbb{R}^2$ such that $D = \operatorname{cl}(K \setminus (L + p))$, where $\operatorname{cl}(\cdot)$ denotes the closure of a set. As a simplification, we may also use the term cap instead of $L$-cap if there is no risk of misunderstanding.

Let $M$ be a smooth convex disc, meaning that it has a unique outer unit normal $u(M, x)$ at each boundary point $x \in \partial M$. If $M$ is strictly convex, then for any unit vector $u$, there exists a unique boundary point $x(M, u)$ of $M$ with the property that $u = u(M, x(u))$. Based on this, we use the notation $\kappa_M(u) = \kappa_M(x(M, u))$ for the curvature if $\partial M$ is $C^2_+$.

We recall [FPV20, Lemma 2.1] (see also [FKV14, Lemma 4.1]) that for each $L$-cap $D = \mathrm{cl}\,(K \setminus (L+p))$, there exists a unique point $x_0 \in D \cap K$ and a number $t \geq 0$ such that $y_0 = x_0 - tu(K, x_0) \in D \cap (\partial L + p)$ and $u(L+p, y_0) = u(K, x_0)$. We call $x_0$ the vertex and $t$ the height of $D$. We denote the unique $L$-cap determined by $u \in S^1$ and $t \geq 0$ by $D(u,t)$. Let $A(u,t) = A(D(u,t))$, and let $l(u,t)$ be the arc length of $D \cap (\partial L + p)$.

By Lemma 2.2 in [FPV20], for a fixed $u \in S^1$

$$\lim_{t \to 0^+} \ell(u,t) t^{-\frac{1}{2}} = 2\gamma(u), \ \lim_{t \to 0^+} A(u,t) t^{-\frac{3}{2}} = \frac{4}{3}\gamma(u), \tag{7}$$

where $\gamma(u) = \sqrt{2/(\kappa_K(u) - \kappa_L(u))}$.

If (1) holds, then for any pair of points $x, y \in K$ there exist exactly two translates of $L$ which contain $x$ and $y$ on it boundary. Each one of these two translates determines an $L$-cap, we denote these caps by $D_-(x,y)$ and $D_+(x,y)$ such that $A(D_-(x,y)) \leq A(D_+(x,y))$. Let $A_-(x,y) = A(D_-(x,y))$ and $A_+(x,y) = A(D_+(x,y))$. There exists a constant $\delta > 0$ depending only on $K$ and $L$, such that $A_+(x,y) > \delta$ holds for any $x, y \in \mathrm{int}\, K$, cf. [FPV20, Lemma 2.3].

Let $0 < \varepsilon_K < \min_{x \in \partial K} \kappa(x)$. By a similar argument as in [FV18], one can show that the Hausdorff distance $d_H(K, K_n^L) \geq \varepsilon_K$ of $K$ and $K_n^L$ is at most $\varepsilon_K$ with high probability. In order to see this, assume that $d_H(K, K_n^L) \geq \varepsilon_K$. Then there exists a point $x$ on the boundary $K_n^L$ such that $\varepsilon_K B^2 + x \subset K$. Thus, there is a translate of $L$ which supports $K_n^L$ in $x$ and defines an $L$-cap $D$ of height at least $\varepsilon_K$. Since the minimum of the areas of $L$-caps of $K$ with height at least $\varepsilon_K$ is positive, there exists a constant $c_0$, depending on $K$, $L$, and $\varepsilon_K$, such that the probability that a random point falls into $D$ is at least $c_0$. Thus

$$\mathbb{P}(d_H(K, K_n^L) \geq \varepsilon_K) \leq (1 - c_0)^n. \tag{8}$$

## 3. Proof of Theorem 1

The first part of the proof is based on an argument of Reitzner [Rei03] and had already been used in [FV18] in the spindle convex setting (the case when $L = B^2$), therefore we only highlight the most important steps which use the $L$-convex property.

