Finite monidal intervals II – A 10-element monoidal interval

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Abstract. This paper is a small contribution to the solution of Szendrei's problem on three-element sets. A monoid on the set $\{0, 1, 2\}$ is presented for which the corresponding monoidal interval has 10 elements.

1. Introduction

Let A be a finite set with at least three elements. It is well known that the set of all clones on A whose set of unary operations coincides with a transformation monoid M on A forms an interval in the lattice of all clones on A (see Á. Szendrei [10, Chapter 3]). An interval of this form is called a monoidal interval. The monoidal intervals partition the clone lattice into finitely many blocks. Since the clone lattice has continuum many elements if $|A| \geqslant 3$, one might expect that 'for most M' the monoidal interval $\mathrm{Int}(M)$ contains uncountably many clones. We remark that this is the case on 3-element sets: there are at least 499 transformation monoids (in 99 $\bowtie$-classes) among the all 699 transformation monoids (in 160 $\bowtie$-classes) for which the corresponding monoidal intervals have cardinality $2^{\aleph_0}$ (cf. Dormán–Makay–Maróti–Vajda [2]). Nevertheless, it turns out that for many interesting transformation monoids the corresponding monoidal intervals are finite.

Á. Szendrei in [10] posed the problem of classifying transformation monoids according to the cardinalities of the corresponding monoidal intervals. A complete classification of transformation monoids according to the sizes of the corresponding monoidal intervals seems a very hard problem at present. However, for certain classes of monoids we can solve this problem.

In this paper we will consider a certain transformation monoid (M_{88}) on the set $\{0, 1, 2\}$ with 10-element monoidal interval.

2. Preliminaries

For a finite set A we will denote the full transformation semigroup, and the set of unary constant operations on A by T_A, and Γ_A, respectively. For an arbitrary element a of A we will use the notation γ_a for the unary constant

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operation on A with value a, and a tuple whose all components are a will be denoted by $\hat{a}$. If $\mathbf{a}$ is an ℓ-tuple ($\ell \in \mathbb{N}$) then $\mathbf{a}_{[i]}$ will refer to its i-th component ($1 \leqslant i \leqslant \ell$).

For the set of positive integers we will use the notation $\mathbb{N}$, and we will refer to them as natural numbers.

Let A be a set and ℓ be a natural number. The set of all finitary operations on A will be denoted by $\mathcal{O}_A$. We call the operation f *essentially* k*-ary* ($k \in \mathbb{N},\ k \geqslant 2$) if it depends on exactly k of its variables. If f depends on at most one of its variables, we call f *essentially unary*. A set $\mathcal{C}$ of finitary operations on a set A is said to be a *clone* if it contains all the projections and is closed under superposition of operations. It is obvious that $\mathcal{O}_A$ and the set $\mathcal{P}_A$ of all projections on A are clones.

For a k-ary relation ϱ on A, a ϱ*-matrix* over A is a matrix whose columns belong to ϱ. An n-ary operation f on A *preserves* the m-ary relation ϱ on A if for every ϱ-matrix $X = (x_{i,j}) \in A^{n \times m}$ we have that

$$f(X) \overset{\text{def.}}{=} \begin{pmatrix} f(x_{1,1}, \ldots, x_{1,n}) \\ \vdots \\ f(x_{m,1}, \ldots, x_{m,n}) \end{pmatrix} \in \varrho.$$

If R is a set of finitary relations on A then $\mathrm{Pol}(R)$ will denote the set of all operations $f \in \mathcal{O}_A$ such that f preserves each relation in R.

It is well-known that a set $\mathcal{C}$ of finitary operations on A forms a clone if and only if $C = \mathrm{Pol}(R)$ holds for some set R of finitary relations on A.

Since the intersection of an arbitrary family of clones on A is also a clone, the set of all clones on A constitutes a complete lattice with respect to the set-theoretic inclusion. Furthermore, we can define the *clone generated by a subset* F of $\mathcal{O}_A$ as the intersection of all clones that contain F. This clone will be denoted by $\langle F \rangle$. For a natural number ℓ, the set of all ℓ-ary operations of a clone $\mathcal{C}$ will be denoted by $\mathcal{C}^{(\ell)}$.

