Finite monoidal intervals

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Abstract. An interval of cardinality $(2^{|A|} - 1)$ of the lattice of all transformation monoids on a finite set A is studied in the paper. We will show that finite monoidal intervals correspond to the transformation monoids in this interval.

1. Introduction

Let A be a finite set with at least three elements. It is well known that the set of all clones on A whose set of unary operations coincides with a transformation monoid M on A forms an interval in the lattice of all clones on A (see Á. Szendrei [8, Chapter 3]). An interval of this form is called a monoidal interval. The monoidal intervals partition the clone lattice into finitely many blocks. Since the clone lattice has continuum many elements if $|A| \geqslant 3$, one might expect that 'for most M' the monoidal interval $\mathrm{Int}(M)$ contains uncountably many clones. We remark that this is the case on 3-element sets: there are at least 499 transformation monoids (in 99 $\bowtie$-classes) among the all 699 transformation monoids (in 160 $\bowtie$-classes) for which the corresponding monoidal intervals have cardinality $2^{\aleph_0}$ (cf. Dormán–Makay–Maróti–Vajda [2]). Nevertheless, it turns out that for many interesting transformation monoids the corresponding monoidal intervals are finite.

Á. Szendrei in [8] posed the problem of classifying transformation monoids according to the cardinalities of the corresponding monoidal intervals. A complete classification of transformation monoids according to the sizes of the corresponding monoidal intervals seems a very hard problem at present. However, for certain classes of monoids we can solve this problem.

On a 3-element set there are 156 monoids (in 42 $\bowtie$-classes) for which the corresponding monoidal intervals wrere unknown, according to [2].

In this paper we will consider a certain class of transformation monoids that constitute an interval in the lattice of all submonoids of the full transformation semigroup with cardinality of $2^{|A|} - 1$.

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2. Preliminaries

For a finite set A we will denote the full transformation semigroup, and the set of unary constant operations on A by T_A, and Γ_A, respectively. For an arbitrary element a of A we will use the notation γ_a for the unary constant operation on A with value a, and a tuple whose all components are a will be denoted by $\hat{a}$. If $\mathbf{a}$ is an ℓ-tuple ($\ell \in \mathbb{N}$) then $\mathbf{a}_{[i]}$ will refer to its i-th component ($1 \leqslant i \leqslant \ell$).

For the set of positive integers we will use the notation $\mathbb{N}$, and we will refer to them as natural numbers.

Let A be a set and ℓ be a natural number. The set of all finitary operations on A will be denoted by $\mathcal{O}_A$. We call the operation f *essentially k-ary* ($k \in \mathbb{N},\ k \geqslant 2$) if it depends on exactly k of its variables. If f depends on at most one of its variables, we call f *essentially unary*. A set $\mathcal{C}$ of finitary operations on a set A is said to be a *clone* if it contains all the projections and is closed under superposition of operations. It is obvious that $\mathcal{O}_A$ and the set $\mathcal{P}_A$ of all projections on A are clones.

For a k-ary relation ϱ on A, a *ϱ-matrix* over A is a matrix whose columns belong to ϱ. An n-ary operation f on A *preserves* the m-ary relation ϱ on A if for every ϱ-matrix $X = (x_{i,j}) \in A^{n\times m}$ we have that

$$f(X) \stackrel{\text{def.}}{=} \begin{pmatrix} f(x_{1,1},\dots,x_{1,n}) \\ \vdots \\ f(x_{m,1},\dots,x_{m,n}) \end{pmatrix} \in \varrho.$$

If R is a set of finitary relations on A then $\operatorname{Pol}(R)$ will denote the set of all operations $f \in \mathcal{O}_A$ such that f preserves each relation in R.

It is well-known that a set $\mathcal{C}$ of finitary operations on A forms a clone if and only if $C = \operatorname{Pol}(R)$ holds for some set R of finitary relations on A.

Since the intersection of an arbitrary family of clones on A is also a clone, the set of all clones on A constitutes a complete lattice with respect to the set-theoretic inclusion. Furthermore, we can define the *clone generated by a subset* F of $\mathcal{O}_A$ as the intersection of all clones that contain F. This clone will be denoted by $\langle F\rangle$. For a natural number ℓ, the set of all ℓ-ary operations of a clone $\mathcal{C}$ will be denoted by $\mathcal{C}^{(\ell)}$.

Let M be a transformation monoid on A, and let $\operatorname{Int}(M)$ denote the collection of all clones $\mathcal{C}$ on A such that the set of unary operations of $\mathcal{C}$ is M. The clone $\langle M\rangle$ of essentially unary operations generated by M is a member of $\operatorname{Int}(M)$, in fact, it is the least member of $\operatorname{Int}(M)$, so $\operatorname{Int}(M)$ is non-empty. Furthermore, it is clear that every clone $\mathcal{C}$ in $\operatorname{Int}(M)$ is contained in the set

$$\begin{aligned}\operatorname{Sta}(M) = \big\{ f(x_1,\dots,x_\ell) \in \mathcal{O}_A \mid \ell \in \mathbb{N},\ \text{and} \\ f(\mu_1,\dots,\mu_\ell) \in M \text{ for all } \mu_1,\dots,\mu_\ell \in M \big\},\end{aligned}$$

which is called the *stabilizer* of the monoid M. It is easy to verify that $\mathrm{Sta}(M)$ is a clone on A, in fact, $\mathrm{Sta}(M) = \mathrm{Pol}(\varrho_M)$, where

$$\varrho_M = \{(\mu(0), \ldots, \mu(n-1)) : \mu \in M\}\,.$$

Therefore $\mathrm{Sta}(M)$ is the largest member of $\mathrm{Int}(M)$. So, a clone $\mathcal{C}$ on A belongs to $\mathrm{Int}(M)$ if and only if $\langle M\rangle \subseteq \mathcal{C} \subseteq \mathrm{Sta}(M)$. Thus $\mathrm{Int}(M)$ is the interval $[\langle M\rangle, \mathrm{Sta}(M)]$ in the lattice $\mathcal{O}_A$ of all clones on A. Such an interval is called a *monoidal interval*.

Define the relation $\bowtie$ on the set of all submonoids of T_A in the following way: the transformation monoids M and M' on A are $\bowtie$-related if there is a permutation $\pi \in S_A$ such that $M' = \pi \circ M \circ \pi^{-1}$. The $\bowtie$-relation is an equivalence relation, moreover, if $M \bowtie M'$ then the monoidal intervals $\mathrm{Int}(M)$ and $\mathrm{Int}(M')$ are isomorphic (as lattices), in particular, they have the same cardinalities.

If $F \subseteq \mathrm{Sta}(M)$ then *the clone generated by F over M* is $\langle F \cup M\rangle$, which will be denoted by $\langle F\rangle_M$. It is obvious that $\langle F\rangle_M$ belongs to $\mathrm{Int}(M)$.

The monoid M will be called *collapsing* if the monoidal interval corresponding to it contains only one element, that is, there is no essentially at least binary operation in the stabilizer of M (cf. Grabowski [4]).

We close this section by a well-known result that will be a useful tool in the sequel. Let $L = \{a, b\}$ be a 2-element set and $(L; \wedge, \vee)$ be a lattice with lattice order $a \leqslant b$.

Theorem 1 (Complete Disjunctive Normal Form)**.** *Let f be an ℓ-ary operation ($\ell \in \mathbb{N}$) on L.*

(a) *Then*

$$f(\mathbf{x}) = \bigvee_{\mathbf{d}\in H_f} \wedge\, \mathbf{dx},$$

where $H_f = f^{-1}(b)$, $\wedge\,\mathbf{dx} = \mathbf{d}_{[1]}\mathbf{x}_{[1]} \wedge \cdots \wedge \mathbf{d}_{[\ell]}\mathbf{x}_{[\ell]}$,

$$\mathbf{d}_{[i]}\mathbf{x}_{[i]} = \begin{cases} \mathbf{x}_{[i]}, & \text{if } \mathbf{d}_{[i]} = b, \\ \pi(\mathbf{x}_{[i]}), & \text{if } \mathbf{d}_{[i]} = a, \end{cases} \qquad (1 \leqslant i \leqslant \ell),$$

and $\pi = (a\ b)$.

(b) *If f is monotone with respect to the lattice order $\leqslant$ then*

$$f(\mathbf{x}) = \bigvee_{\mathbf{d}\in K_f} \wedge\, \mathbf{dx},$$

where $K_f = \min f^{-1}(b)$ and $\wedge\,\mathbf{dx} = \wedge_{i\in\{j:\mathbf{d}_{[j]}=b\}}\, \mathbf{x}_{[i]}$.

3. The transformation monoids and the main result

Let $n \geqslant 2$ be a natural number, $A = \{0, 1, \ldots, n-1\}$, $\mathbb{I} = \{n-1\}$, and $\mathbb{O} = A \setminus \mathbb{I}$. For $k \in \mathbb{O}$ define transformations σ_k and τ_k on A in the following way:

$$\sigma_k\colon A \to A,\ \sigma_k(x) = \begin{cases} k, & \text{if } x \in \mathbb{O}, \\ n-1, & \text{otherwise,} \end{cases}$$

$$\tau_k\colon A \to A,\ \tau_k(x) = \begin{cases} n-1, & \text{if } x \in \mathbb{O}, \\ k, & \text{otherwise.} \end{cases}$$

It is straightforward to check that $\tau_k \circ \tau_k = \sigma_k$ is an idempotent transformation for every k in $\mathbb{O}$. For subsets $\mathfrak{S}$ and $\mathfrak{T}$ of $\mathbb{O}$ let

$$M_{\mathfrak{S},\mathfrak{T}} = \Gamma_A \cup \{\mathrm{id}_A\} \cup \{\sigma_k : k \in \mathfrak{S}\} \cup \{\tau_k : k \in \mathfrak{T}\}\,.$$

Then

$$M_{\emptyset,\emptyset} = \Gamma_A \cup \{\mathrm{id}_A\}$$

and

$$M_{\mathbb{O},\mathbb{O}} = M_{\emptyset,\emptyset} \cup \{\sigma_k : k \in \mathbb{O}\} \cup \{\tau_k : k \in \mathbb{O}\}\,.$$

are transformation monoids. In the article we will study those monoidal intervals that correspond to the transformation monoids in the interval $I_n = [M_{\emptyset,\emptyset}, M_{\mathbb{O},\mathbb{O}}]$.

If $n = 2$ then $A = \{0,1\}$ and the interval I_2 consists of two transformation monoids:

$$M_{\emptyset,\emptyset} = \Gamma_A \cup \{\mathrm{id}_A\} \qquad \text{and} \qquad M_{\mathbb{O},\mathbb{O}} = T_A.$$

The corresponding monoidal intervals are finite, more precisely,

$$\begin{aligned} \mathrm{Int}(M_{\emptyset,\emptyset}) &= \{\langle M_{\emptyset,\emptyset}\rangle, \langle\wedge\rangle_{M_{\emptyset,\emptyset}}, \langle\vee\rangle_{M_{\emptyset,\emptyset}}, \langle\wedge,\vee\rangle_{M_{\emptyset,\emptyset}}\}, \\ \mathrm{Int}(M_{\mathbb{O},\mathbb{O}}) &= \{\langle M_{\mathbb{O},\mathbb{O}}\rangle, \langle+\rangle_{M_{\mathbb{O},\mathbb{O}}}, \mathcal{O}_A\}, \end{aligned}$$

where $\wedge$ and $\vee$ are the lattice operations with respect to the lattice order $0 \leqslant 1$ on $A = \{0,1\}$, furthermore, $+$ is the addition modulo 2. (cf., Post [7]).

In the rest of the paper, we will always assume that $n \geqslant 3$. First, we describe the transformation monoids in the interval I_n.

Proposition 2. *Let $\mathfrak{S}$ and $\mathfrak{T}$ be subsets of $\mathbb{O}$. Then*

$$M_{\mathfrak{S},\mathfrak{T}} = M_{\emptyset,\emptyset} \cup \{\sigma_k : k \in \mathfrak{S}\} \cup \{\tau_k : k \in \mathfrak{T}\}$$

is a transformation monoid if and only if either $\mathfrak{T} = \emptyset$ or $\mathfrak{S} = \mathfrak{T}$. Furthermore, the interval I_n contains $2^n - 1$ transformation monoids and it is isomorphic to the lattice ordered set $(\{0,1\} \times P(\mathbb{O})) \setminus \{(1,\emptyset)\}; \sqsubseteq)$, where $(a,H) \sqsubseteq (a',H')$ if and only if $a \leqslant a'$ and $H \subseteq H'$ $(a,a' \in \{0,1\},\ H,H' \subseteq \mathbb{O})$.

Proof. The first statement of the proposition follows from the following equalities:

$$\begin{aligned} \sigma_{k'} \circ \sigma_k &= \sigma_{k'}, & \tau_{k'} \circ \tau_k &= \sigma_{k'}, \\ \sigma_{k'} \circ \tau_k &= \tau_{k'}, & \tau_{k'} \circ \sigma_k &= \tau_{k'}, \end{aligned}$$

where k and k' are arbitrary elements in $\mathbb{O}$.

To prove the second statement, we remark that the mapping

$$\begin{aligned} & I_n \to (\{0,1\} \times P(\mathbb{O})) \setminus \{(1,\emptyset)\}, \\ M_{\emptyset,\emptyset} \cup \{\sigma_k : k \in \mathfrak{S}\} \cup \{\tau_k : k \in \mathfrak{T}\} \mapsto & \begin{cases} (0,\mathfrak{S}), & \text{if } \mathfrak{T} = \emptyset, \\ (1,\mathfrak{S}), & \text{if } \mathfrak{S} = \mathfrak{T} \neq \emptyset \end{cases} \end{aligned}$$

is an order preserving bijection for which its inverse is also order preserving. $\square$

The aim of the paper is to proof the following statement.

Theorem 3. *Let $n \geqslant 3$ be a positive integer. Then for each transformation monoid M in the interval I_n the monoidal interval $\mathrm{Int}(M)$ is finite.*

Detailed information about the monoidal intervals can be found in Theorems 13, 17, 24, and 25.

If $\mathfrak{S} = \mathfrak{T} = \emptyset$ then the transformation monoid $M_{\emptyset,\emptyset} = \Gamma_A \cup \{\mathrm{id}_A\}$ is collapsing (cf., P. P. Pálfy [6]). Therefore, in the sequel, we will assume that $|\mathfrak{S}| \geqslant 1$. Without loss of generality, we may also assume that

$$\mathfrak{S} = \{0, \dots, p-1\},$$

where $1 \leqslant p \leqslant n-1$ is an integer.