Let $x \in K$ arbitrary, and let $x_i x_j$ be an edge of $K_n^L$. We say that an edge $x_i x_j$ is visible from $x$ if $x$ is not contained in the translate of $L$ that determines $x_i x_j$. For $x \in K$, we denote the set of edges of $K_n^L$ visible from $x$ by $\mathcal{F}_n(x)$, and the cardinality of $\mathcal{F}_n(x)$ by $F_n = F_n(x)$. Let $x_{n+1}$ be a uniform random point from $K$, independent of $x_1, \dots, x_n$. If $x_{n+1} \notin K_n^L$, then the same argument as in [FV18] and the Efron-Stein jackknife inequality [ES81] yield that

$$\mathrm{Var}\, f_0(K_n^L) \leq (n+1)\mathbb{E}(f_0(K_{n+1}^L) - f_0(K_n^L))^2 \ll n\mathbb{E}(F_n^2(x_{n+1})).$$

Let $I = (i_1, i_2)$ be an ordered pair of indices from $\{1, \dots, n\}$ with $i_1 \neq i_2$. Denote by $F_I$ the shorter boundary arc of the unique translate $L+p$ with $x_{i_1}, x_{i_2} \in \partial L + p$ on which $x_{i_1}$ and $x_{i_2}$ follow each other in positive order. Let $\mathbf{1}(\cdot)$ denote the indicator function of an event.

We know from (8), that assuming that $d_H(K, K_n^L) < \varepsilon_K$ introduces an error $O((1 - c_K)^n)$. Therefore, we obtain, similarly as in [FV18], that

$$\mathbb{E}(F_n(x_{n+1})^2) \ll \frac{1}{A(K)^{n+1}} \sum_I \sum_J \int_K \int_K \dots \int_K \mathbf{1}(F_I \in \mathcal{F}_n(x_{n+1}))\mathbf{1}(F_J \in \mathcal{F}_n(x_{n+1}))$$

$$\times\, \mathbf{1}(d_H(K, K_n^L) \leq \varepsilon_K)\, \mathrm{d}x_1 \ldots \mathrm{d}x_n \mathrm{d}x_{n+1} + O((1-c_K)^n), \tag{9}$$

where summation is over all ordered pairs $I$ and $J$ from $\{1, \ldots, n\}$.

We can choose $\varepsilon_K$ so small that $A(K \setminus K_n^L) < \delta$, so only the arc defined by $x_i$ and $x_j$ can be an edge of $K_n^L$.

Let $D$ be an $L$-cap with vertex $x$. For a line $e$ perpendicular to $u(K, x)$, denote by $e_+$ the closed half-plane containing $x$. Then there exists a maximal (classical) cap $C_-(D) = K \cap e_+$, which is contained in $D$, and a minimal cap $C_+(D) = e'_+ \cap K$, which contains $D$.

Let $0 \leq k \leq 2$ be an integer. We denote the shorter arc defined by $x_1$ and $x_2$ by $F_1$, and the shorter arc defined by $x_{3-k}$ and $x_{4-k}$ by $F_2$, and $D_1 = D_-(x_1, x_2)$ and $D_2 = D_-(x_{3-k}, x_{4-k})$, respectively. Let $\operatorname{diam}(\cdot)$ denote the diameter of a set. Thus, using the same argument (this part does not use $L$-convexity) as on pages 1148–1149 of [FV18], we obtain that

$$(9) \ll \sum_{k=0}^{2} n^{4-k} \int_K \cdots \int_K \left(1 - \frac{A(D_1)}{A(K)}\right)^{n-4+k} A(D_1) \mathbf{1}(D_1 \cap D_2 \neq \emptyset)$$
$$\times\, \mathbf{1}(\operatorname{diam} C_+(D_1) \geq \operatorname{diam} C_+(D_2)) \mathbf{1}(d_H(K, K_n^L) \leq \varepsilon_K)\, \mathrm{d}x_1 \ldots \mathrm{d}x_{4-k}. \tag{10}$$

**Lemma 1.** *If $D_1 \cap D_2 \neq \emptyset$, $d_H(K, K_n^L) \leq \varepsilon_K$ and $\operatorname{diam} C_+(D_1) \geq \operatorname{diam} C_+(D_2)$, then there exists a constant $c_1$ depending only on $K$ and $L$ such that*

$$D_2 \subset c_1(D_1 - x_{D_1}) + x_{D_1}.$$

*Proof.* We first show that there exists a constant $\tilde{c}$, which depends only on $K$ and $L$, such that if $D$ is an $L$-cap of sufficiently small height, then the following holds:

$$\tilde{c}(C_-(D) - x) \supset C_+(D) - x.$$

For this, let us denote the heights of $C_-(D)$ and $C_+(D)$ by $h_-$ and $h_+$. Due to convexity, it is sufficient to find a constant $\tilde{c} > 0$ such that $h_+/h_- < \tilde{c}$ holds for every $L$-cap of sufficiently small height in $K$.