Let M be a transformation monoid on A, and let $\mathrm{Int}(M)$ denote the collection of all clones $\mathcal{C}$ on A such that the set of unary operations of $\mathcal{C}$ is M. The clone $\langle M \rangle$ of essentially unary operations generated by M is a member of $\mathrm{Int}(M)$, in fact, it is the least member of $\mathrm{Int}(M)$, so $\mathrm{Int}(M)$ is non-empty. Furthermore, it is clear that every clone $\mathcal{C}$ in $\mathrm{Int}(M)$ is contained in the set

$$\begin{aligned}\mathrm{Sta}(M) = \big\{ f(x_1, \ldots, x_\ell) \in \mathcal{O}_A \mid \ell \in \mathbb{N},\ \text{and} \\ f(\mu_1, \ldots, \mu_\ell) \in M \text{ for all } \mu_1, \ldots, \mu_\ell \in M \big\},\end{aligned}$$

which is called the *stabilizer* of the monoid M. It is easy to verify that $\mathrm{Sta}(M)$ is a clone on A, in fact, $\mathrm{Sta}(M) = \mathrm{Pol}(\varrho_M)$, where

$$\varrho_M = \{(\mu(0), \ldots, \mu(n-1)) : \mu \in M\}\,.$$

Therefore $\mathrm{Sta}(M)$ is the largest member of $\mathrm{Int}(M)$. So, a clone $\mathcal{C}$ on A belongs to $\mathrm{Int}(M)$ if and only if $\langle M \rangle \subseteq \mathcal{C} \subseteq \mathrm{Sta}(M)$. Thus $\mathrm{Int}(M)$ is the interval

$[\langle M\rangle, \mathrm{Sta}(M)]$ in the lattice $\mathcal{O}_A$ of all clones on A. Such an interval is called a *monoidal interval.*

If $F \subseteq \mathrm{Sta}(M)$ then *the clone generated by F over M* is $\langle F \cup M\rangle$, which will be denoted by $\langle F\rangle_M$. It is obvious that $\langle F\rangle_M$ belongs to $\mathrm{Int}(M)$.

3. The monoidal interval corresponding to M_{88}

Set $M = M_{88}$. The transformation monoid M consists of the following transformations: γ_a ($a \in A$), id_A, $\varepsilon_0 = \tau_{002}$, $\varepsilon_1 = \tau_{112}$, and $\pi = \tau_{102}$. We remark that $M = M_{64} \cup \{\pi\}$. Set $A_k = A \setminus \{k\}$ ($k \in A$).

There are 18 essentially binary operations in $\mathrm{Sta}(M)$:

$$\hat{g}\colon A^2 \to A,\ \begin{cases} g(\mathbf{a}), & \text{if } \mathbf{a} \in B^2,\\ 2, & \text{otherwise,}\end{cases}$$

where g is an arbitrary binary on B,

$$0_{\wedge''}\colon A^2 \to A,\ (x,y) \mapsto \begin{cases} 2, & \text{if } x = y = 2,\\ 0, & \text{otherwise,}\end{cases}$$

and $\pi \circ 0_{\wedge''}$.

The aim of this paper is to prove the following statement.

Theorem 1. *The monoidal interval* $\mathrm{Int}(M_{88})$ *consists of exactly ten clones.*

We first study the the operations in the stabilizer of M_{88}.

Proposition 2. *Let $\tau \in M$ be an arbitrary unary operation and $a \in A_2$. Then*

(a) *if $\tau(2) = a$ then $\tau = \gamma_a$,*
(b) *if $\tau(a) \neq \pi(\tau(a))$ then $\tau(2) = 2$.*

Proposition 3. (a) *$\varrho = A_2^2 \cup \{2\}^2 \in \mathrm{Con}(A;M)$, hence, ϱ is a congruence of $(A;\mathrm{Sta}(M))$;*
(b) *for every operation $f \in \mathrm{Sta}(M)$, the operation f/α is a monotone operation with respect to the lattice order $\mathbf{0} \leqslant \mathbf{2}$, where $\mathbf{0} = A_2$ and $\mathbf{2} = \{2\}$.*
(c) *if $f \in \mathrm{Sta}(M)$ is not a constant operation then $f(A_2^n) \subseteq A_2$, furthermore, if $f(\hat{0}) = 0$ then $f(A_1^n) \subseteq A_1$,*
(d) *if $f \in \mathrm{Sta}(M)$ is not a constant operation and $f(\hat{0}) = 0$ then $f|_{A_1}$ is a monotone operation with respect to the lattice order $0 \leqslant 2$.*