Let $M = M_{\mathfrak{S},\mathfrak{T}}$, where $\mathfrak{S}, \mathfrak{T} \subseteq \mathbb{O}$ and either $\mathfrak{T} = \emptyset$ or $\mathfrak{S} = \mathfrak{T} \neq \emptyset$. Let α be the equivalence relation $\mathbb{O}^2 \cup \mathbb{I}^2$ on A. Then α is a congruence of $(A; \mathrm{Sta}(M))$, since α is a congruence of $(A; M)$. The factor transformation monoid $\bar{M} = M/\alpha$ on $\bar{A} = A/\alpha$ is

$$\bar{M} = \begin{cases} C_{\bar{A}} \cup \{\mathrm{id}_{\bar{A}}\}, & \text{if } \mathfrak{T} = \emptyset, \\ T_{\bar{A}}, & \text{otherwise,} \end{cases}$$

and

$$\mathrm{Int}(\bar{M}) = \begin{cases} \{\langle \bar{M} \rangle, \langle \bar{\wedge} \rangle_{\bar{M}}, \langle \bar{\vee} \rangle_{\bar{M}}, \langle \bar{\wedge}, \bar{\vee} \rangle_{\bar{M}}\}, & \text{if } \mathfrak{T} = \emptyset, \\ \{\langle \bar{M} \rangle, \langle \bar{+} \rangle_{\bar{M}}, \mathcal{O}_{\bar{A}}\}, & \text{otherwise,} \end{cases}$$

where $\bar{\wedge}$ and $\bar{\vee}$ are the lattice operations with respect to the lattice order $\mathbb{O} \leqslant \mathbb{I}$.

4. The fine structure of operations in $\mathrm{Sta}(M)$

Let $M = M_{\mathfrak{S},\mathfrak{T}}$, where $\mathfrak{S}, \mathfrak{T} \subseteq \mathbb{O}$ and either $\mathfrak{T} = \emptyset$ or $\mathfrak{S} = \mathfrak{T}$. Let f be an ℓ-ary operation in $\mathrm{Sta}(M)$ ($\ell \in \mathbb{N}$). For an ℓ-tuple $\mathbf{a} \in A^\ell$ with property $f(\mathbf{a}) \in \mathbb{O}$ let $W_{\mathbf{a}} = \big\{i : \mathbf{a}_{[i]} \in \mathbb{O}\big\}$. For an arbitrary ℓ-tuple $\mathbf{e} \in A^\ell$ let $\mathbf{e}_{\downarrow\mathbf{a}}$ be the ℓ-tuple whose i-th component ($1 \leqslant i \leqslant \ell$) is

$$(\mathbf{e}_{\downarrow\mathbf{a}})_{[i]} = \begin{cases} \mathbf{e}_{[i]}, & \text{if } i \in W_{\mathbf{a}}, \\ n-1, & \text{otherwise.} \end{cases}$$

Define the operation $f_{\mathbf{a}}$ in the following way:

$$f_{\mathbf{a}} \colon A^\ell \to A,\ \mathbf{e} \mapsto f(\mathbf{e}_{\downarrow\mathbf{a}}).$$

Then $f_{\mathbf{a}} \in \mathrm{Sta}(M)$ and $f_{\mathbf{a}}(\mathbb{O}^\ell) = f(\mathbf{a}/\alpha) \subseteq \mathbb{O}$, hence, the operation $(f_{\mathbf{a}})|_{\mathbb{O}}$ is in $\mathrm{Sta}(M_{\mathbb{O}})$, where $M_{\mathbb{O}} = \{\mu|_{\mathbb{O}} : \mu \in M,\ \mu(\mathbb{O}) \subseteq \mathbb{O}\}$ is a transformation monoid on $\mathbb{O}$ and

$$\mathrm{Sta}(M_{\mathbb{O}}) = \begin{cases} \langle \wedge, \vee \rangle_{M_{\mathbb{O}}}, & \text{if } n = 3, \\ \langle M_{\mathbb{O}} \rangle, & \text{if } n \geqslant 4, \end{cases} \tag{1}$$

(cf., E. L. Post [7] for $n = 3$ and P. P. Pálfy [6] for $n \geqslant 4$).

Proposition 4. *Let f be an ℓ-ary operation in* $\mathrm{Sta}(M)$ $(\ell \in \mathbb{N})$*. Then for arbitrary ℓ-tuples $\mathbf{b}, \mathbf{c} \in A^\ell$ if $f(\mathbf{b}), f(\mathbf{c}) \in \mathbb{O}$ then $(f_{\mathbf{b}})|_{\mathbb{O}} = (f_{\mathbf{c}})|_{\mathbb{O}}$.*

Corollary 5. *Let f and g be ℓ-ary operations in* $\mathrm{Sta}(M)$ $(\ell \in \mathbb{N})$*. If*

$$f^{-1}(n-1) = g^{-1}(n-1)$$

and there is an ℓ-tuple $\mathbf{b}$ such that $(f_{\mathbf{b}})|_{\mathbb{O}} = (g_{\mathbf{b}})|_{\mathbb{O}}$ then $f = g$.

Proof. Suppose that f and g satisfy the requirements of the statement. Let $\mathbf{a}$ be an ℓ-tuple in A^ℓ. By the assumption

$$f(\mathbf{a}) = n - 1 \iff g(\mathbf{a}) = n - 1,$$

hence, we may assume that $f(\mathbf{a}), g(\mathbf{a}) \in \mathbb{O}$. Let $\mathbf{a}'$ be the following ℓ-tuple:

$$\mathbf{a}'_{[i]} = \begin{cases} \mathbf{a}_{[i]}, & \text{if } i \in W_{\mathbf{a}}, \\ 0, & \text{otherwise.} \end{cases}$$

Then $\mathbf{a}' \in \mathbb{O}^\ell$ and, by Proposition 4,

$$f(\mathbf{a}) = f_{\mathbf{a}}(\mathbf{a}') = f_{\mathbf{b}}(\mathbf{a}') = g_{\mathbf{b}}(\mathbf{a}') = g_{\mathbf{a}}(\mathbf{a}') = g(\mathbf{a}),$$

which completes the proof. □

In the proof of Proposition 4 we will use several simple statements that will be summarized in the next two lemmas.

Let f be an ℓ-ary operation in $\mathrm{Sta}(M)$ $(\ell \in \mathbb{N})$. For an ℓ-tuple $\mathbf{a}$ and $j \in A$ define transformations $\omega_1^{(\mathbf{a},j)}, \dots, \omega_\ell^{(\mathbf{a},j)} \in M$ as follows:

$$\omega_i^{(\mathbf{a},j)} = \begin{cases} \mathrm{id}_A, & \text{if } \mathbf{a}_{[i]} = j \\ c_{\mathbf{a}_{[i]}}, & \text{otherwise,} \end{cases}$$

and set $\omega^{(\mathbf{a},j)} = f(\omega_1^{(\mathbf{a},j)}, \dots, \omega_\ell^{(\mathbf{a},j)})$. Then $\omega^{(\mathbf{a},j)} \in M$.

Lemma 6. *Let f be an ℓ-ary operation in* $\mathrm{Sta}(M) \setminus \langle M \rangle$.

(a) *If either $f(\hat{0}) = 0$ and f is not surjective or $f(\hat{0}) \in \mathbb{O} \setminus \{0\}$ then $f(\mathbb{O}^\ell) = \{f(\hat{0})\}$ and the range of f is $\{f(\hat{0}), n-1\}$.*
(b) *If $f(\hat{0}) = n - 1$ and either f is not surjective or $\mathfrak{T} = \emptyset$ then f is the constant operation with value $n - 1$.*

Proof. (a) Suppose that either $f(\hat{0}) = 0$ and f is not surjective or $f(\hat{0}) \in \mathbb{O} \setminus \{0\}$. We will prove that for every integer k ($0 \leqslant k < n-1$) equality $f(\{0,\dots,k\}^\ell) = \{f(\hat{0})\}$ holds. We proceed by induction on k. For $k=0$ the statement is true. Suppose that $f(\{0,\dots,j\}^\ell) = \{f(\hat{0})\}$ holds for $0 \leqslant j < k$. Let $\mathbf{b}$ be an arbitrary ℓ-tuple in $\{0,\dots,k\}^\ell$. Then

$$(\omega_1^{(\mathbf{b},k)}(0),\dots,\omega_\ell^{(\mathbf{b},k)}(0)) \in \{0,\dots,k-1\}^\ell.$$

Hence, by the induction hypothesis, $\omega^{(\mathbf{b},k)}(0) = f(\omega_1^{(\mathbf{b},k)}(0),\dots,\omega_\ell^{(\mathbf{b},k)}(0)) = f(\hat{0})$. If $f(\hat{0}) = 0$ then because of the non-surjectivity of f, $\omega^{(\mathbf{b},k)}$ can not be id_A. Therefore, $\omega^{(\mathbf{b},k)} \in \{\gamma_{f(\hat{0})}, \sigma_{f(\hat{0})}\}$, which implies that $f(\mathbf{b}) = \omega^{(\mathbf{b},k)}(k) = f(\hat{0})$.

To prove that the range of f is $\{f(\hat{0}), n-1\}$, choose an arbitrary ℓ-tuple $\mathbf{a}$ in $A^\ell \setminus \mathbb{O}^\ell$. Then $\omega^{(\mathbf{a},n-1)}(0) = f(\omega_1^{(\mathbf{a},n-1)}(0),\dots,\omega_\ell^{(\mathbf{a},n-1)}(0)) = f(\hat{0})$ implies that $\omega^{(\mathbf{a},n-1)} \in \{\gamma_{f(\hat{0})}, \sigma_{f(\hat{0})}\}$, hence $f(\mathbf{a}) = \omega^{(\mathbf{a},n-1)}(n-1) \in \{f(\hat{0}), n-1\}$. We note that $\sigma_{f(\hat{0})}$ must belong to M since otherwise f would be a constant operation.

(b) Suppose that $f(\hat{0}) = n-1$. Then $\bar{f}(\hat{\mathbb{O}}) = \mathbb{I}$. Let $\mathbf{a} \in A^\ell \setminus \mathbb{O}^\ell$. If $\mathfrak{T} = \emptyset$ then $\omega^{(\mathbf{a},n-1)}(0) = \cdots = \omega^{(\mathbf{a},n-1)}(n-2) = n-1$ imply that $\omega^{(\mathbf{a},n-1)} = \gamma_{n-1}$, hence, $f(\mathbf{a}) = \omega^{(\mathbf{a},n-1)}(n-1) = n-1$, that is, f is a constant operation with value $n-1$. Suppose that f is not surjective and it is not constant. Then there is an ℓ-tuple $\mathbf{a} \in A^\ell$ such that $f(\mathbf{a}) \in \mathbb{O}$. Then $(f_\mathbf{a})|_\mathbb{O}$ is essentially unary operation, and so, there is a transformation $\mu \in M_\mathbb{O}$ and an index i for which $f_\mathbf{a}(\mathbf{b}) = \mu(\mathbf{b}_{[i]})$ holds for all ℓ-tuples $\mathbf{b} \in \mathbb{O}^\ell$. If $\mu = \mathrm{id}_\mathbb{O}$ then f would be surjective, hence, μ is a constant transformation, say $\mu = \gamma_b$, and so, $\tau_b \in M$. Therefore, $\mathfrak{T}$ is not empty.

The statements are proved. □

Lemma 7. *Let k be an arbitrary element in $\mathbb{O}$.*

(a) *If $\mathbf{a} \in \{0, n-1\}^\ell$ then there is a transformation $\mu \in M$ such that $\mu(0) = f(\hat{0})$ and $\mu(n-1) = f(\mathbf{a})$.*
(b) *If $k \in \mathfrak{S}$ and $\mathbf{a} \in \{k, n-1\}^\ell$ then there is a transformation $\mu \in M$ such that $\mu(0) = f(\hat{k})$ and $\mu(n-1) = f(\mathbf{a})$.*
(c) *If $k \in \mathfrak{T}$ and $\mathbf{a}, \mathbf{a}' \in \{k, n-1\}^\ell$ then there is a transformation $\mu \in M$ such that $\mu(0) = f(\mathbf{a}')$ and $\mu(n-1) = f(\mathbf{a})$.*

Proof. (a) Let $\mathbf{a}$ be an arbitrary ℓ-tuple in $\{0, n-1\}^\ell$. For every index i ($1 \leqslant i \leqslant \ell$) define transformation μ_i in the following way:

$$\mu_i = \begin{cases} \gamma_0, & \text{if } i \in W_\mathbf{a}, \\ \mathrm{id}_A, & \text{otherwise.} \end{cases}$$

Then for the transformation $\mu = f(\mu_1,\dots,\mu_\ell) \in M$, we have that $\mu(0) = f(\hat{0})$ and $\mu(n-1) = f(\mathbf{a})$.

(b) Suppose that $k \in \mathfrak{S}$. Let $\mathbf{a}$ be an arbitrary ℓ-tuple in $\{k, n-1\}^\ell$. For every index i ($1 \leqslant i \leqslant \ell$) define transformation μ_i in the following way:

$$\mu_i = \begin{cases} \gamma_k, & \text{if } i \in W_{\mathbf{a}}, \\ \sigma_k, & \text{otherwise.} \end{cases}$$

Then for the transformation $\mu = f(\mu_1, \dots, \mu_\ell) \in M$, we have that $\mu(0) = f(\hat{k})$ and $\mu(n-1) = f(\mathbf{a})$.

(c) Suppose that $k \in \mathfrak{T}$. Then $\mathfrak{S} = \mathfrak{T}$, hence, $k \in \mathfrak{S}$ also holds. Let $\mathbf{a}$ and $\mathbf{a}'$ be arbitrary ℓ-tuples in $\{k, n-1\}^\ell$. For every index i ($1 \leqslant i \leqslant \ell$) define transformation μ_i in the following way:

$$\mu_i = \begin{cases} \gamma_k, & \text{if } i \in W_{\mathbf{a}} \cap W_{\mathbf{a}'}, \\ \sigma_k, & \text{if } i \in W_{\mathbf{a}} \setminus W_{\mathbf{a}'}, \\ \tau_k, & \text{if } i \in W_{\mathbf{a}'} \setminus W_{\mathbf{a}}, \\ \gamma_{n-1}, & \text{otherwise.} \end{cases}$$

Then for the transformation $\mu = f(\mu_1, \dots, \mu_\ell) \in M$, we have that $\mu(0) = f(\mathbf{a}')$ and $\mu(n-1) = f(\mathbf{a})$. $\square$

Now we are in a position to prove Proposition 4. The proof will fall naturally into two parts.

Proof of Proposition 4. Suppose for a contradiction that there are ℓ-tuples $\mathbf{b}, \mathbf{c} \in A^\ell$ such that $f(\mathbf{b}), f(\mathbf{c}) \in \mathbb{O}$ and $(f_{\mathbf{b}})|_{\mathbb{O}} \neq (f_{\mathbf{c}})|_{\mathbb{O}}$. Then there is an ℓ-tuple $\mathbf{d} \in \mathbb{O}^\ell$ for which $f_{\mathbf{b}}(\mathbf{d}) \neq f_{\mathbf{c}}(\mathbf{d})$ holds.

We will distinguish two cases according to $n = 3$ or $n \geqslant 4$ hold.

Case 1: $n = 3$.