Let $R_1 \in (1/\kappa_K^m, 1)$, and consider the closed circular disc $\hat{B} = R_1 B^2 + x - R_1 u_K(x)$, of radius $R_1$, which is the support circle at point $x$ of the disc $K$. Then $\hat{B} \supset K$ and so $D = \operatorname{cl}(K \setminus (L+p)) \subset \operatorname{cl}(\hat{B} \setminus (L+p)) = \hat{D}$. For the heights $\hat{h}_-$ and $\hat{h}_+$ of the classical caps $C_-(\hat{D})$ and $C_+(\hat{D})$, $\hat{h}_- = h_-$ and $\hat{h}_+ > h_+$ hold.

Now consider the closed circular disc $\tilde{B} = B^2 + x - (h_- + 1)u_K(x)$ of radius 1, supported by $L$ at the point $x - h_- u_K(x)$. Then $\tilde{B} \subset L + p$ and $\hat{D} = \operatorname{cl}(\hat{B} \setminus (L+p)) \subset \operatorname{cl}(\hat{B} \setminus \tilde{B}) = \tilde{D}$. Furthermore, for the heights $\tilde{h}_-$ and $\tilde{h}_+$ of the caps $C_-(\tilde{D})$ and $C_+(\tilde{D})$, $\tilde{h}_- = \hat{h}_- = h_-$ and $\tilde{h}_+ > \hat{h}_+ > h_+$ hold. From elementary geometry it is clear that there exists a constant $\tilde{c}$ such that $\tilde{h}_+/\tilde{h}_- < \tilde{c}$, and so

$$\frac{h_+}{h_-} < \frac{\tilde{h}_+}{\tilde{h}_-} < \tilde{c}$$

holds as well.

Reitzner proved (see [Rei03, pp. 2149–2150]) that if $D_1 \cap D_2 \neq \emptyset$, $d_H(K, K_n^L) \leq \varepsilon_K$ and $\operatorname{diam} C_+(D_1) \geq \operatorname{diam} C_+(D_2)$, then there exists a constant $\bar{c}$ depending only on $K$, for which $C_+(D_2) \subset \bar{c}(C_+(D_1) - x_{D_1}) + x_{D_1}$, where $x_{D_1}$ is the vertex of the cap $D_1$. The two claims together prove the lemma. □

Thus $A(D_2) \leq c_1^2 A(D_1)$, and therefore

$$\int_K \cdots \int_K \mathbf{1}(D_1 \cap D_2 \neq \emptyset)\mathbf{1}(\operatorname{diam} C_+(D_1) \geq \operatorname{diam} C_+(D_2)) \\ \times \mathbf{1}(d_H(K, K_n^L) \leq \varepsilon_K)\,\mathrm{d}x_3 \ldots \mathrm{d}x_{4-k} \ll A(D_1)^{2-k},$$

and

$$(10) \ll \sum_{k=0}^{2} n^{4-k} \int_K \int_K \left(1 - \frac{A(D_1)}{A(K)}\right)^{n-4+k} A(D_1)^{3-k}\mathbf{1}(d_H(K, K_n^L) \leq \varepsilon_K)\,\mathrm{d}x_1\mathrm{d}x_2. \tag{11}$$

The pair of points $(x_1, x_2)$ is parameterized in the same way as [FPV20]:

$$(x_1, x_2) = \Phi(u, t, u_1, u_2),$$

where $u \in S^1$ and $t \leq t_0$ such that $D(u,t) = D_-(x_1, x_2)$. Let $L(u,t)$ denote the arc $D(u,t) \cap (\partial L + x(K,u) - x(L,u) - tu)$, then $x_1, x_2 \in L(u,t)$. The outer unit normals associated to $L + x(K,u) - x(L,u) - tu$ on the arc $L(u,t)$ define a connected arc of $S^1$. Denote this arc by $L^*(u,t)$. At points $x_1$ and $x_2$, denote the outer normal vectors of $L + x(K,u) - x(L,u) - tu$ by $u_1$ and $u_2$, i.e.

$$x_i = x(K,u) - x(L,u) - tu + x(L, u_i), \quad i = 1, 2,$$

where $u_1, u_2 \in L^*(u,t)$.