Lemma 4. *Let N be a transformation monoid on a finite set A and let $\epsilon \in N$ be an idempotent transformation. Then for every operation $f \in \mathrm{Sta}(N)$ we have that $(\varepsilon \circ f)|_{\varepsilon(A)} \in \mathrm{Sta}(N_\varepsilon)$, where $N_\varepsilon = \big\{(\varepsilon \circ \tau)|_{\varepsilon(A)} : \tau \in N\big\}$ is a transformation monoid on $\varepsilon(A)$.*

Proof. Let f be an arbitrary n-ary operation in $\mathrm{Sta}(N)$ and set $f_\varepsilon = (\varepsilon \circ f)|_{\varepsilon(A)}$. Let $\tau_1, \ldots, \tau_n$ be arbitrary transformations in N_ε. Then there are transformations $\mu_1, \dots, \mu_n$ in N such that $\tau_k = (\varepsilon \circ \mu_k)|_{\varepsilon(A)}$ $(1 \leqslant k \leqslant n)$, furthermore,

$$\begin{aligned} f_\varepsilon(t_1, \dots, t_n) &= f_\varepsilon((\varepsilon \circ \mu_1)|_{\varepsilon(A)}, \dots, (\varepsilon \circ \mu_n)|_{\varepsilon(A)}) \\ &= (\varepsilon \circ f)(\varepsilon \circ \mu_1, \dots, \varepsilon \circ \mu_n)|_{\varepsilon(A)} \\ &= (\varepsilon \circ (f(\varepsilon \circ \mu_1, \dots, \varepsilon \circ \mu_n)))|_{\varepsilon(A)} \\ &\in N_\varepsilon, \end{aligned}$$

which proves the statement of the lemma. □

Proof of Proposition 3. (a) It is straightforward to check that ϱ is a congruence of $(A; M)$.

(b) Let f be an arbitrary operation in $\mathrm{Sta}(M)$, say n-ary. Then $f/\varrho \in \mathrm{Sta}(M/\varrho) = \mathrm{Sta}(\Gamma_{\{\mathbf{0},\mathbf{2}\}} \cup \{\mathrm{id}_{\{\mathbf{0},\mathbf{2}\}}\})$ is a monotone operation.

(c) Suppose that $f \in \mathrm{Sta}(M)$ is an n-ary operation for which $f(A_2^n) \nsubseteq A_2$ holds. Then $(f/\varrho)(\mathbf{0}) = \mathbf{2}$, which implies that f/ϱ is a constant operation with value $\mathbf{2}$. Hence, f is a constant operation with value 2.

(d) Let $f \in \mathrm{Sta}(M)$ be a non-constant operation with $f(\hat{0}) = 0$. Let $\mathbf{a}$ be an arbitrary n-tuple in A_2^n. Then $X = (\hat{0}^T\ \hat{0}^T\ \mathbf{a}^T) \in A^{3\times n}$ is a ϱ_M-matrix, hence, $(0, 0, f(\mathbf{a}))^T = (f(\hat{0}), f(\hat{0}), f(\mathbf{a}))^T = f(X) \in \varrho$ shows that $f(\mathbf{a}) \in A_1$. The monotonicity of $f|_{A_1}$ follows from Lemma 4 with $\varepsilon = \varepsilon_0$. □

Proposition 5. *Let $f \in \mathrm{Sta}(M)$ be a non-constant operation with $f(\hat{0}) = 0$. Then $f(\hat{2}) = 2$ and f is determined by its values on the set $A_1^n \cup A_2^n$.*