Case 1.1: $\mathfrak{T} = \emptyset$. We may assume that $f_{\mathbf{b}}(\mathbf{d}) = 0$ and $f_{\mathbf{c}}(\mathbf{d}) = 1$. Let $\mathbf{d}'$ and $\mathbf{d}''$ be the following ℓ-tuples:

$$\mathbf{d}'_{[i]} = \begin{cases} \mathbf{d}_{[i]}, & \text{if } i \in W_{\mathbf{b}} \cap W_{\mathbf{c}}, \\ 0, & \text{if } i \in W_{\mathbf{b}} \setminus W_{\mathbf{c}}, \\ 2, & \text{if } i \notin W_{\mathbf{b}}, \end{cases} \qquad \mathbf{d}''_{[i]} = \begin{cases} \mathbf{d}_{[i]}, & \text{if } i \in W_{\mathbf{b}} \cap W_{\mathbf{c}}, \\ 1, & \text{if } i \in W_{\mathbf{c}} \setminus W_{\mathbf{b}}, \\ 2, & \text{if } i \notin W_{\mathbf{c}}, \end{cases}$$

($1 \leqslant i \leqslant \ell$). Then $\mathbf{d}' \, \varrho^\ell \, \mathbf{d}_{\downarrow\mathbf{b}}$ and $\mathbf{d}_{\downarrow\mathbf{c}} \, \varrho^\ell \, \mathbf{d}''$, where

$$\varrho = \pi_{1,2}(\varrho_M) = \{(0,0), (0,1), (1,1), (2,2)\}.$$

Hence $f(\mathbf{d}') \, \varrho \, f(\mathbf{d}_{\downarrow\mathbf{b}}) = 0$ and $1 = f(\mathbf{d}_{\downarrow\mathbf{c}}) \, \varrho \, f(\mathbf{d}'')$, which imply that $f(\mathbf{d}') = 0$ and $f(\mathbf{d}'') = 1$. For the ℓ-tuple $\mathbf{d}^\dagger$, where

$$(\mathbf{d}^\dagger)_{[i]} = \begin{cases} \mathbf{d}_{[i]}, & \text{if } i \in W_{\mathbf{b}} \cap W_{\mathbf{c}}, \\ 0, & \text{if } i \in W_{\mathbf{b}} \setminus W_{\mathbf{c}}, \\ 1, & \text{if } i \in W_{\mathbf{c}} \setminus W_{\mathbf{b}}, \\ 2, & \text{otherwise,} \end{cases}$$

we have that $\mathbf{d}^\dagger = (\mu_1(1), \dots, \mu_\ell(1))$, moreover, $\mathbf{d}' = (\mu_1(2), \dots, \mu_\ell(2))$, where

$$\mu_i = \begin{cases} \gamma_{\mathbf{d}_{[i]}}, & \text{if } i \in W_{\mathbf{b}} \cap W_{\mathbf{c}}, \\ \gamma_0, & \text{if } i \in W_{\mathbf{b}} \setminus W_{\mathbf{c}}, \\ \mathrm{id}_A, & \text{if } i \in W_{\mathbf{c}} \setminus W_{\mathbf{b}}, \\ \gamma_2, & \text{otherwise,} \end{cases} \qquad (1 \leqslant i \leqslant \ell).$$

Hence $f(\mathbf{d}^\dagger) = 0$ since $f(\mathbf{d}') = 0$. Define transformations $\nu_1, \dots, \nu_\ell \in M$ as follows:

$$\nu_i = \begin{cases} \gamma_{\mathbf{d}_{[i]}}, & \text{if } i \in W_{\mathbf{b}} \cap W_{\mathbf{c}}, \\ \mathrm{id}_A, & \text{if } i \in W_{\mathbf{b}} \setminus W_{\mathbf{c}}, \\ \gamma_1, & \text{if } i \in W_{\mathbf{c}} \setminus W_{\mathbf{b}}, \\ \gamma_2, & \text{otherwise,} \end{cases} \qquad (1 \leqslant i \leqslant \ell)$$

and set $\nu = f(\nu_1, \dots, \nu_\ell)$. Then $\nu \in M$, $\nu(0) = f(\mathbf{d}^\dagger) = 0$ and $\nu(2) = f(\mathbf{d}'') = 1$, we get a contradiction.

Case 1.2: $\mathfrak{T} \neq \emptyset$. Then $0 \in \mathfrak{T}$ since $0 \in \mathfrak{S} = \mathfrak{T}$. If the operation f is not surjective then the statement of the proposition follows from Lemma 6, hence we may suppose that f is surjective.

If $f(\hat{0}) = n-1$ then $f(\mathbb{O}^\ell) = \mathbb{I}$, and so, $\mathfrak{S} = \mathfrak{T} = \mathbb{O}$ and $M = M_{\mathbb{O},\mathbb{O}}$. Define transformations $\mu_1, \dots, \mu_\ell \in M$ as follows:

$$\mu_i = \begin{cases} \gamma_{\mathbf{d}_{[i]}}, & \text{if } i \in W_{\mathbf{b}} \cap W_{\mathbf{c}}, \\ \sigma_{\mathbf{d}_{[i]}}, & \text{if } i \in W_{\mathbf{b}} \setminus W_{\mathbf{c}}, \\ \tau_{\mathbf{d}_{[i]}}, & \text{if } i \in W_{\mathbf{c}} \setminus W_{\mathbf{b}}, \\ \gamma_2, & \text{otherwise,} \end{cases} \qquad (1 \leqslant i \leqslant \ell)$$

and set $\mu = f(\mu_1, \dots, \mu_\ell)$. Then $\mu \in M$, $\mu(0) = f_{\mathbf{b}}(\mathbf{d}) = 0$ and $\mu(2) = f_{\mathbf{c}}(\mathbf{d}) = 1$, we get a contradiction.

Suppose that $f(\hat{0}) \in \mathbb{O}$. Then we may also assume that $\hat{0} \in \{\mathbf{b}, \mathbf{c}\}$ since $f_{\hat{0}}(\mathbf{d}) = f(\mathbf{d})$ and $f_{\mathbf{b}}(\mathbf{d}) \neq f_{\mathbf{c}}(\mathbf{d})$. Set $\{\hat{0}, \mathbf{x}\} = \{\mathbf{b}, \mathbf{c}\}$. Let $\mathbf{d}'$ be the following ℓ-tuple:

$$\mathbf{d}'_{[i]} = \begin{cases} \mathbf{d}_{[i]}, & \text{if } i \in W_{\mathbf{x}} \text{ or } i \notin W_{\mathbf{x}},\ \mathbf{d}_{[i]} = 0, \\ 0, & \text{if } i \notin W_{\mathbf{x}},\ \mathbf{d}_{[i]} = 1. \end{cases}$$

Then for every index i $(1 \leqslant i \leqslant \ell)$ we have that

$$(\mathbf{d}'_{[i]}, \mathbf{d}_{[i]}, (\mathbf{d}_{\downarrow\mathbf{x}})_{[i]}) = \begin{cases} (\mathbf{d}_{[i]}, \mathbf{d}_{[i]}, \mathbf{d}_{[i]}), & \text{if } i \in W_{\mathbf{x}}, \\ (0,0,2), & \text{if } i \notin W_{\mathbf{x}} \text{ and } \mathbf{d}_{[i]} = 0, \\ (0,1,2), & \text{if } i \notin W_{\mathbf{x}} \text{ and } \mathbf{d}_{[i]} = 1, \end{cases}$$

hence,

$$X = \begin{pmatrix} \mathbf{d}'_{[1]} & \dots & \mathbf{d}'_{[\ell]} \\ \mathbf{d}_{[1]} & \dots & \mathbf{d}_{[\ell]} \\ (\mathbf{d}_{\downarrow\mathbf{x}})_{[1]} & \dots & (\mathbf{d}_{\downarrow\mathbf{x}})_{[\ell]} \end{pmatrix}$$

is a ϱ_M-matrix. However, $f(X) = \begin{pmatrix} f(\mathbf{d}') \\ f(\mathbf{d}) \\ f(\mathbf{d}_{\downarrow\mathbf{x}}) \end{pmatrix}$ does not belong to ϱ_M since $f(\mathbf{d}) = f(\mathbf{d}_{\downarrow\hat{0}})$ and $f(\mathbf{d}_{\downarrow\mathbf{x}})$ are distinct elements in $\mathbb{O}$, which is a contradiction.

Case 2: $n \geqslant 4$. Then the operations $(f_{\mathbf{b}})|_{\mathbb{O}}$ and $(f_{\mathbf{c}})|_{\mathbb{O}}$ are essentially unary operations, hence, there are indexes $i_0, j_0 \in \{1, \dots, \ell\}$ and transformations $\mu, \nu \in M_{\mathbb{O}}$ such that

$$f_{\mathbf{b}}(\mathbf{a}) = \mu(\mathbf{a}_{[i_0]}),$$
$$f_{\mathbf{c}}(\mathbf{a}) = \nu(\mathbf{a}_{[j_0]})$$

hold for every ℓ-tuple $\mathbf{a} \in \mathbb{O}^\ell$.

Case 2.1: $\mu = \gamma_y$ and $\nu = \gamma_z$. Then y and z are distinct elements in $\mathbb{O}$, furthermore, we may assume that $\mathbf{d} = \hat{0}$. Therefore, $\mathbf{d}_{\downarrow\mathbf{b}}, \mathbf{d}_{\downarrow\mathbf{c}} \in \{0, n-1\}^\ell$. Define transformations $\xi_1, \dots, \xi_\ell$, $\xi'_1, \dots, \xi'_\ell$ in the following way:

$$\xi_i = \begin{cases} \gamma_0, & \text{if } i \in W_{\mathbf{b}}, \\ \mathrm{id}_A, & \text{otherwise}, \end{cases} \qquad (1 \leqslant i \leqslant \ell),$$

$$\xi'_i = \begin{cases} \gamma_0, & \text{if } i \in W_{\mathbf{c}}, \\ \mathrm{id}_A, & \text{otherwise}, \end{cases} \qquad (1 \leqslant i \leqslant \ell),$$

and set $\xi = f(\xi_1, \dots, \xi_\ell)$, $\xi' = f(\xi'_1, \dots, \xi'_\ell)$. Then $\xi, \xi' \in M$, $\xi(0) = \xi'(0) = f(\hat{0})$ and $\xi(n-1) = f(\mathbf{d}_{\downarrow\mathbf{b}}) = y$, $\xi'(n-1) = f(\mathbf{d}_{\downarrow\mathbf{c}}) = z$. Hence, $f(\hat{0}) = n-1$. Thus, τ_y and τ_z belong to M, and so, $\mathfrak{T}$ is not empty. We get a contradiction as follows. Let $\varphi = f(\varphi_1, \dots, \varphi_\ell)$, where

$$\varphi_i = \begin{cases} \mathrm{id}_A, & \text{if } i \in W_{\mathbf{b}} \setminus W_{\mathbf{c}}, \\ \tau_0, & \text{if } i \in W_{\mathbf{c}} \setminus W_{\mathbf{b}}, \\ \gamma_0, & \text{if } i \in W_{\mathbf{b}} \cap W_{\mathbf{c}}, \\ \gamma_{n-1}, & \text{otherwise}, \end{cases} \qquad (1 \leqslant i \leqslant \ell).$$

Then $\varphi \in M$, $\varphi(0) = f(\mathbf{d}_{\downarrow\mathbf{b}}) = y$, and $\varphi(n-1) = f(\mathbf{d}_{\downarrow\mathbf{c}}) = z$, which is a contradiction.

Case 2.2: $\mu = c_y$ and $\nu = \mathrm{id}_{\mathbb{O}}$ with $y \in \mathbb{O}$. Since $\hat{0}_{\downarrow\mathbf{b}} \in \{0, n-1\}^\ell$ for the transformations

$$\xi_i = \begin{cases} \gamma_0, & \text{if } 0 \in W_{\mathbf{b}}, \\ \mathrm{id}_A, & \text{otherwise}, \end{cases} \qquad (1 \leqslant i \leqslant \ell)$$

we have that $(\xi_1(0), \dots, \xi_\ell(0)) = \hat{0}$ and $(\xi_1(n-1), \dots, \xi_\ell(n-1)) = \hat{0}_{\downarrow\mathbf{b}}$. Then for the transformation $\xi = f(\xi_1, \dots, \xi_\ell) \in M$ we obtain that

$$\xi(n-1) = f(\hat{0}_{\downarrow\mathbf{b}}) = y \in \mathbb{O},$$

which implies that $f(\hat{0}) = \xi(0) \in \{y, n-1\}$.

Suppose that $f(\hat{0}) = y \in \mathbb{O}$. Then f can be restricted to $\mathbb{O}$, hence, $f|_{\mathbb{O}} \in \langle M_{\mathbb{O}} \rangle$. Fix an element $y' \in \mathbb{O} \setminus \{y\}$. Let $\varphi, \varphi_1, \dots, \varphi_\ell$ be the following transformations:

$$\varphi_i = \begin{cases} \gamma_{y'}, & \text{if } i = j_0, \\ \gamma_0, & \text{if } i \in W_{\mathbf{c}},\ i \neq j_0, \\ \mathrm{id}_A, & \text{otherwise,} \end{cases} \qquad (1 \leqslant i \leqslant \ell),$$

and $\varphi = f(\varphi_1, \ldots, \varphi_\ell)$. Then for the ℓ-tuple $\mathbf{e} = (0, \ldots, 0, \overbrace{\quad y' \quad}^{j_0\text{-th comp.}}, 0, \ldots, 0)$ we get that

$$\varphi(n-1) = f(\mathbf{e}_{\downarrow\mathbf{c}}) = f_{\mathbf{c}}(\mathbf{e}) = v(\mathbf{e}_{[j_0]}) = y',$$

from which $f(\mathbf{e}) = \varphi(0) \in \{y', n-1\}$ follows, thus, $f(\mathbf{e}) = y'$ since $f(\mathbf{e}) \in \mathbb{O}$. As a consequence of this result, we obtain that $f|_{\mathbb{O}}(a_1, \ldots, a_\ell) = a_{j_0}$ holds for each ℓ-tuple $(a_1, \ldots, a_\ell) \in \mathbb{O}^\ell$. Thus, $y = 0$. Let $\mathbf{a}$ be the ℓ-tuple such that $\mathbf{a}_{[k]} = n-2$ if $k \in W_{\mathbf{b}}$ and $\mathbf{a}_{[k]} = n-1$ if $k \notin W_{\mathbf{b}}$. Then

$$f(\mathbf{a}) = f(\mathbf{a}_{\downarrow\mathbf{b}}) = f(\widehat{n-2}_{\downarrow\mathbf{b}}) = f_{\mathbf{b}}(\widehat{n-2}) = 0.$$

Choose transformations $\psi_1, \ldots, \psi_\ell$ in the following way:

$$\psi_k = \begin{cases} \gamma_{n-2}, & \text{if } k \in W_{\mathbf{b}}, \\ \mathrm{id}_A, & \text{otherwise,} \end{cases} \qquad (1 \leqslant k \leqslant \ell),$$

and set $\psi = f(\psi_1, \ldots, \psi_\ell)$. Then $\psi(n-1) = f(\mathbf{a})$ and $\psi(n-2) = f(\widehat{n-2})$. This leads to a contradiction, since $f(\widehat{n-2}) = n-2$ and $f(\mathbf{a}) = 0$. Hence, $f(\hat{0}) = n-1$. Let k be an arbitrary element in $\mathbb{O}$, and let $\chi_1, \ldots, \chi_\ell$ be the following transformations in M:

$$\chi_i = \begin{cases} \gamma_k, & \text{if } i \in W_{\mathbf{c}}, \\ \mathrm{id}_A, & \text{otherwise,} \end{cases} \qquad (1 \leqslant i \leqslant \ell),$$

and set $\chi = f(\chi_1, \ldots, \chi_\ell)$. Then $\chi \in M$ and $\chi(n-1) = f_{\mathbf{c}}(\hat{k}) = k \in \mathbb{O}$, $\chi(0) = n-1$, hence, $\chi = \tau_k$. Therefore, $\mathfrak{T} = \mathbb{O}$. Let $z \in \mathbb{O} \setminus \{y\}$. Then $f_{\mathbf{b}}(\hat{z}) = y$ and $f_{\mathbf{c}}(\hat{z}) = z$ and $\hat{z}_{\downarrow\mathbf{b}}, \hat{z}_{\downarrow\mathbf{c}} \in \{z, n-1\}^\ell$. By Lemma 7 (c), there is a transformation $\omega \in M$ such that $\omega(0) = f(\hat{z}_{\downarrow\mathbf{b}}) = f_{\mathbf{b}}(\hat{z}) = y$ and $\omega(n-1) = f(\hat{z}_{\downarrow\mathbf{c}}) = f_{\mathbf{c}}(\hat{z}) = z$, we get a contradiction.