Since every $L$-cap has a unique vertex and height, $\Phi$ is well defined, and according to [FPV20], it is bijective and differentiable on a domain $(u, t, u_1, u_2)$, with the possible exception of a set of measure zero. The Jacobian of $\Phi$ is

$$|J\Phi| = \frac{|u_1 \times u_2|}{\kappa_L(u_1)\kappa_L(u_2)} \left(\frac{1}{\kappa_L(u)} - \frac{1}{\kappa_K(u)} + t\right) = \frac{|u_1 \times u_2|}{\kappa_L(u_1)\kappa_L(u_2)} k(u,t),$$

where $|u_1 \times u_2|$ is the vector product of $u_1$ and $u_2$. From a simple compactness argument, we obtain that $k(u,t) \ll 1$. Here, the notation $\ll$ is used in the sense of $t \to 0$, which is implied by $n \to \infty$.

Let $L(u,t) = \partial D_1 \cap \operatorname{int} K$ and $A(u,t) = A(D_1)$, then we get the following for the integral with a suitable $t^*(u)$ depending only on $K$ and $L$:

$$(11) \quad \ll \quad \sum_{k=0}^{2} n^{4-k} \int_{S^1} \int_0^{t^*(u)} \left(1 - \frac{A(u,t)}{A(K)}\right)^{n-4+k} A(u,t)^{3-k} I^*(u,t)\,\mathrm{d}t\mathrm{d}u \tag{12}$$

with

$$I^*(u) = \int_{L^*(u)} \int_{L^*(u)} \frac{|u_1 \times u_2|}{\kappa_L(u_1)\kappa_L(u_2)}\,\mathrm{d}u_1\mathrm{d}u_2,$$

where $L^*(u) \subset S^1$ is a half circle with midpoint $u \in S^1$. Using the conditions on the curvature,

$$I^*(u,t) \ll \ell^*(u,t) - \sin \ell^*(u,t) \ll t^{3/2},$$

where $\ell^*(u,t)$ is the length of the arc $L^*(u,t) \subset S^1$. The last inequality follows from the Taylor expansion of $\sin x$ around 0, the limit $\lim_{t \to 0^+} \ell^*(u,t)/\ell(u,t) = \kappa_L(u)$ and (7). Thus,

$$(12) \ll \sum_{k=0}^{2} n^{4-k} \int_{S^1} \int_0^{t^*(u)} \left(1 - \frac{A(u,t)}{A(K)}\right)^{n-4+k} A(u,t)^{3-k} t^{\frac{3}{2}}\,\mathrm{d}t\mathrm{d}u.$$

We split the domain of integration in $t$ into two parts in a standard way. Let $h(n) = (c_2 \log n/n)^{2/3}$ with some sufficiently large constant $c_2 > 0$. By (7), for a suitable $\gamma_1$, $A(u,t) \geq \gamma_1 t^{3/2}$ holds for all $u \in S^1$. Furthermore, $A(u,t) \leq A(K)$, and $t$ can also be bounded from above by a constant, hence

$$\begin{aligned}
&\sum_{k=0}^{2} n^{4-k} \int_{S^1} \int_{h(n)}^{t^*(u)} \left(1 - \frac{A(u,t)}{A(K)}\right)^{n-4+k} A(u,t)^{3-k} t^{\frac{3}{2}} \,\mathrm{d}t\mathrm{d}u \\
&\ll \sum_{k=0}^{2} n^{4-k} \int_{S^1} \int_{h(n)}^{t^*(u)} \left(1 - \frac{\gamma_1 t^{\frac{3}{2}}}{A(K)}\right)^{n-4+k} \mathrm{d}t\mathrm{d}u \\
&\ll \sum_{k=0}^{2} n^{4-k} \left(1 - \frac{\gamma_1 (c_2 \log n)}{nA(K)}\right)^{n-4+k} \ll n^{-\frac{2}{3}},
\end{aligned}$$

where the last inequality requires $\gamma_1 c_2/A(K)$ to be large enough.