Proof. Let f and g be n-ary operations in $\mathrm{Sta}(M)$ such that $f(\mathbf{a}) = g(\mathbf{a})$ holds for every n-tuple $\mathbf{a} \in A_1^n \cup A_2^n$. Let $\mathbf{b}$ be an arbitrary n-tuple in $A^n \setminus (A_1^n \cup A_2^n)$. Then $\mathbf{b}\ \varrho^n\ \varepsilon_0(\mathbf{b}) \in A_1^n$, and so,

$$g(\mathbf{b})\ \varrho\ g(\varepsilon_0(\mathbf{b})) = f(\varepsilon_0(\mathbf{b}))\ \varrho\ f(\mathbf{b}),$$

which implies that

$$f(\mathbf{b})\ \varrho\ g(\mathbf{b}). \tag{1}$$

If $f(\mathbf{b}) = 2$ then $g(\mathbf{b}) = 2$ by (1). If $f(\mathbf{b}) \in A_2$ then $g(\mathbf{b}) \in A_2$ by (1). Set $\mu = f(\mu_1, \dots, \mu_n)$, where

$$\mu_k = \begin{cases} \gamma_{\mathbf{b}_{[k]}}, & \text{if } \mathbf{b}_{[k]} \in A_2, \\ \varepsilon_0, & \text{otherwise.} \end{cases}$$

Then $\mu \in M$, $(\mu_1(2), \dots, \mu_n(2)) = \mathbf{b}$, and so, $\mu(2) = f(\mathbf{b})$ and

$$(\mu_1(0), \dots, \mu_n(0)) = (\mu_1(1), \dots, \mu_n(1)) \in A_2^n.$$

Therefore,

$$\begin{aligned} g(\mu_1(0), \dots, \mu_n(0)) &= f(\mu_1(0), \dots, \mu_n(0)) \\ &= f(\mu_1(1), \dots, \mu_n(1)) \\ &= g(\mu_1(1), \dots, \mu_n(1)), \end{aligned}$$

hence, $g(\mathbf{b}) = g(\mu_1(2), \ldots, \mu_n(2)) = f(\mu_1(2), \ldots, \mu_n(2)) = f(\mathbf{b})$.

This completes the proof of the statement. □

Let $f \in \mathrm{Sta}(M)$ be a non-constant n-ary operation. We may assume that $f(\hat{0}) = 0$, since otherwise we may take $\pi \circ f$ for which $(\pi \circ f)(\hat{0}) = 0$ and $\langle \pi \circ f \rangle_M = \langle f \rangle_M$ hold. Let $g_f \colon A_2^n \to A_2$ and $h_f \colon A_1^n \to A_1$ be the operations $f|_{A_2}$ and $f|_{A_1}$, respectively. Then h_f is a monotone operation with respect to the partial order $0 \leqslant 2$ by Proposition 3 (b).

Suppose that g_f depends on its i-th variable. Then there are elements $a_1, \ldots, a_n, a_i' \in A_2$ such that $g_f(\mathbf{a}) \neq g_f(\mathbf{a}')$, where $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{a}' = (a_1, \ldots, a_{i-1}, a_i', a_{i+1}, \ldots, a_n)$. Moreover, $f(\mathbf{a}) = g_f(\mathbf{a}) \neq g_f(\mathbf{a}') = f(\mathbf{a}')$. Since $(a_j, a_j, a_j)^T \in \varrho_M$ $(1 \leqslant j \leqslant n,\ j \neq i)$ and $(a_i, a_i', 2)^T \in \varrho_M$, we get that

$$(f(\mathbf{a}), f(\mathbf{a}'), f(a_1, \ldots, a_{i-1}, 2, a_{i+1}, \ldots, a_n)) \in \varrho_M,$$

and so $f(a_1, \ldots, a_{i-1}, 2, a_{i+1}, \ldots, a_n) = 2$. Then for arbitrary elements $\mathbf{b} \in A^n$ with $\mathbf{b}_{[i]} = 2$ and $\mathbf{b}_{[j]} \in \{0,1\}^n$ $(j \neq i)$, the matrix

$$\begin{pmatrix} a_1 & \ldots & a_{i-1} & 2 & a_{i+1} & \ldots & a_n \\ \mathbf{b}_{[1]} & \ldots & \mathbf{b}_{[i-1]} & 2 & \mathbf{b}_{[i+1]} & \ldots & \mathbf{b}_{[n]} \\ 2 & \ldots & 2 & 2 & 2 & \ldots & 2 \end{pmatrix}$$

is a ϱ_M-matrix, which yields that $f(\mathbf{b}) = 2$. Let $\mathbf{d}$ be an n-tuple in A^n such that $\mathbf{d}_{[i]} = 2$. For an element $x \in A$ let