Case 2.3: Finally, suppose that $\mu = \nu = \mathrm{id}_{\mathbb{O}}$. Then $i_0 \in W_{\mathbf{b}}$, $j_0 \in W_{\mathbf{c}}$ and $i_0 \neq j_0$. Since $\hat{0}$ and $\hat{0}_{\downarrow\mathbf{b}}$ belong to $\{0, n-1\}^\ell$, by Lemma 7 (a), there is a transformation $\xi \in M$ such that $\xi(0) = f(\hat{0})$ and $\xi(n-1) = f(\hat{0}_{\downarrow\mathbf{b}}) = 0$. Thus, we have that $f(\hat{0}) \in \{0, n-1\}$. Suppose that $f(\hat{0}) = 0$. Then either $f_{\hat{0}}(\mathbf{d}) \neq f_{\mathbf{b}}(\mathbf{d})$ or $f_{\hat{0}}(\mathbf{d}) \neq f_{\mathbf{c}}(\mathbf{d})$. We may assume that the latter inequality holds, that is, $f_{\hat{0}}(\mathbf{d}) \neq f_{\mathbf{c}}(\mathbf{d})$. Under the above assumption, $W_{\mathbf{c}} \subsetneq A = W_{\hat{0}}$.

If $i_0 \notin W_{\mathbf{c}}$ then let $\mathbf{a}$ be the ℓ-tuple with components $\mathbf{a}_{[k]} = 0$ if $k \in W_{\mathbf{c}}$ and $\mathbf{a}_{[k]} = n-1$ if $k \in A \setminus W_{\mathbf{c}}$. Then $f(\mathbf{a}) = f(\hat{0}_{\downarrow\mathbf{c}}) = f_{\mathbf{c}}(\hat{0}) = 0$. Let

$\varphi = f(\varphi_1, \dots, \varphi_\ell)$, where

$$\varphi_k = \begin{cases} \gamma_0, & \text{if } k \in W_{\mathbf{c}}, \\ \mathrm{id}_A, & \text{otherwise}, \end{cases} \qquad (1 \leqslant k \leqslant \ell).$$

Then $\varphi \in M$,

$$\varphi(z) = f(\varphi_1(z), \dots, \varphi_\ell(z)) = f_{\hat{0}}(\varphi_1(z), \dots, \varphi_\ell(z)) = \varphi_{i_0}(z) = z \ (z \in \mathbb{O})$$

and $\varphi(n-1) = f(\varphi_1(n-1), \dots, \varphi_\ell(n-1)) = f(\mathbf{a}) = 0$, which is impossible.

If $i_0 \in W_{\mathbf{c}}$ then let $\mathbf{a}$ be the ℓ-tuple with components $\mathbf{a}_{[i_0]} = 1$, $\mathbf{a}_{[k]} = 0$ if $k \in W_{\mathbf{c}} \setminus \{i_0\}$, and $\mathbf{a}_{[k]} = n-1$ if $k \in A \setminus W_{\mathbf{c}}$. Let $\psi = f(\psi_1, \dots, \psi_\ell)$, where

$$\psi_k = \begin{cases} \gamma_1, & \text{if } k = i_0, \\ \gamma_0, & \text{if } k \in W_{\mathbf{c}} \setminus \{i_0\}, \\ \mathrm{id}_A, & \text{otherwise}, \end{cases} \qquad (1 \leqslant k \leqslant \ell).$$

Then $\psi \in M$,

$$\psi(z) = f(\psi_1(z), \dots, \psi_\ell(z)) = f_{\hat{0}}(\psi_1(z), \dots, \psi_\ell(z)) = \psi_{i_0}(z) = 1 \ (z \in \mathbb{O})$$

and $\psi(n-1) = f(\psi_1(n-1), \dots, \psi_\ell(n-1)) = f(\mathbf{a}) = f_{\mathbf{c}}(\mathbf{a}') = 0$, where $\mathbf{a}'_{[i_0]} = 1$ and $\mathbf{a}'_{[k]} = 0$ $(k \neq i_0)$. But this is a contradiction.

Then $f(\hat{0}) = n-1$ holds, which implies that $f(\mathbb{O}) = \mathbb{I}$. For $z \in \mathbb{O}$ let $\mathbf{z}$ be the ℓ-tuple

$$(0, \dots, \overbrace{z}^{i_0\text{-th comp.}}, \dots, 0).$$

Let $\chi = f(\chi_1, \dots, \chi_\ell)$, where

$$\chi_k = \begin{cases} \gamma_z, & \text{if } k = i_0, \\ \gamma_0, & \text{if } k \in W_{\mathbf{c}} \setminus \{i_0\}, \\ \mathrm{id}_A, & \text{otherwise}, \end{cases} \qquad (1 \leqslant k \leqslant \ell).$$

Then $\chi \in M$, $\chi(0) = f(\mathbf{z}) = n-1$ and $\chi(n-1) = f(\mathbf{z}_{\downarrow\mathbf{b}}) = z$. Thus, $\tau_z = \chi \in M$, hence $\mathfrak{S} = \mathfrak{T} = \mathbb{O}$. Let a and b be distinct elements in $\mathbb{O}$, furthermore, let $\mathbf{e}'$ and $\mathbf{e}''$ be the following ℓ-tuples in $\mathbb{O}^\ell$:

$$(\mathbf{e}')_{[k]} = \begin{cases} a, & \text{if } k = i_0, \\ 0, & \text{if } k \in W_{\mathbf{b}} \setminus \{i_0\}, \\ n-1, & \text{otherwise}, \end{cases}$$

$$(\mathbf{e}'')_{[k]} = \begin{cases} b, & \text{if } k = j_0, \\ 0, & \text{if } k \in W_{\mathbf{c}} \setminus \{j_0\}, \\ n-1, & \text{otherwise}. \end{cases}$$

Define the transformations $\omega, \omega_1, \ldots, \omega_\ell$ in the following way:

$$\omega_k = \begin{cases} \gamma_0, & \text{if } \mathbf{e}'_{[k]} = \mathbf{e}''_{[k]} = 0, \\ \sigma_0, & \text{if } \mathbf{e}'_{[k]} = 0 \text{ and } \mathbf{e}''_{[k]} = n-1, \\ \tau_0, & \text{if } \mathbf{e}'_{[k]} = n-1 \text{ and } \mathbf{e}''_{[k]} = 0, \\ \gamma_{n-1}, & \text{if } \mathbf{e}'_{[k]} = \mathbf{e}''_{[k]} = n-1, \\ \gamma_a, & \text{if } \mathbf{e}'_{[k]} = \mathbf{e}''_{[k]} = a, \\ \sigma_a, & \text{if } \mathbf{e}'_{[k]} = a \text{ and } \mathbf{e}''_{[k]} = n-1, \\ \gamma_b, & \text{if } \mathbf{e}'_{[k]} = \mathbf{e}''_{[k]} = b, \\ \tau_a, & \text{if } \mathbf{e}'_{[k]} = n-1 \text{ and } \mathbf{e}''_{[k]} = b, \end{cases} \qquad (1 \leqslant k \leqslant \ell).$$

and set $\omega = f(\omega_1, \ldots, \omega_\ell)$. Then $\omega \in M$, $\omega(0) = f(\mathbf{e}'_{\downarrow\mathbf{b}}) = a$ and $\omega(n-1) = f(\mathbf{e}''_{\downarrow\mathbf{c}}) = b$, which is a contradiction.

This finishes the proof of Proposition 4. □

5. Non-surjective operations in the stabilizer

For an arbitrary transformation monoid M, let $\mathrm{Sta}_{\mathrm{ns}}(M)$ be the collection of all operations in $\mathrm{Sta}(M)$ that are either essentially unary or non-surjective. It is straightforward to check that $\mathrm{Sta}_{\mathrm{ns}}(M)$ is a clone, in particular,

$$\mathrm{Sta}_{\mathrm{ns}}(M) = \mathrm{Sl}_A \cap \mathrm{Sta}(M) = \langle \{ f \in \mathrm{Sta}(M) : f \text{ is not surjective} \} \rangle_M,$$

where Sl_A is the Słupecki clone on A. Furthermore, if $\mathcal{C} \in \mathrm{Int}(M)$ is a clone such that $\mathcal{C} \setminus \langle M \rangle$ contains only non-surjective operations then $\mathcal{C} \subseteq \mathrm{Sta}_{\mathrm{ns}}(M)$.

A clone in $\mathrm{Int}(M)$ is said to be *non-surjective* if $\mathcal{C} \setminus \langle M \rangle$ contains only non-surjective operations.

The above can be summarized in the following way.

Proposition 8. *Let M be a transformation monoid on a finite set A. Then the set of all non-surjective clones in* $\mathrm{Int}(M)$ *forms an interval with least and largest elements* $\langle M \rangle$ *and* $\mathrm{Sta}_{\mathrm{ns}}(M)$*, respectively.*

The interval of non-surjective clones in $\mathrm{Int}(M)$ will be denoted by $\mathrm{Int}_{\mathrm{ns}}(M)$. By Proposition 8, $\mathrm{Int}_{\mathrm{ns}}(M) = [\langle M \rangle, \mathrm{Sta}_{\mathrm{ns}}(M)]$.

For an operation $h \in \mathrm{Sta}(\bar{M})$, say h is ℓ-ary, define the operation $h^\diamond$ as follows:

$$h^\diamond \colon A^\ell \to A,\ (a_1, \ldots, a_\ell) \mapsto \begin{cases} n-1, & \text{if } g(\bar{a}_1, \ldots, \bar{a}_\ell) = \mathbb{I}, \\ 0, & \text{otherwise.} \end{cases}$$

If $C \subseteq \mathrm{Sta}(\bar{M})$ then $C^\diamond$ will denote the set $\{h^\diamond : h \in C\}$.

Proposition 9. *Let $f \in \mathrm{Sta}(M)$ be a non-surjective operation and $h \in \mathrm{Sta}(\bar{M})$ be an arbitrary operation.*

(a) *The operation $h^\diamond$ belongs to* $\mathrm{Sta}(M)$.
(b) *If the range of f is $\{0, n-1\}$ then $f = (\bar{f})^\diamond$.*

(c) *The operation f depends on its i-th variable iff $\bar{f}$ does.*

Proof. (a) Let $h \in \mathrm{Sta}(\bar{M})$, say ℓ-ary. Let $\mu_1, \dots, \mu_\ell$ be arbitrary transformations in M, and set $\mu = h^\diamond(\mu_1, \dots, \mu_\ell)$. Since for every element $i \in \{1, \dots, \ell\}$ we have that either $\{\mu_i(0), \dots, \mu_i(n-2)\} \subseteq \mathbb{O}$ or $\{\mu_i(0), \dots, \mu_i(n-2)\} = \mathbb{I}$, hence, the ℓ-tuples $(\mu_1(b)/\alpha, \dots, \mu_\ell(b)/\alpha)$ $(b \in \mathbb{O})$ coincide and $\mu(n-1) \in \{0, n-1\}$. Thus, $\mu \in \{\gamma_0, \gamma_{n-1}, \sigma_0, \tau_0\}$. If $\mu \in \{\gamma_0, \gamma_{n-1}, \sigma_0\}$ then $\mu \in M$. Suppose that $\mu = \tau_0$ and $\tau_0 \notin M$. Then $\mathfrak{T} = \emptyset$ and

$$\begin{aligned} n-1 = \tau_0(0) &= \mu(0) = h^\diamond(\mu_1(0), \dots, \mu_\ell(0)), \\ 0 = \tau_0(n-1) &= \mu(n-1) = h^\diamond(\mu_1(n-1), \dots, \mu_\ell(n-1)), \end{aligned}$$

which imply that h is not monotone: $\mu_1(0)/\alpha \leqslant \mu_1(n-1)/\alpha, \dots, \mu_\ell(0)/\alpha \leqslant \mu_\ell(n-1)/\alpha$ and $h(\mu_1(0)/\alpha, \dots, \mu_\ell(0)/\alpha) \nleqslant h(\mu_1(n-1)/\alpha, \dots, \mu_\ell(n-1)/\alpha)$. We got a contradiction, which proves that $\mu \in M$. Hence, $h^\diamond \in \mathrm{Sta}(M)$.

(b) Let $f \in \mathrm{Sta}(M)$ be an operation with range $\{0, n-1\}$, say f is ℓ-ary. Then for arbitrary ℓ-tuple $\mathbf{a} \in A^\ell$ we have that

$$f(\mathbf{a}) = n-1 \quad \Longleftrightarrow \quad \bar{f}(\bar{\mathbf{a}}) = \mathbb{I} \quad \Longleftrightarrow \quad \bar{f}^\diamond(\mathbf{a}) = n-1,$$

therefore, $f = \bar{f}^\diamond$.