Now only the part of (12) where $0 \leq t \leq h(n)$ remains to be estimated. We will use the following statement from [BFRV09, Formula (11)]: For any $\beta \geq 0$, $\omega > 0$ and $\alpha > 0$, it holds that

$$\int_0^{g(n)} t^{\beta} (1 - \omega t^{\alpha})^n \,\mathrm{d}t \sim \frac{1}{\alpha \omega^{\frac{\beta+1}{\alpha}}} \cdot \Gamma\left(\frac{\beta+1}{\alpha}\right) \cdot n^{-\frac{\beta+1}{\alpha}}, \tag{13}$$

as $n \to \infty$, assuming

$$\left(\frac{(\beta + \alpha + 1)\log n}{\alpha \omega n}\right)^{\frac{1}{\alpha}} < g(n) < \omega^{-\frac{1}{\alpha}},$$

for sufficiently large $n$.

For $0 \leq k \leq 2$, using (7) yields

$$\begin{aligned}
&n^{4-k} \int_{S^1} \int_0^{h(n)} \left(1 - \frac{A(u,t)}{A(K)}\right)^{n-4+k} A(u,t)^{3-k} t^{\frac{3}{2}} \,\mathrm{d}t\mathrm{d}u \\
&\ll n^{4-k} \int_0^{h(n)} \left(1 - \frac{\gamma_1 t^{\frac{3}{2}}}{A(K)}\right)^{n-4+k} t^{\frac{12-3k}{2}} \,\mathrm{d}t \ll n^{-\frac{2}{3}},
\end{aligned}$$

where the last inequality is the consequence of (13) with $\alpha = 3/2$, $\beta = (12-3k)/2$, $\omega = \gamma_1/A(K)$, and the inequality on $h(n)$ holds if $c_2 > (17-3k)A(K)/3\gamma_1$. Since $k$ was arbitrary, this finishes the proof of (3).

The proof of (4) is very similar, for the necessary adjustments see, for example, [FV18, p. 1151].

## 4. Proof of Theorem 2

We only prove (5), the upper bound on the variance of the number of vertices. The statement concerning the area (6) is very similar.

We use the following facts from [FPV20, Lemma 4.1, p. 509]. If $L$ has $C^2_+$ boundary, then

$$\lim_{t \to 0^+} \ell^*(u,t) = \pi, \text{ and } \lim_{t \to 0^+} A(u,t) \cdot t^{-1} = w(u), \tag{14}$$

where $w(u)$ is the width of the disc $L$ in the direction perpendicular to the vector $u \in S^1$.

The Efron–Stein inequality for the number of vertices of $L_n$ yields that

$$\operatorname{Var}(f_0(L_n)) \ll n\mathbb{E}(F_n(x_{n+1}))^2.$$

With the same notation as in the proof Theorem 1 in Section 3, and using essentially the same argument as on pages 1152–1153 of [FV18], we get that

$$n\mathbb{E}(F_n(x_{n+1}))^2 \ll \sum_{k=0}^{2} n^{5-k} \int_L \dots \int_L \left(1 - \frac{A(D_1)}{A(L)}\right)^{n-4+k} \frac{A(D_1)}{A(L)} \\ \times \mathbf{1}(A(D_1) \geq A(D_2))\, \mathrm{d}x_1 \dots \mathrm{d}x_{4-k}, \quad (15)$$

**Lemma 2.** *For sufficiently small $A(D_1)$, if $A(D_1) \geq A(D_2)$, then the set of possible locations of the points $x_3, \dots x_{4-k}$ has area at most $\gamma_2 A(D_1)$, for some constant $\gamma_2 > 0$ depending only on $L$.*