$$x^\diamond = \begin{cases} 1, & \text{if } x = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\begin{pmatrix} \mathbf{d}_{[1]}^\diamond & \ldots \mathbf{d}_{[i-1]}^\diamond & 2 & \mathbf{d}_{[i+1]}^\diamond & \ldots & \mathbf{d}_{[n]}^\diamond \\ \mathbf{d}_{[1]}^\diamond & \ldots \mathbf{d}_{[i-1]}^\diamond & 2 & \mathbf{d}_{[i+1]}^\diamond & \ldots & \mathbf{d}_{[n]}^\diamond \\ \mathbf{d}_{[1]} & \ldots \mathbf{d}_{[i-1]} & 2 & \mathbf{d}_{[i+1]} & \ldots & \mathbf{d}_{[n]} \end{pmatrix}$$

is a ϱ_M-matrix and $f(\mathbf{d}_{[1]}^\diamond, \ldots, \mathbf{d}_{[i-1]}^\diamond, 2, \mathbf{d}_{[i+1]}^\diamond, \ldots, \mathbf{d}_{[n]}^\diamond) = 2$ by the previous argument, hence, $f(\mathbf{d}) = 2$.

The previous argument yields the following statement.

Proposition 6. *Let $f \in \mathrm{Sta}(M)$ be a non-constant n-ary operation with $f(\hat{0}) = 0$. If g_f depends on its i-th variable then $f(\mathbf{a}) = 2$ holds for every n-tuple $\mathbf{a} \in A^n$ with $\mathbf{a}_{[i]} = 2$.*

To describe the clones in $\mathrm{Int}(M)$ we will use the following binary operations:

$0_{\vee''}$	0	1	2
0	0	0	2
1	0	0	2
2	2	2	2

,

$\rightrightarrows_{\vee''}$	0	1	2
0	0	0	2
1	1	1	2
2	2	2	2

,

$+_{\vee''}$	0	1	2
0	0	1	2
1	1	0	2
2	2	2	2

,

$\vee_{\vee''}$	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

.

Proposition 7. *The stabilizer of M is $\langle \vee_{\vee''}, 0_{\vee''}\rangle_M$.*

Proof. Let f be an n-ary operation in $\mathrm{Sta}(M)\setminus\langle M\rangle$ such that $f(\hat{0})=0$. We may assume that g_f depends exactly on its first k variables. Let

$$\begin{aligned} P_f &= \{\mathbf{a}\in A_2^n : g_f(\mathbf{a}) = f(\mathbf{a}) = 1\}\,,\\ Q_f &= \min\{\mathbf{b}\in A_1^n : h_f(\mathbf{b}) = f(\mathbf{b}) = 2\} = \{\mathbf{b}_1,\dots,\mathbf{b}_q\}. \end{aligned}$$

Then

$$g(\mathbf{x}) = \bigvee_{\mathbf{a}\in P_f} \wedge\,\mathbf{a}\mathbf{x} \qquad (\mathbf{x}\in A_2^n),$$

and

$$h(\mathbf{x}) = \bigvee_{\mathbf{b}\in Q_f}'' \wedge''\,\mathbf{b}\mathbf{x} \qquad (\mathbf{x}\in A_1^n),$$

where

$$\begin{aligned} \wedge\,\mathbf{a}\mathbf{x} &= \pi^{\mathbf{a}_{[1]}}(\mathbf{x}_{[1]}) \wedge\cdots\wedge \pi^{\mathbf{a}_{[k]}}(\mathbf{x}_{[k]}),\\ \wedge''\,\mathbf{b}\mathbf{x} &= \wedge''\left\{\mathbf{x}_{[i]} : 1\leqslant i\leqslant n \text{ and } \mathbf{b}_{[i]} = 2\right\}, \end{aligned}$$

and $\pi^0 = \mathrm{id}_A$ and $\pi^1 = \pi$. By Proposition 6, the n-tuples $\{0\}^{i-1}\times\{1\}\times\{0\}^{n-i}$ $(1\leqslant i\leqslant k)$ belong to the set Q_f. We will assume that $\mathbf{b}_i = \{0\}^{i-1}\times\{1\}\times\{0\}^{n-i}$ $(1\leqslant i\leqslant k)$. Let