(c) Suppose that f depends on its i-th variable. Then there is an ℓ-tuple $(a_1, \dots, a_\ell) \in A^\ell$ such that $\nu = f(a_1, \dots, a_{i-1}, \mathrm{id}_A, a_{i+1}, \dots, a_\ell) \in M$ is not constant, and so, $\nu \in \{\sigma_k, \tau_k\}$. Thus,

$$\begin{aligned} \bar{\nu} &= \overline{f(a_1, \dots, a_{i-1}, \mathrm{id}_A, a_{i+1}, \dots, a_\ell)} \\ &= \bar{f}(\overline{a_1}, \dots, \overline{a_{i-1}}, \mathrm{id}_{\bar{A}}, \overline{a_{i+1}}, \dots, \overline{a_\ell}) \end{aligned}$$

is not a constant transformation on $\bar{A}$, which implies that $\bar{f}$ depends on its i-th variable. □

For an ℓ-tuple $\mathbf{b} \in \{\mathbb{O}, \mathbb{I}\}^\ell$ define the ℓ-tuples $\mathbf{b}^\diamond$ as follows:

$$(\mathbf{b}^\diamond)_{[i]} = \begin{cases} n-1, & \text{if } \mathbf{b}_{[i]} = \mathbb{I}, \\ 0, & \text{otherwise.} \end{cases}$$

Now we can describe the interval $\mathrm{Int}_{\mathrm{ns}}(M)$.

Proposition 10. *Let $n \geqslant 3$ be a natural number and let $M = M_{\mathfrak{S},\mathfrak{T}}$ be a transformation monoid in the interval $[M_{\emptyset,\emptyset}, M_{\mathbb{O},\mathbb{O}}]$. Then*

$$\mathrm{Int}_{\mathrm{ns}}(M) = \begin{cases} \{\langle M \rangle, \langle \bar{\wedge}^\diamond \rangle_M, \langle \bar{\vee}^\diamond \rangle_M, \langle \bar{\wedge}^\diamond, \bar{\vee}^\diamond \rangle_M\}, & \text{if } \mathfrak{T} = \emptyset, \\ \{\langle M \rangle, \langle \bar{+}^\diamond \rangle_M, \langle (\mathcal{O}_{\bar{A}})^\diamond \rangle_M\}, & \text{if } \mathfrak{S} = \mathfrak{T} \neq \emptyset. \end{cases}$$

Proof. Let f be an ℓ-ary ($\ell \geqslant 2$) non-surjective operation in $\mathrm{Sta}(M) \setminus \langle M \rangle$. Then the range of f is $\{k, n-1\}$ for an element $k \in \mathbb{O}$ by Lemma 6 (a). Hence, there is an ℓ-tuple $(a_1, \dots, a_\ell) \in A^\ell$ and there is an index $i \in \{1, \dots, \ell\}$

such that $\mu = f(a_1, \dots, a_{i-1}, \mathrm{id}_A, a_{i+1}, \dots, a_\ell) \in M$ is not constant, and so, $\mu \in \{\sigma_k, \tau_k\}$. Since $\tau_k^2 = \sigma_k$, $\sigma_k \in M$. Then $\sigma_k \circ (\sigma_0 \circ f) = f$, hence,

$$\langle f \rangle_M = \langle \sigma_0 \circ f \rangle_M. \tag{2}$$

Let Φ and Ψ be the following mappings:

$$\Phi\colon \mathrm{Int}_{\mathrm{ns}}(M) \to \mathrm{Int}(\bar{M}),\ \mathcal{C} \mapsto \bar{\mathcal{C}},$$
$$\Psi\colon \mathrm{Int}(\bar{M}) \to \mathrm{Int}_{\mathrm{ns}}(M),\ \mathcal{H} \mapsto \langle \mathcal{H}^\diamond \rangle_M,$$

where $\bar{\mathcal{C}} = \{\bar{g} : g \in \mathcal{C}\}$ and $\mathcal{H}^\diamond = \{h^\diamond : h \in \mathcal{H}\}$. Let $\mathcal{C}$ and $\mathcal{H}$ be clones in $\mathrm{Int}_{\mathrm{ns}}(M)$ and $\mathrm{Int}(\bar{M})$, respectively. Then

$$\langle \bar{\mathcal{C}}^\diamond \rangle_M = \langle \{f \in \mathcal{C} : \text{the range of } f \text{ is } \{0, n-1\}\} \rangle_M = \mathcal{C},$$

where, in the second equality, we used (2). Furthermore,

$$\overline{\langle \mathcal{H}^\diamond \rangle_M} = \langle \overline{\mathcal{H}^\diamond} \rangle_{\bar{M}} = \mathcal{H},$$

where, in the second equality, we used Proposition 9 (b). Therefore, Φ and Ψ are mutually inverse bijections. It is obvious that Φ and Ψ are monotone mappings with respect to (set theoretic) inclusion. Hence, Φ is an isomorphism between the intervals $\mathrm{Int}_{\mathrm{ns}}(M)$ and $\mathrm{Int}(\bar{M})$. Then

$$\mathrm{Int}_{\mathrm{ns}}(M) \cong \mathrm{Int}(\bar{M}) = \begin{cases} \{\langle \bar{M} \rangle, \langle \bar{\wedge} \rangle_{\bar{M}}, \langle \bar{\vee} \rangle_{\bar{M}}, \langle \bar{\wedge}, \bar{\vee} \rangle_{\bar{M}}\}, & \text{if } \mathfrak{T} = \emptyset, \\ \{\langle \bar{M} \rangle, \langle \bar{+} \rangle_{\bar{M}}, \mathcal{O}_{\bar{A}}\}, & \text{if } \mathfrak{S} = \mathfrak{T} \neq \emptyset, \end{cases}$$

and so,

$$\mathrm{Int}_{\mathrm{ns}}(M) = \begin{cases} \{\langle M \rangle, \langle \bar{\wedge}^\diamond \rangle_M, \langle \bar{\vee}^\diamond \rangle_M, \langle \bar{\wedge}^\diamond, \bar{\vee}^\diamond \rangle_M\}, & \text{if } \mathfrak{T} = \emptyset, \\ \{\langle M \rangle, \langle \bar{+}^\diamond \rangle_M, \langle (\mathcal{O}_{\bar{A}})^\diamond \rangle_M\}, & \text{if } \mathfrak{S} = \mathfrak{T} \neq \emptyset. \end{cases}$$

This finishes the proof of the proposition. □

6. When the base set A has more than three elements

In this section we will make the assumption that $n = |A| \geqslant 4$. Then $|\mathbb{O}| \geqslant 3$, and so, by (1), the monoid

$$M_{\mathbb{O}} = \{\mu|_{\mathbb{O}} : \mu \in M \text{ and } \mu(\mathbb{O}) \subseteq \mathbb{O}\} = \Gamma_{\mathbb{O}} \cup \{\mathrm{id}_{\mathbb{O}}\}$$

is collapsing.

Our main tool in the investigation of operations in $\mathrm{Sta}(M)$ is the following simple observation, besides Proposition 4.

Proposition 11. *Let f be an ℓ-ary operation in* $\mathrm{Sta}(M)$. *If $f(\mathbb{O}^\ell) \subseteq \mathbb{O}$ and $f|_{\mathbb{O}}$ depends on its i-th variable then $f(\mathbf{a}) = n-1$ holds for every ℓ-tuple $\mathbf{a} \in A^\ell$ with $\mathbf{a}_{[i]} = n-1$.*

Proof. Suppose that $f(\mathbb{O}^\ell) \subseteq \mathbb{O}$ and $f|_{\mathbb{O}}$ depends on its i-th variable. We may assume that $i = 1$. Since $f|_{\mathbb{O}} \in \mathrm{Sta}(M_{\mathbb{O}})$ is an essentially unary operation, we have that $f(\mathbf{b}) = \mathbf{b}_{[1]}$ holds for every ℓ-tuple $\mathbf{b} \in \mathbb{O}^\ell$. Let $\mathbf{a}$ be an arbitrary element in A^ℓ with $\mathbf{a}_{[1]} = n-1$, furthermore, let $\mu_1, \dots, \mu_\ell \in M$ be the following transformations:

$$\mu_k = \begin{cases} \gamma_{\mathbf{a}_{[k]}}, & \text{if } \mathbf{a}_{[k]} \in \mathbb{O}, \\ \mathrm{id}_A, & \text{otherwise,} \end{cases} \qquad (1 \leqslant k \leqslant \ell)$$

and set $\mu = f(\mu_1, \dots, \mu_\ell) \in M$. Then $\mu(k) = k$ $(k \in \mathbb{O})$ imply that $f(\mathbf{a}) = \mu(n-1) = n-1$. ☐

Let $\mathbf{a}$ be an arbitrary element in A^ℓ and let $\mu_1, \dots, \mu_\ell$ be the following transformations in M:

$$\mu_k = \begin{cases} \gamma_{\mathbf{a}_{[k]}}, & \text{if } \mathbf{a}_{[k]} \in \mathbb{O}, \\ \mathrm{id}_A, & \text{otherwise,} \end{cases} \qquad (1 \leqslant k \leqslant \ell).$$

Then the transformation $f(\mu_1, \dots, \mu_\ell) \in M$ will be denoted by $\mu^{(\mathbf{a})}$. Then $\mu^{(\mathbf{a})}(n-1) = f(\mathbf{a})$ and $(\mu_1(k), \dots, \mu_\ell(k)) \in \mathbb{O}$ $(k \in \mathbb{O})$.

Proposition 12. *Let $n \geqslant 4$ be a natural number. If $\mathfrak{S} \neq \mathbb{O}$ then $\mathrm{Sta}(M_{\mathfrak{S},\mathfrak{T}}) \setminus \langle M_{\mathfrak{S},\mathfrak{T}} \rangle$ does not contain surjective operations. Hence,*

$$\mathrm{Int}(M_{\mathfrak{S},\mathfrak{T}}) = \mathrm{Int}_{\mathrm{ns}}(M_{\mathfrak{S},\mathfrak{T}}).$$

The interval $\mathrm{Int}_{\mathrm{ns}}(M_{\mathfrak{S},\mathfrak{T}})$ of non-surjective clones can be seen in Figure 1.

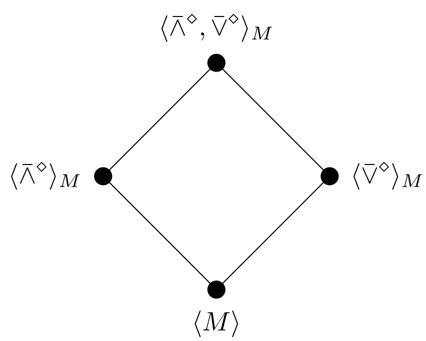

FIGURE 1. The interval $\mathrm{Int}_{\mathrm{ns}}(M)$ of non-surjective clones

Proof. Let $M = M_{\mathfrak{S},\mathfrak{T}}$, where $\mathfrak{S}$ is a nonempty subset of $\mathbb{O}$ that does not contain $n-2$. Suppose that there is an ℓ-ary surjective operation f in $\mathrm{Sta}(M)$. If $f(\hat{0}) = n-1$ then $f(\mathbb{O}^\ell) = \mathbb{I}$. Let $\mathbf{a} \in A^\ell$ such that $f(\mathbf{a}) = n-2$. Then $\mu^{(\mathbf{a})} \in M$ and $\mu^{(\mathbf{a})}(0) = n-1$. Therefore, $\tau_{n-2} = \mu^{(\mathbf{a})} \in M$, hence, $n-2 \in \mathfrak{T} = \mathfrak{S}$ holds, contradicting to our assumptions.

Then $f(\mathbb{O}^\ell) \subseteq \mathbb{O}$, and so, $f|_{\mathbb{O}}$ is an essentially unary operation, hence, there is an index i_0 in $\{1, \dots, \ell\}$ such that $f|_{\mathbb{O}}(\mathbf{b}) = \mathbf{b}_{[i_0]}$ $(\mathbf{b} \in \mathbb{O}^\ell)$, since otherwise

f would be non-surjective. Then $f(\mathbf{a}) = n-1$ holds if $\mathbf{a} \in A^\ell$ and $\mathbf{a}_{[i_0]} = n-1$ by Proposition 11.

Let $\mathbf{b} \in A^\ell$ such that $\mathbf{b}_{[i]} = n-2$. Then for the transformation $\mu^{(\mathbf{b})}$ we get that $\mu^{(\mathbf{b})}(0) = \cdots = \mu^{(\mathbf{b})}(n-2) = n-2$, and so, $n-2 \notin \mathfrak{S}$ implies that $f(\mathbf{b}) = n-2$. Let $\mathbf{a}$ be an arbitrary element in A^ℓ such that $\mathbf{a}_{[i_0]} \in A \setminus \{n-1, n-2\}$. Let $\varphi_1, \dots, \varphi_\ell$ and $\psi_1, \dots, \psi_\ell$ be the following transformations in M:

$$\varphi_k = \begin{cases} \mathrm{id}_A, & \text{if } \mathbf{a}_{[k]} = n-1, \\ \gamma_{\mathbf{a}_{[k]}}, & \text{otherwise,} \end{cases}$$
$$\psi_k = \begin{cases} \mathrm{id}_A, & \text{if } k = i_0, \\ \gamma_{\mathbf{a}_{[k]}}, & \text{otherwise,} \end{cases}$$

$(1 \leqslant k \leqslant \ell)$. Set $\varphi = f(\varphi_1, \dots, \varphi_\ell)$ and $\psi = f(\psi_1, \dots, \psi_\ell)$. Then for every j $(1 \leqslant j \leqslant n-2)$ we have that $(\varphi_1(j), \dots, \varphi_\ell(j)) \in \mathbb{O}^\ell$, which implies that $\varphi(j) = \varphi_{i_0}(j) = \mathbf{a}_{[i_0]}$. Hence,

$$f(\mathbf{a}) = \varphi(n-1) \in \{\mathbf{a}_{[i_0]}, n-1\}. \tag{3}$$

Furthermore, $\psi(n-1) = n-1$ and $\psi(n-2) = n-2$ since $\psi_{i_0}(n-1) = n-1$ and $\psi_{i_0}(n-2) = n-2$, respectively, which implies that $\psi \in \{\mathrm{id}_A, \gamma_{n-2}\}$. Therefore,

$$f(\mathbf{a}) \in \{\mathbf{a}_{[i_0]}, n-2\} \tag{4}$$

since $\mathbf{a} = (\psi_1(\mathbf{a}_{[i_0]}), \dots, \psi_\ell(\mathbf{a}_{[i_0]}))$. From (3) and (4) we get that $f(\mathbf{a}) = \mathbf{a}_{[i_0]}$, hence, f is in $\langle M \rangle$. This contradicts to our assumption.

This completes the proof of Proposition 12. $\square$

Define the following binary operations:

$$\sqcup\colon A^2 \to A,\ x \sqcup y = \begin{cases} n-1, & \text{if } n-1 \in \{x, y\}, \\ x, & \text{otherwise,} \end{cases}$$
$$\sqcap\colon A^2 \to A,\ x \sqcap y = \begin{cases} n-1, & \text{if } x = y = n-1, \\ 0, & \text{otherwise.} \end{cases}$$

Then $\sqcap \in \mathrm{Sta}(M_{\mathfrak{S},\mathfrak{T}})$ for every nonempty subset $\mathfrak{S}$ of $\mathbb{O}$, moreover $\sqcup \in \mathrm{Sta}(M_{\mathfrak{S},\mathfrak{T}})$ if and only if $\mathfrak{S} = \mathbb{O}$. (We remark that $\sqcap = \bar{\wedge}^\diamond$.)