*Proof.* Let $r_m = \min_{x \in \partial L}(1/\kappa(x))$. Then a circular disc of radius $r_m$ rolls freely in $L$, i.e., for each $x \in \partial L$, there exists a $p \in \mathbb{R}^2$ such that $x \in r_m B^2 + p \subset L$. For $t \leq r_m$, let the inner parallel body of depth $t$ of $L$ be the set

$$\widetilde{L}_t := \{x \in L : x + tB^2 \subset L\},$$

which is also convex and a Minkowski summand of $L$. Let the area of $A(D_1)$ be sufficiently small that (by (14)) there is a $t_M < r_m/2$ such that all caps of area at most $A(D_1)$ are completely contained in the set $L \setminus \widetilde{L}_{t_M}$. By (1.3) of [HCS10], similar to Steiner's formula, one obtains that there exists a positive constant $c_L$, depending only on $L$, such that $A(L \setminus \widetilde{L}_{t_M}) = c_L t_M + O(t_M^2)$. Using the fact that $t_M$ is bounded above by a constant multiple of the height of the cap $D_1$, we obtain the claim of the lemma. □

Thus, using Lemma 2 and integrating with respect to $x_3, \dots x_{4-k}$, we get

$$(15) \ll \sum_{k=0}^{2} n^{5-k} \int_L \int_L \left(1 - \frac{A(D_1)}{A(L)}\right)^{n-4+k} A(D_1)^{3-k}\, \mathrm{d}x_1 \mathrm{d}x_2.$$

We may assume that there exists a constant $c_3 > 0$ such that $A(D_1)/A(L) < c_3 \log n/n$. If, on the contrary, $A(D_1)/A(L) \geq c_3 \log n/n$, then

$$\left(1 - \frac{A(D_1)}{A(L)}\right)^{n-4+k} \cdot A(D_1)^{3-k} \leq \exp\left(-\frac{c_3(n-4+k)\log n}{n}\right) \cdot A(L)^{3-k} \ll n^{-c_3}.$$

So for a sufficiently large $c_3$, the contribution of the part $A(D_1)/A(L) \geq c_3 \log n/n$ is $O(n^{-1})$. Thus,

$$n\mathbb{E}(F_n(x_{n+1})) \ll \sum_{k=0}^{2} n^{5-k} \int_L \int_L \left(1 - \frac{A(D_1)}{A(L)}\right)^{n-4+k} A(D_1)^{3-k} \\ \times \mathbf{1}\left(A(D_1) \leq \frac{c_3 \log n}{n}\right) \mathrm{d}x_1 \mathrm{d}x_2 + O(n^{-1}). \quad (16)$$

We use the parametrization (3) as above, which in this case has the Jacobian

$$|J\Phi| = \frac{|u_1 \times u_2|}{\kappa_L(u_1)\kappa_L(u_2)} t.$$

By (14), there exists a constant $c^*$ such that for all $t > c^* \log n/n$, $A(u,t)/A(L) > c_3 \log n/n$. Thus, after the integral transformation, we get

$$(16) \ll \sum_{k=0}^{2} n^{5-k} \int_{S^1} \int_0^{c^* \frac{\log n}{n}} I^*(u,t) \left(1 - \frac{A(u,t)}{A(L)}\right)^{n-4+k} A(u,t)^{3-k} t \,\mathrm{d}t\mathrm{d}u + O(n^{-1}). \tag{17}$$

Now we use that the quantity $I^*(u,t)$ is bounded, and by (14) $A(u,t) \approx t$, so there exists a constant $\omega > 0$ for which

$$(17) \ll \sum_{k=0}^{2} n^{5-k} \int_0^{c^* \frac{\log n}{n}} (1 - \omega t)^{n-4+k} t^{4-k} \,\mathrm{d}t + O(n^{-1}), \tag{18}$$

to which we can again apply (13) with $\alpha = 1$, $\beta = 4 - k$. Thus

$$(18) \ll \sum_{k=0}^{2} n^{5-k} n^{-(5-k)} + O(n^{-1}) \ll 1.$$

This gives the upper bound (5) on the number of vertices.

## 5. Circumscribed $L$-polygons

In this section, we consider a circumscribed model of random $L$-polygons. Let $K$ and $L$ be two convex discs with $C^2_+$ smooth boundaries. Assume that $K$ is $L$-convex and $L$ contains the origin in its interior. Let the set

$$K^* := L \div K = \bigcap_{y \in K} (L - y),$$

be called the *$L$-convex dual* of $K$, where $L \div K$ denotes the Minkowski-difference of $L$ and $K$, cf. [Sch14, Chapter 3]. By definition, $K^*$ is $L$-convex since it is the intersection of translates of $L$. Equivalently, the set $K^*$ can also be written as

$$K^* = \left\{x \in \mathbb{R}^2 \,|\, K \subset L - x\right\}. \tag{19}$$

The $L$-convex dual may also be defined by (19). We note that, for instance, in [NV23], the spindle convex dual is $(-K)^*$. Here, we use the notion of duality the same – somewhat more natural – way as in [MM22], where $K^*$ is the Minkowski-difference of $L$ and $K$.