Theorem 13. *Let $n \geqslant 4$ be a natural number. If $\mathfrak{S} = \mathbb{O}$, then $\mathrm{Int}(M_{\mathfrak{S},\mathfrak{T}})$ is the lattice that can be seen in Figure 2 if $\mathfrak{T} = \emptyset$ and in Figure 3 if $\mathfrak{T} = \mathbb{O}$.*

Proof of Theorem 13. Let $f \in \mathrm{Sta}(M) \setminus \langle M \rangle$ be an ℓ-ary surjective operation $(\ell \in \mathbb{N},\ \ell \geqslant 2)$ that depends on all of its variables. Let $\mathbf{a}$ be an arbitrary but fixed ℓ-tuple in A^ℓ such that $f(\mathbf{a}) \in \mathbb{O}$. Then $(f_{\mathbf{a}})|_{\mathbb{O}}$ is an essentially unary operation by (1). Proposition 4 and the surjectivity of f imply that $(f_{\mathbf{a}})|_{\mathbb{O}}$

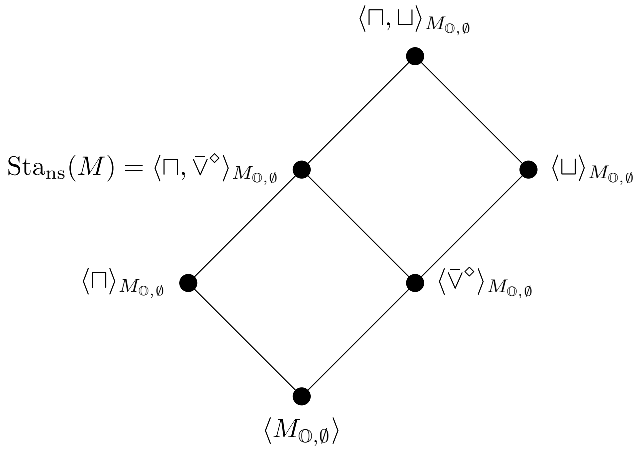

FIGURE 2. The monoidal interval $\mathrm{Int}(M_{\mathbb{O},\emptyset})$

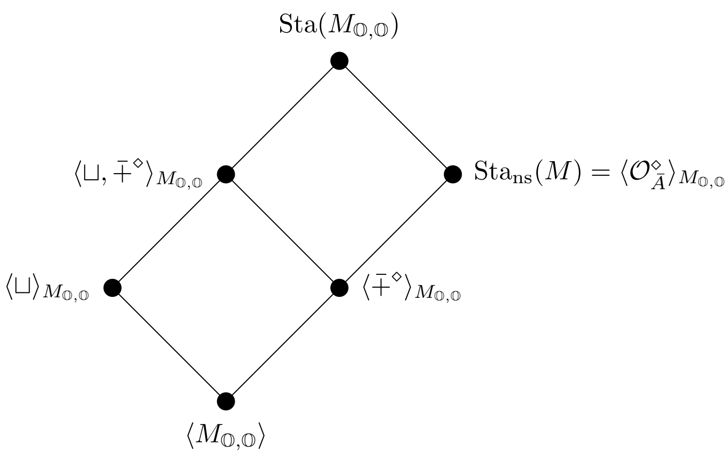

FIGURE 3. The monoidal interval $\mathrm{Int}(M_{\mathbb{O},\mathbb{O}})$

is not a constant operation. We may assume that $(f_{\mathbf{a}})|_{\mathbb{O}}$ depends on its first variable. Hence, $1 \in W_{\mathbf{a}}$ and

$$f_{\mathbf{a}}(\mathbf{b}) = (f_{\mathbf{a}})|_{\mathbb{O}}(\mathbf{b}) = \mathbf{b}_{[1]} \quad (\mathbf{b} \in \mathbb{O}^{\ell}).$$

Let $\mathbf{c} \in A^{\ell}$ be an ℓ-tuple such that $f(\mathbf{c}) \in \mathbb{O}$. Then $1 \in W_{\mathbf{c}}$ and for the ℓ-tuple $\mathbf{c}' \in \mathbb{O}^{\ell}$ with

$$(\mathbf{c}')_{[i]} = \begin{cases} \mathbf{c}_{[i]}, & \text{if } i \in W_{\mathbf{c}}, \\ 0, & \text{otherwise}, \end{cases} \quad (1 \leqslant i \leqslant \ell)$$

we have that

$$f(\mathbf{c}) = f_{\mathbf{c}}(\mathbf{c}') = f_{\mathbf{a}}(\mathbf{c}') = (\mathbf{c}')_{[1]} = \mathbf{c}_{[1]}, \tag{5}$$

where in the second equality we used Proposition 4. As a consequence of (5), the operation $\bar{f}$ determines f in the following way, for every ℓ-tuple $\mathbf{x} \in A^{\ell}$:

$$f(\mathbf{x}) = \begin{cases} n-1, & \text{if } \bar{f}(\bar{\mathbf{x}}) = \mathbb{I}, \\ \mathbf{x}_{[1]}, & \text{if } \bar{f}(\bar{\mathbf{x}}) = \mathbb{O}. \end{cases}$$

It follows that

$$f(\mathbf{x}) = n-1 \text{ if } \mathbf{x}_{[1]} = n-1 \quad (\mathbf{x} \in A^{\ell}). \tag{6}$$

By Theorem 1 (a),

$$\bar{f}(\mathbf{x}) = \mathop{\bar{\bigvee}}_{\mathbf{b} \in H_{\bar{f}}} \bar{\wedge}\, \mathbf{b}\mathbf{x} \quad (\mathbf{x} \in \bar{A}^{\ell}),$$

where $H_{\bar{f}} = \bar{f}^{-1}(\mathbb{I})$. Moreover, by (6),

$$\bar{f}(\mathbf{x}) = \mathbf{x}_{[1]} \mathbin{\bar{\vee}} \left(\mathop{\bar{\bigvee}}_{\mathbf{b} \in H'_{\bar{f}}} (\mathbf{b}_{[2]}\mathbf{x}_{[2]} \mathbin{\bar{\wedge}} \ldots \mathbin{\bar{\wedge}} \mathbf{b}_{[\ell]}\mathbf{x}_{[\ell]}) \right) \quad (\mathbf{x} \in \bar{A}^{\ell}),$$

where $H'_{\bar{f}} = \left\{ \mathbf{b} \in H_{\bar{f}} : \mathbf{b}_{[1]} = \mathbb{O} \right\}$. Then

$$f(\mathbf{x}) = \mathbf{x}_{[1]} \sqcup \left(\bigsqcup_{\mathbf{b} \in H'_{\bar{f}}} (\mathbf{b}^{\diamond}_{[2]}\mathbf{x}_{[2]} \sqcap \cdots \sqcap \mathbf{b}^{\diamond}_{[\ell]}\mathbf{x}_{[\ell]}) \right) \quad (\mathbf{x} \in A^{\ell}),$$

where ... $\square$

7. When the base set A has three elements

In this section we will make the assumption that $n = |A| = 3$. Then $|\mathbb{O}| = 2$, and so, the stabilizer of $M_{\mathbb{O}}$ consists of the monotone operations on $\mathbb{O}$ and the monoidal interval $\mathrm{Int}(M_{\mathbb{O}})$ has four elements: $\langle M_{\mathbb{O}} \rangle$, $\langle \wedge \rangle_{M_{\mathbb{O}}}$, $\langle \vee \rangle_{M_{\mathbb{O}}}$, and $\langle \wedge, \vee \rangle_{M_{\mathbb{O}}}$.

The interval I_3 consists of seven monoids (see Figure 4, where '?' indicates monoids for which the corresponding monoidal intervals have been unknown so far, the numbering of the monoids is according to the article [2]):

$$\begin{aligned} M_{24} &= M_{\emptyset,\emptyset}, & & \\ M_{38} &= M_{\{0\},\emptyset}, & M'_{38} &= M_{\{1\},\emptyset}, \\ M_{64} &= M_{\mathbb{O},\emptyset}, & & \\ M_{65} &= M_{\{0\},\{0\}}, & M'_{65} &= M_{\{1\},\{1\}}, \\ M_{109} &= M_{\mathbb{O},\mathbb{O}}, & & \end{aligned}$$

moreover, $M_{38} \bowtie M'_{38}$ and $M_{65} \bowtie M'_{65}$; the monoid M_{24} is collapsing and $\mathrm{Int}(M_{38})$ is a 6-element interval (cf., Dormán [3]).

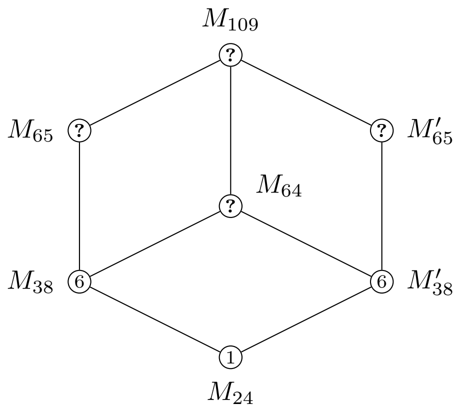

FIGURE 4. The interval $[M_{24}, M_{109}]$

7.1. The transformation monoid M_{64}. Set $M = M_{64} = M_{\mathbb{O},\emptyset}$. The sets $\{0,1\}$ and $\{0,2\}$ will be denoted by $\mathbb{O}$ and $\mathbb{D}$, respectively.

Proposition 14. *Let f be an arbitrary ℓ-ary operation ($\ell \in \mathbb{N}$) in* $\mathrm{Sta}(M)$*.*

(a) *If $f(\hat{0}) = 0$ then for every element b ($b \in \{1,2\}$) $f(\{0,b\}^n) \subseteq \{0,b\}$ and the operation $f|_{\{0,b\}}$ is monotone with respect to the lattice order $0 \leqslant b$.*
(b) *If $f(\hat{0}) = 1$ then $f(A^\ell) \subseteq \{1,2\}$.*
(c) *If $f(\hat{0}) = 2$ then $f(A^\ell) = \{2\}$.*

Proof. (a) Suppose that $f(\hat{0}) = 0$ then $\bar{f}(\hat{\mathbb{O}}) = \mathbb{O}$. Then $f(\mathbb{O}^\ell) \subseteq \mathbb{O}$, hence, f preserves the relation $\pi_{1,2}(\varrho_M) \setminus \{(2,2)\} = \{(0,0),(0,1),(1,1)\}$. Thus, $f|_{\mathbb{O}}$ is monotone with respect to the lattice order $0 \leqslant 1$.

Let $\mathbf{a}$ be an arbitrary element in $\mathbb{D}^\ell$. Then $(\hat{0}\,\hat{0}\,\mathbf{a})^T$ is a ϱ_M-matrix and $f\begin{pmatrix}\hat{0}\\ \hat{0}\\ \mathbf{a}\end{pmatrix} = \begin{pmatrix}0\\ 0\\ f(\mathbf{a})\end{pmatrix}$, which imply that $f(\mathbf{a}) \in \{0,2\}$. Then f preserves the relation $\pi_{1,3}(\varrho_M) \setminus \{(1,1)\} = \{(0,0),(0,2),(2,2)\}$. Thus, $f|_{\{0,2\}}$ is monotone with respect to the lattice order $0 \leqslant 2$.

(b) Suppose that $f(\hat{0}) = 1$. Then $f(\mathbb{O}^\ell) \subseteq \mathbb{O}$ and $f|_{\mathbb{O}}$ is monotone with respect to $0 \leqslant 1$, hence, $f(\mathbb{O}^\ell) = \{1\}$. Let $\mathbf{a} \in A^\ell$ and for every index i ($1 \leqslant i \leqslant \ell$) set

$$\mu_i = \begin{cases} \gamma_{\mathbf{a}_{[i]}}, & \text{if } \mathbf{a}_{[i]} \in \mathbb{O},\\ \mathrm{id}_A, & \text{otherwise.}\end{cases}$$

Then $\mu = f(\mu_1,\dots,\mu_\ell) \in M$ and $(\mu_1(0),\dots,\mu_\ell(0)),(\mu_1(1),\dots,\mu_\ell(1)) \in \mathbb{O}^\ell$. Therefore, $\mu(0) = \mu(1) = 1$, and so, $f(\mathbf{a}) = \mu(2) \in \{1,2\}$.

(c) Suppose that $f(\hat{0}) = 2$. Then $f(\mathbb{O}^\ell) = \{2\}$. Let $\mathbf{a}$ be an arbitrary ℓ-tuple in A^ℓ. Using the transformations $\mu,\mu_1,\dots,\mu_\ell$ that were defined in part (b), we get that $\mu(0) = 2$, hence, $f(\mathbf{a}) = \mu(2) = 2$ since $\mu = \gamma_2$. $\square$

The following statement is a consequence of Proposition 4.

Corollary 15. *Let $f \in \mathrm{Sta}(M)$ be an ℓ-ary non-constant operation for which $f(\hat{0}) = 0$ holds. Then $f(\hat{2}) = 2$ and f is determined by its values on the set $\mathbb{O}^\ell \cup \mathbb{D}^\ell$.*

Let f be an ℓ-ary ($\ell \geqslant 2$) surjective operation in $\mathrm{Sta}(M)$ that depends on all of its variables. Then $f(\hat{0}) = 0$ by Proposition 14, furthermore, the operation $f_b := f|_{\{0,b\}}$ is monotone with respect to the partial order $0 \leqslant b$ ($b \in \{1, 2\}$). For $b \in \{1, 2\}$ let $N_b = \{\mathrm{id}_{\{0,b\}}, \gamma_0, \gamma_b\} \leqslant T_{\{0,b\}}$. The lattice operations with respect to the lattice order $0 \leqslant 2$ on $\mathbb{D}$ will be denoted by $\curlywedge$ and $\curlyvee$.

Proposition 16. *If f_1 depends on its i-th variable then $f(\mathbf{a}) = 2$ holds for every ℓ-tuple $\mathbf{a} \in A^\ell$ with the property $\mathbf{a}_{[i]} = 2$.*

Proof. Suppose that f_1 depends on its i-th variable. We may assume, without loss of generality, that $i = 1$. Then there are elements $b_2, \ldots, b_\ell \in \mathbb{O}$ such that the transformation $\nu \colon \mathbb{O} \to \mathbb{O},\ x \mapsto f_1(x, b_2, \ldots, b_\ell)$ is not constant and belongs to N_1, hence, $\nu = \mathrm{id}_{\mathbb{O}}$. Set $\xi = f(\mathrm{id}_A, \gamma_{b_2}, \ldots, \gamma_{b_\ell})$. Then $\xi \in M$ and

$$\xi(b) = f(b, b_2, \ldots, b_\ell) = f_1(b, b_2, \ldots, b_\ell) = \nu(b) = b \quad (b \in \mathbb{O}),$$

which imply that $\xi = \mathrm{id}_A$, hence, $f(2, b_2, \ldots, b_\ell) = \xi(2) = 2$. Moreover, $f(\mathbb{I} \times \mathbb{O}^{\ell-1}) = \mathbb{I}$ holds since $\alpha = \mathbb{O}^2 \cup \mathbb{I}^2$ is a congruence of $(A; \mathrm{Sta}(M))$. Let $\mathbf{a}$ be an arbitrary element in A^ℓ with $\mathbf{a}_{[1]} = 2$. Then for the transformation $\varphi = f(\varphi_1, \ldots, \varphi_\ell)$, where for $k \in \{1, \ldots, \ell\}$

$$\varphi_k = \begin{cases} \gamma_{\mathbf{a}_{[k]}}, & \text{if } k = 1 \text{ or } \mathbf{a}_{[k]} \in \mathbb{O}, \\ \mathrm{id}_A, & \text{otherwise,} \end{cases}$$

we get that $\varphi(0) = \varphi(1) = 2$ and $\varphi(2) = f(\mathbf{a})$. Since $\varphi \in M$ and $\mathfrak{T} = \emptyset$, it follows that $\varphi = \gamma_2$, hence, $f(\mathbf{a}) = 2$. This completes the proof. ☐

To describe the clones in the interval $\mathrm{Int}(M)$ we will need the following binary operations in $\mathrm{Sta}(M)$:

$0_\curlywedge$	0	1	2
0	0	0	0
1	0	0	0
2	0	0	2

$0_\curlyvee$	0	1	2
0	0	0	2
1	0	0	2
2	2	2	2

$\to_\curlyvee$	0	1	2
0	0	0	2
1	1	1	2
2	2	2	2

$\wedge_\curlyvee$	0	1	2
0	0	0	2
1	0	1	2
2	2	2	2

$\wedge_\curlyvee$	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

.

Theorem 17. *The monoidal interval that corresponds to the transformation monoid M_{64} consists of twelve clones that can be seen in Figure 5.*

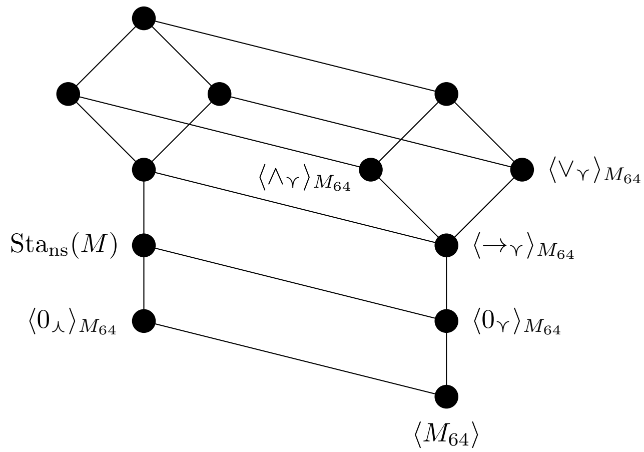

Figure 5. The monoidal interval $\mathrm{Int}(M_{64})$

Proof. Let $f \in \mathrm{Sta}(M) \setminus \langle M \rangle$ be an ℓ-ary surjective operation that depends on all of its variables. By Corollary 15, f is determined by the restrictions $f_1 = f|_{\mathbb{O}}$ and $f_2 = f|_{\mathbb{D}}$. The surjectivity of f implies that f_1 is not a constant operation. We may assume that f_1 depends on its first k variables ($k \in \mathbb{N}$). By Proposition 16, for every ℓ-tuple $\mathbf{a} \in A^\ell$ with the property that $2 \in \{\mathbf{a}_{[1]}, \ldots, \mathbf{a}_{[k]}\}$ we have that $f(\mathbf{a}) = 2$. Let $H_b = \min f_b^{-1}(b)$ ($b \in \{1, 2\}$). Then $H_1 \neq \emptyset$,

$$\{(0, \ldots, 0, \underbrace{2}_{i\text{-th comp.}}, 0, \ldots, 0) : 1 \leqslant i \leqslant k\} \subseteq H_2,$$

$\mathbf{b}^{(1)}_{[j]} = 0$ ($\mathbf{b}^{(1)} \in H_1, k+1 \leqslant j \leqslant \ell$), and

$$f_1(\mathbf{x}) = \bigvee_{\mathbf{b}^{(1)} \in H_1} \wedge\, \mathbf{b}^{(1)} \mathbf{x},$$
$$f_2(\mathbf{x}) = \curlyvee_{\mathbf{b}^{(2)} \in H_2} \curlywedge\, \mathbf{b}^{(2)} \mathbf{x} = \mathbf{x}_{[1]} \curlyvee \ldots \curlyvee \mathbf{x}_{[k]} \curlyvee \left(\curlyvee_{\mathbf{b} \in H_2'} \curlywedge\, \mathbf{b}^{(2)} \mathbf{x} \right),$$

where

$$H_2' = H_2 \setminus \{(0, \ldots, 0, \underbrace{2}_{i\text{-th comp.}}, 0, \ldots, 0) : 1 \leqslant i \leqslant k\}.$$

Claim 18. *If $H_2' = \emptyset$ then*

$$f_1(\mathbf{x}) = \bigvee_{\mathbf{b}^{(1)} \in H_1} \wedge\, \mathbf{b}^{(1)} \mathbf{x},$$
$$f_2(\mathbf{x}) = \mathbf{x}_{[1]} \curlyvee \ldots \curlyvee \mathbf{x}_{[k]},$$

and

$$\langle f\rangle_M = \begin{cases} \langle \to_\curlyvee\rangle_M, & \text{if } \langle f_1\rangle_{M_B} = \langle M_{\mathbb{O}}\rangle,\\ \langle \wedge_\curlyvee\rangle_M, & \text{if } \langle f_1\rangle_{M_B} = \langle \wedge\rangle_{M_{\mathbb{O}}},\\ \langle \vee_\curlyvee\rangle_M, & \text{if } \langle f_1\rangle_{M_B} = \langle \vee\rangle_{M_{\mathbb{O}}},\\ \langle \wedge_\curlyvee, \vee_\curlyvee\rangle_M, & \text{if } \langle f_1\rangle_{M_B} = \langle \wedge, \vee\rangle_{M_{\mathbb{O}}}. \end{cases}$$

The statement of Claim 18 follows from the fact that

$$f(\mathbf{x}) = \underset{\mathbf{b}^{(1)}\in H_1}{\vee_\curlyvee} \wedge_\curlyvee (\mathbf{b}^{(1)})^\diamond \mathbf{x},$$

by Corollary 5.

Let $f^\flat$ and $f^\natural$ be the ℓ-ary operations

$$f^\flat\colon A^\ell \to A,\ (x_1,\dots,x_\ell) \to f(x_1,\dots,x_k,0,\dots,0),$$
$$f^\natural\colon A^\ell \to A,\ (x_1,\dots,x_\ell) \to f(0,\dots,0,x_{k+1},\dots,x_\ell).$$

Since $\langle\wedge_\curlyvee\rangle_M \cap \langle\vee_\curlyvee\rangle_M = \langle\to_\curlyvee\rangle_M$, we get that $f = f^\flat \to_\curlyvee f^\natural$, and so,

$$\langle f\rangle_M = \langle f^\flat, f^\natural, \to_\curlyvee\rangle_M = \langle f^\flat\rangle_M \vee \langle f^\natural\rangle_M. \tag{7}$$

Claim 19. *If $H_2' \neq \emptyset$ then*

$$\langle f^\natural\rangle_M = \begin{cases} \langle 0_\curlywedge\rangle_M, & \text{if } \langle \curlyvee_{\mathbf{b}\in H_2'} \curlywedge \mathbf{b}^{(2)}\mathbf{x}\rangle_{M_{\mathbb{D}}} = \langle \curlywedge\rangle_{M_{\mathbb{D}}},\\ \langle 0_\curlyvee\rangle_M, & \text{if } \langle \curlyvee_{\mathbf{b}\in H_2'} \curlywedge \mathbf{b}^{(2)}\mathbf{x}\rangle_{M_{\mathbb{D}}} = \langle \curlyvee\rangle_{M_{\mathbb{D}}},\\ \langle 0_\curlywedge, 0_\curlyvee\rangle_M, & \text{if } \langle \curlyvee_{\mathbf{b}\in H_2'} \curlywedge \mathbf{b}^{(2)}\mathbf{x}\rangle_{M_{\mathbb{D}}} = \langle \curlywedge, \vee\rangle_{M_{\mathbb{D}}}, \end{cases}$$

where $M_{\mathbb{D}} = \{\mu|_{\mathbb{D}} : \mu \in M,\ \mu(\mathbb{D}) \subseteq \mathbb{D}\} = \{\gamma_1, \gamma_2, \mathrm{id}_{\mathbb{D}}\}$.

Then, by Claim 18, Claim 19, and (7), the 1-generated surjective clones in $\mathrm{Int}(M)$ are

$$\langle\to_\curlyvee\rangle_M,\ \langle\wedge_\curlyvee\rangle_M,\ \langle\vee_\curlyvee\rangle_M,\ \langle\wedge_\curlyvee,\vee_\curlyvee\rangle_M,$$
$$\langle\to_\curlyvee\rangle_M \vee \mathcal{C},\ \langle\wedge_\curlyvee\rangle_M \vee \mathcal{C},\ \langle\vee_\curlyvee\rangle_M \vee \mathcal{C},\ \langle\wedge_\curlyvee,\vee_\curlyvee\rangle_M \vee \mathcal{C},$$

where $\mathcal{C}$ is an arbitrary non-surjective clone in $\mathrm{Int}(M)$. Therefore, the lattice of clones in $\mathrm{Int}(M)$ coincides with the lattice in Figure 5. □

7.2. The transformation monoid M_{65}. Set $M = M_{65} = M_{\{0\},\{0\}}$.

The equivalence relation $\beta = \{0,2\}^2 \cup \{1\}^2$ is a congruence of $(A; M)$, hence, of the algebra $(A; \mathrm{Sta}(M))$, as well. Let $\bar{\bar{0}} = 0/\beta$, $\bar{\bar{1}} = 1/\beta$, $\bar{\bar{A}} = A/\beta$, and $\bar{\bar{M}} = M/\beta = C(\bar{\bar{A}}) \cup \mathrm{id}_{\bar{\bar{A}}}$. Then $\mathrm{Sta}(\bar{\bar{M}})$ coincides with the set of all monotone operations with respect to the lattice order $\bar{\bar{0}} \leqslant \bar{\bar{1}}$ and $\mathrm{Int}(\bar{\bar{M}})$ consists of the following four clones:

$$\langle\bar{\bar{M}}\rangle, \langle\bar{\bar{\wedge}}\rangle_{\bar{\bar{M}}}, \langle\bar{\bar{\vee}}\rangle_{\bar{\bar{M}}}, \langle\bar{\bar{\wedge}},\bar{\bar{\vee}}\rangle_{\bar{\bar{M}}}. \tag{8}$$

Let f be an ℓ-ary operation in $\mathrm{Sta}(M)$. Then

- $\bar{\bar{f}}$ is a monotone operation,
- if $\bar{\bar{f}}$ is not the constant operation with value $\bar{\bar{1}}$ then $\bar{\bar{f}}(\bar{\bar{0}},\dots,\bar{\bar{0}}) = \bar{\bar{0}}$, which means that if f is not the constant operation with value 1 then $f(\mathbb{D}^\ell) \subseteq \mathbb{D}$.

In the sequel we will assume that $f \notin \langle M \rangle$.

The clone $\langle f \rangle_M/\beta = \langle \bar{\bar{f}} \rangle_{\bar{\bar{M}}}$ coincides with one of the clones in (8). Since, there is no operation $g \in \mathrm{Sta}(M)$ such that $\bar{\bar{g}} = \bar{\bar{\vee}}$, we have that $\langle f \rangle_M/\beta$ coincides with either $\langle \bar{\bar{M}} \rangle$ or $\langle \bar{\bar{\wedge}} \rangle_{\bar{\bar{M}}}$. Moreover, it is straightforward to check that if $\bar{\bar{f}} = \bar{\bar{\wedge}}$ then f is the following operation:

$$\begin{array}{c|ccc} \wedge_{\curlyvee} & 0 & 1 & 2 \\ \hline 0 & 0 & 0 & 2 \\ 1 & 0 & 1 & 2 \\ 2 & 2 & 2 & 2 \end{array}.$$

Proposition 20. *If f is not surjective and it is not a constant operation then the range of is $\mathbb{D}$.*

Proof. Let f be a non-surjective operation that is not a constant operation. By Lemma 6, the range of f is either $\mathbb{D} = \{0, 2\}$ or $\{1, 2\}$. It is obvious that the latter case is impossible. □

Proposition 21. *If f depends on at least two of its variables and it is surjective then*

$$\langle f \rangle_M = \langle \wedge_{\curlyvee} \rangle_M.$$

Proof. Assume f to be an ℓ-ary surjective operation in $\mathrm{Sta}(M) \setminus \langle M \rangle$. Suppose that $\bar{\bar{f}}$ is an essentially unary operation. Then there is an index $i \in \{1, \ldots, \ell\}$ such that equality $\bar{\bar{f}}(\mathbf{a}) = \mathbf{a}_{[i]}$ holds for every ℓ-tuple $\mathbf{a} \in \bar{\bar{A}}$ since f is surjective. Therefore, $f(\mathbf{c}) = 1$ if and only if $\mathbf{c}_{[i]} = 1$ ($\mathbf{c} \in A^\ell$). Let $\mathbf{d}$ be an arbitrary ℓ-tuple in A^ℓ with the property that $\mathbf{d}_{[i]} \in \{0, 2\}$. Let $\mu = f(\mu_1, \ldots, \mu_\ell)$, where

$$\mu_k = \begin{cases} \mathrm{id}_A, & \text{if } k = i, \\ \gamma_{\mathbf{d}_{[k]}}, & \text{otherwise.} \end{cases}$$

Then $\mu \in M$, $\mu(1) = 1$ and $f(\mathbf{d}) \in \{\mu(0), \mu(2)\}$ implies that $\mu = \mathrm{id}_A$, and so, $f(\mathbf{d}) = \mathbf{d}_{[i]}$. Thus, f is the ℓ-ary i-th projection that contradicts to our assumption. Then $\langle \bar{\bar{\wedge}} \rangle_M = \langle \bar{\bar{f}} \rangle_{\bar{\bar{M}}}$. From which, it follows that equality $\bar{\bar{f}}(\mathbf{x}) = \mathbf{x}_{[i_1]} \,\bar{\bar{\wedge}} \ldots \bar{\bar{\wedge}}\, \mathbf{x}_{[i_k]}$ holds for a suitable natural number $k \geqslant 2$ and indexes $1 \leqslant i_1 < \cdots < i_k \leqslant \ell$. We may assume that $i_j = j$ ($1 \leqslant j \leqslant k$). Then

$$\begin{aligned} f(\mathbf{a}) = 1 &\iff \bar{\bar{f}}(\bar{\bar{\mathbf{a}}}) = \bar{\bar{1}} \\ &\iff \bar{\bar{\mathbf{a}}}_{[1]} = \cdots = \bar{\bar{\mathbf{a}}}_{[k]} = \bar{\bar{1}} \\ &\iff \mathbf{a}_{[1]} = \cdots = \mathbf{a}_{[k]} = 1. \end{aligned}$$

Moreover, $f(\mathbf{b}) = 0$ if $\mathbf{b} \in \mathbb{O}^\ell \setminus (\{1\}^k \times \mathbb{O}^{\ell-k})$ since α is a congruence of $(A; \mathrm{Sta}(M))$.