We recall parts of Lemma 4.1 from [NV23]:

**Lemma 3.** *For any $L$-convex disc $K$ and $u \in S^1$, the following hold:*
*(i)* $h_K(u) + h_{K^*}(u) = h_L(u)$,
*(ii)* $r_K(u) + r_{K^*}(u) = r_L(u)$,
*(iii)* $A(K^*) = A(K) - 2A(K,L) + A(L)$,
*where $A(K,L)$ is the mixed area of $K$ and $L$.*

We consider the following probability model. Let $K$ and $L$ be two convex discs with $C^2_+$ smooth boundaries, and assume that $K$ is $L$-convex. Let $x_1, \ldots, x_n$ be $n$ i.i.d. random points in $K^*$ chosen according to the uniform probability distribution, and observe

$$K^*_{(n)} := [x_1, \ldots, x_n]^* = \bigcap_{x \in \{x_1, \ldots, x_n\}} (L - x).$$

By definition, $K^*_{(n)}$ contains $K$ and $K^*_{(n)} = ([x_1, \ldots, x_n]_L)^*$. Thus, $f_0(K^*_{(n)}) = f_0(([x_1, \ldots, x_n]_L)^*) = f_0([x_1, \ldots, x_n]_L)$.

**Theorem 4.** *Let $K$ and $L$ be two convex discs with $C^2_+$ smooth boundary and suppose that* (1) *holds. Then, with the notation above,*

$$\lim_{n\to\infty} \mathbb{E}(f_0(K^*_{(n)}))\cdot n^{-\frac{1}{3}} = \sqrt[3]{\frac{2}{3(A(K)-2A(K,L)+A(K)}}\,\Gamma\left(\frac{5}{3}\right)$$
$$\times\int_{S^1}\left(\frac{1}{\kappa_L(u)}\right)^{\frac{1}{3}}\frac{(\kappa_K(u)-\kappa_L(u))^{\frac{2}{3}}}{\kappa_K(u)}\,\mathrm{d}u.$$

We note that for $L=rB^2$ with $r>\max_{x\in\partial K}\kappa_K^{-1}(x)$, Theorem 4 gives back the result of [FV18, Theorem 6].

*Proof.* Since $K^*$ is $L$-convex and has $C^2_+$ boundary by Lemma 3 and $f_0(K^*_{(n)}) = f_0([x_1,\dots,x_n]_L)$, we can apply (2). Lemma 3 yields $\kappa_{K^*}(u) = \frac{\kappa_K(u)}{\frac{\kappa_K(u)}{\kappa_L(u)}-1}$ for any $u\in S^1$. Thus,

$$\int_{S^1}\frac{(\kappa_{K^*}(u)-\kappa_L(u))^{\frac{1}{3}}}{\kappa_{K^*(u)}}\,\mathrm{d}u = \int_{S^1}\frac{\left(\frac{\kappa_K(u)}{\frac{\kappa_K(u)}{\kappa_L(u)}-1}-\kappa_L(u)\right)^{\frac{1}{3}}}{\frac{\kappa_K(u)}{\frac{\kappa_K(u)}{\kappa_L(u)}-1}}\,\mathrm{d}u$$
$$=\int_{S^1}\frac{(\kappa_K(u)-\kappa_L(u))^{\frac{2}{3}}}{\kappa_K(u)\cdot\kappa_L(u)^{\frac{1}{3}}}\,\mathrm{d}u.$$

□

We note that in the spindle convex case similar statements were proved about the excess area and the difference of the perimeters in [FV18, Theorem 6].

By Theorem 1 and the definition of $K^*_{(n)}$, using Lemma 3, we get the following asymptotic upper bound on the variance of the number of vertices of $K^*_{(n)}$.

**Theorem 5.** *With the same assumptions as in Theorem 4,*

$$\mathrm{Var}(f_0(K^*_{(n)}))\ll n^{\frac{1}{3}}.$$

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