Claim 22. *If $\mathbf{a} \in \{1\}^k \times A^{\ell-k}$ then $f(\mathbf{a}) = 1$.*

Let $\nu_1, \ldots, \nu_\ell$ be the following transformations:

$$\nu_i = \begin{cases} \gamma_{\mathbf{a}_{[i]}}, & \text{if } \mathbf{a}_{[i]} \in \mathbb{O}, \\ \mathrm{id}_A, & \text{if } \mathbf{a}_{[i]} = 2, \end{cases}$$

and set $\nu = f(\nu_1, \dots, \nu_\ell) \in M$. Then $(\nu_1(0), \dots, \nu_\ell(0)), (\nu_1(1), \dots, \nu_\ell(1)) \in \{1\}^k \times \mathbb{O}^{\ell-k}$ implies that $\nu(0) = \nu(1) = 1$, hence, $\nu = \gamma_1$. Thus, $f(\mathbf{a}) = \nu(2) = 1$.

Claim 23. *If* $\mathbf{a} \in (\mathbb{O}^k \setminus \{1\}^k) \times A^{\ell-k}$ *then* $f(\mathbf{a}) = 0$.

Since $\mathbf{a}\ \alpha^\ell\ (1, \dots, 1, \mathbf{a}_{[k+1]}, \dots, \mathbf{a}_{[\ell]})$, we have that $f(\mathbf{a}) \in \mathbb{O}$ by Claim 22. Suppose that $f(\mathbf{a}) = 1$. Let $\xi_1, \dots, \xi_\ell$ be the following transformations in M:

$$\xi_i = \begin{cases} \gamma_{\mathbf{a}_{[i]}}, & \text{if } \mathbf{a}_{[i]} \in \mathbb{O}, \\ \tau_0, & \text{if } \mathbf{a}_{[i]} = 2, \end{cases}$$

and set $\xi = f(\xi_1, \dots, \xi_\ell) \in M$. Then $\xi(0) = \xi(1) = f(\mathbf{a}) = 1$ and

$$(\xi_1(2), \dots, \xi_\ell(2)) \in (\mathbb{O}^k \setminus \{1\}^k) \times \mathbb{O}^{\ell-k},$$

which yields a contradiction since $\xi(2) = 0$ implies that $\xi \notin M$.

Suppose that we have an ℓ-tuple $\mathbf{a} \in (A^k \setminus \mathbb{O}^k) \times A^{\ell-k}$ such that $f(\mathbf{a}) \in \mathbb{D}$. Let $\mathbf{a}'$ be the ℓ-tuple with the following components $(1 \leqslant i \leqslant \ell)$:

$$\mathbf{a}'_{[i]} = \begin{cases} \mathbf{a}_{[i]}, & \text{if } \mathbf{a}_{[i]} \in \mathbb{O}, \\ 0, & \text{otherwise.} \end{cases}$$

Then $\mathbf{a}' \in \mathbb{O}^\ell$, 0 occurs among its first k components and $\mathbf{a}'\ \beta^\ell\ \mathbf{a}$, which imply that $f(\mathbf{a}) = 0$. Let $\mathbf{a}''$ be the ℓ-tuple with the following components $(1 \leqslant i \leqslant \ell)$:

$$\mathbf{a}''_{[i]} = \begin{cases} \mathbf{a}_{[i]}, & \text{if } \mathbf{a}_{[i]} \in \{1,2\}, \\ 1, & \text{otherwise.} \end{cases}$$

Then $\mathbf{a}'' \in \{1,2\}^\ell$, 2 occurs among its first k components and $\mathbf{a}''\ \alpha^\ell\ \mathbf{a}$, which imply that $f(\mathbf{a}'') = 0$, since $f(\mathbf{a}'') = 1$ is not possible. Let $\varphi_1, \dots \varphi_\ell$ be the following transformations in M:

$$\varphi_i = \begin{cases} \gamma_1, & \text{if } \mathbf{a}''_{[i]} = 1, \\ \mathrm{id}_A, & \text{if } \mathbf{a}_{[i]} = 2, \end{cases}$$

and set $\varphi = f(\varphi_1, \dots, \varphi_\ell)$. Then $\varphi \in M$, $\varphi(2) = f(\mathbf{a}) = 0$ and $\varphi(1) = f(\hat{1}) = 1$, which yields the contradiction $\varphi \notin M$.

Therefore, $f(x_1, \dots, x_\ell) = x_1 \wedge_\curlyvee \dots \wedge_\curlyvee x_k \in \langle \wedge_\curlyvee \rangle_M$. Since $k \geqslant 2$, the inclusion $\wedge_\curlyvee \in \langle f \rangle_M$ also holds. This completes the proof of the proposition. □

Theorem 24. *The monoidal interval that corresponds to the transformation monoid* M_{65} *consists of the following clones*

$$\langle M_{65} \rangle \subseteq \langle 0_+ \rangle_{M_{65}} \subseteq \langle 0_\curlywedge, 0_\curlyvee \rangle_{M_{65}} \subseteq \langle \wedge_\curlyvee \rangle_{M_{65}},$$

where

$$\begin{array}{c|ccc} 0_+ & 0 & 1 & 2 \\ \hline 0 & 0 & 0 & 2 \\ 1 & 0 & 0 & 2 \\ 2 & 2 & 2 & 0 \end{array}, \quad \begin{array}{c|ccc} 0_\curlywedge & 0 & 1 & 2 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 2 \end{array}, \quad \begin{array}{c|ccc} 0_\curlyvee & 0 & 1 & 2 \\ \hline 0 & 0 & 0 & 2 \\ 1 & 0 & 0 & 2 \\ 2 & 2 & 2 & 2 \end{array}.$$

Proof. Let f be an n-ary operation in $\mathrm{Sta}(M) \setminus \langle M \rangle$. If f is not surjective then $f(A^n) \subseteq \mathbb{D}$ and f is determined by $\bar{f}$. If f is surjective then $\langle f \rangle_M = \langle \wedge_{\Upsilon} \rangle_M$. □

7.3. The transformation monoid M_{109}. Set $M = M_{109} = M_{\mathbb{O},\mathbb{O}}$.

Since $\mathfrak{T} \neq \emptyset$, the interval $\mathrm{Int}_{\mathrm{ns}}(M)$ is the 3-element chain

$$\langle M \rangle \subseteq \langle 0_+ \rangle_M \subseteq \langle (\mathcal{O}_{\bar{A}})^{\diamond} \rangle_M . \tag{9}$$

Let f be a surjective ℓ-ary operation ($\ell \geqslant 2$) in $\mathrm{Sta}(M)$. Choose and fix an ℓ-tuple $\mathbf{a} \in A^\ell$ such that $f(\mathbf{a}) \in \mathbb{O}$. Then $f_{\mathbf{a}}$ is defined, moreover, $f_{\mathbf{a}}|_{\mathbb{O}}$ is a monotone operation on $\mathbb{O}$. Without loss of generality we may assume that $f_{\mathbf{a}}$ depends on its first k variables. Let $\mathbf{d}$ be an ℓ-tuple with the property that 2 appears among its first k components, say $\mathbf{d}_{[i]} = 2$ for an index $i \in \{1, \ldots, k\}$. Suppose that $f(\mathbf{d}) \in \mathbb{O}$. Then the operation $f_{\mathbf{d}}$ is defined and by Proposition 4

$$f_{\mathbf{d}}|_{\mathbb{O}} = f_{\mathbf{a}}|_{\mathbb{O}}.$$

But this is impossible because $f_{\mathbf{d}}$ does not depend on its i-th variable if $\mathbf{d}_{[i]} = 2$. Therefore, $f(d_1, \ldots, d_\ell) = 2$ if $2 \in \{d_1, \ldots, d_k\}$, and so, $\bar{f}(\overline{d_1}, \ldots, \overline{d_\ell}) = \mathbb{I}$ if $\mathbb{I} \in \{\overline{d_1}, \ldots, \overline{d_k}\}$. Then

$$\bar{f}(\mathbf{x}) = \mathop{\bar{\vee}}_{\mathbf{b} \in H_{\bar{f}}} \bar{\wedge}\, \mathbf{bx} = (\mathbf{x}_{[1]} \mathbin{\bar{\vee}} \ldots \mathbin{\bar{\vee}} \mathbf{x}_{[k]}) \mathbin{\bar{\vee}} \mathop{\bar{\vee}}_{\mathbf{b} \in H'_{\bar{f}}} (\mathbf{b}_{[k+1]}\mathbf{x}_{[k+1]} \mathbin{\bar{\wedge}} \ldots \mathbin{\bar{\wedge}} \mathbf{b}_{[\ell]}\mathbf{x}_{[\ell]}),$$

where $H_{\bar{f}} = \bar{f}^{-1}(\mathbb{I})$ and

$$H'_{\bar{f}} = \left\{ (\mathbf{b}_{[k+1]}, \ldots, \mathbf{b}_{[\ell]}) : \mathbf{b} \in H_{\bar{f}} \text{ and } \mathbf{b}_{[1]} = \cdots = \mathbf{b}_{[k]} = \mathbb{O} \right\}.$$

Let $H = H_{f_{\mathbf{a}}|_{\mathbb{O}}} = \min(f_{\mathbf{a}}|_{\mathbb{O}})^{-1}(1)$, then

$$(f_{\mathbf{a}}|_{\mathbb{O}})(\mathbf{x}) = \bigvee_{\mathbf{b} \in H} \wedge\, \mathbf{bx} \quad (\mathbf{x} \in \mathbb{O}^\ell),$$

moreover, if $\mathbf{b} \in H$ then $\mathbf{b}_{[i]} = 0$ ($k+1 \leqslant i \leqslant \ell$). The operations $\bar{f}$ and $f_{\mathbf{a}}|_{\mathbb{O}}$ completely determine the operation f, by Corollary 5, in the following way:

$$f(\mathbf{x}) = (\mathop{\vee_{\bar{\triangledown}}}_{\mathbf{b} \in H} \wedge_{\bar{\triangledown}}\, \mathbf{bx}) \vee_{\bar{\triangledown}} (\mathop{0_{\bar{\triangledown}}}_{\mathbf{b} \in H'_{\bar{f}}} (0_{\bar{\wedge}}\, \mathbf{bx})') \quad (\mathbf{x} \in A^\ell), \tag{10}$$

where

$0_{\bar{\wedge}}$	0	1	2
0	0	0	0
1	0	0	0
2	0	0	2

,

$0_{\bar{\triangledown}}$	0	1	2
0	0	0	2
1	0	0	2
2	2	2	2

,

$\wedge_{\bar{\triangledown}}$	0	1	2
0	0	0	2
1	0	1	2
2	2	2	2

,

$\vee_{\bar{\triangledown}}$	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

.

Then by (10)

$$f(\mathbf{x}) = f^\flat(\mathbf{x}_{[1]}, \ldots, \mathbf{x}_{[k]}) \vee_{\bar{\triangledown}} f^\natural(\mathbf{x}_{[k+1]}, \ldots, \mathbf{x}_{[\ell]}),$$

where

$$f^\flat(\mathbf{x}) = \mathop{\vee_{\bar{\triangledown}}}_{\mathbf{b} \in H} \wedge_{\bar{\triangledown}}\, \mathbf{bx} \quad \text{and} \quad f^\natural(\mathbf{x}) = \mathop{0_{\bar{\triangledown}}}_{\mathbf{b} \in H'_{\bar{f}}} (0_{\bar{\wedge}}\, \mathbf{bx})'.$$

Furthermore,

$$f^{\flat}(\mathbf{x}) = f(\mathbf{x}_{[1]}, \dots, \mathbf{x}_{[k]}, a_{k+1}, \dots, a_\ell),$$
$$f^{\natural}(\mathbf{x}) = f(0, \dots, 0, \mathbf{x}_{[k+1]}, \dots, \mathbf{x}_{[\ell]})$$

for elements $a_{k+1}, \dots, a_\ell \in A$ with the property that

$$f^{\natural}(0, \dots, 0, a_{k+1}, \dots, a_\ell) = 0.$$

Hence, $\langle f\rangle_M = \langle f^{\flat}\rangle_M$ and the 1-generated surjective clones in $\mathrm{Int}(M)$ are

$$\langle \rightarrow_{\bar{\triangledown}}\rangle_M,\ \langle \wedge_{\bar{\triangledown}}\rangle_M,\ \langle \vee_{\bar{\triangledown}}\rangle_M,\ \langle \wedge_{\bar{\triangledown}}, \vee_{\bar{\triangledown}}\rangle_M. \tag{11}$$

Theorem 25. *The monoidal interval that corresponds to the transformation monoid M_{109} consists of seven clones that can be seen in Figure 6.*

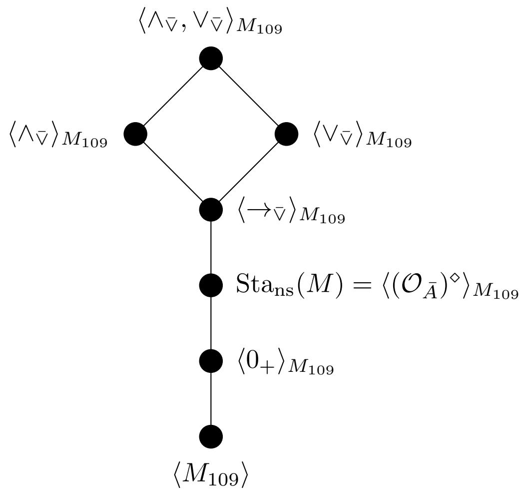

FIGURE 6. The monoidal interval $\mathrm{Int}(M_{109})$

Proof. Let f be an arbitrary operation in $\mathrm{Sta}(M)$. If f is a non-surjective operation then $\langle f\rangle_M$ coincides with of the following clones:

$$\langle M\rangle, \quad \langle (0_+)^{\diamond}\rangle_M, \quad \langle (\mathcal{O}_{\bar{A}})^{\diamond}\rangle_M,$$

by (9). Assume $\mathcal{C}$ to be a surjective clone in $\mathrm{Int}(M)$. Since $\mathcal{C} = \vee_{f\in\mathcal{C}}\langle f\rangle_M$, $\mathcal{C}$ is one of the clones in (11), hence, $\mathrm{Int}(M)$ is the lattice that can be seen in Figure 6. □

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