$$\widetilde{f}(\mathbf{x}) = \left(\bigvee_{\mathbf{a}\in P_f}{}_{\vee''}\,(\wedge_{\vee''}\,\mathbf{a}\mathbf{x})\right) \rightrightarrows_{\vee''} \left(\mathop{\rightrightarrows_{\vee''}}_{j=k+1}^{q} (0_{\wedge''}\,\mathbf{b}_j\mathbf{x})\right) \quad (\mathbf{x}\in A^n),$$

where $\wedge_{\vee''}\,\mathbf{a}\mathbf{x} = \pi^{\mathbf{a}_{[1]}}(\mathbf{x}_{[1]}) \vee_{\vee''} \ldots \vee_{\vee''} \pi^{\mathbf{a}_{[n]}}(\mathbf{x}_{[n]})$. Then equalities $\tilde{f}|_{A_2} = g_f$ and $\tilde{f}|_{A_1} = h_f$ proves that $f = \tilde{f}$. Since $a\wedge_{\vee''} b = \pi(\pi(a)\vee_{\vee''}\pi(b))$ holds for arbitrary elements $a,b\in A$, we get that $\mathrm{Sta}(M)$ is generated by $\vee_{\vee''}$ and $0_{\wedge''}$ over M. □

Let f be a non-surjective operation in $\mathrm{Sta}(M)\setminus\langle M\rangle$ with $f(\hat{0})=0$. Then the range of f is A_1 and g_f is the constant operation with value 0, furthermore, the clone generated by h_f over $M_{A_1} = \{(\varepsilon_0\circ\mu)|_{A_1} : \mu\in M\} = M_{\varepsilon_0}$ is one of the following clones on A_1:

$$\mathcal{I}'' = \langle M_{A_1}\rangle,\quad \mathcal{E}'' = \langle\wedge''\rangle_{M_{A_1}},\quad \mathcal{V}'' = \langle\vee''\rangle_{M_{A_1}},\quad \mathcal{M}'' = \langle\wedge'',\vee''\rangle_{M_{A_1}}.$$

Thus, $\langle f\rangle_M \in \{\langle M\rangle, \langle 0_{\wedge''}\rangle_M, \langle 0_{\vee''}\rangle_M, \langle 0_{\wedge''}, 0_{\vee''}\rangle_M\}$, in fact,

$$\mathrm{Int}_{\mathrm{ns}}(M) = \{\langle M\rangle, \langle 0_{\wedge''}\rangle_M, \langle 0_{\vee''}\rangle_M, \langle 0_{\wedge''}, 0_{\vee''}\rangle_M\}.$$

Let f be a surjective n-ary operation in $\mathrm{Sta}(M) \setminus \langle M\rangle$ with $f(\hat{0}) = 0$. Then f is idempotent and g_f is not a constant operation, hence, $\langle g_f\rangle_{M_{A_2}} \in \{\mathcal{N}, \mathcal{L}, \mathcal{BF}\}$, where

$$\mathcal{N} = \langle M_{A_2}\rangle, \quad \mathcal{L} = \langle +\rangle_{M_{A_2}}, \quad \mathcal{BF} = \mathcal{O}_{A_2}.$$

We may assume that g_f depends on its first variable. The operation h_f is a monotone operation that belongs to $\langle \wedge'', \vee''\rangle_{M_{A_1}}$. By Proposition 6,

$$h_f(\mathbf{x}) = \bigvee''_{\mathbf{b}\in H_h} \wedge'' \mathbf{b}\cdot\mathbf{x} = \mathbf{x}_{[1]} \vee'' \bigvee''_{\mathbf{b}\in H_{h_f}\setminus\{(2,0,\ldots,0)\}} \wedge'' \mathbf{b}\cdot\mathbf{x} \ (\mathbf{x}\in A^n),$$

where $H_{h_f} = \min\{\mathbf{b}\in A_1^n : h_f(\mathbf{b}) = f(\mathbf{b}) = 2\}$, hence,

$$h_f(x, y, \ldots, y) = x \vee'' y \quad \text{and} \quad f(x, y, \ldots, y) = x \rightrightarrows_{\vee''} y.$$

If $\wedge'' \in \langle g\rangle_{M_{A_2}}$ then $0_{\wedge''} \in \langle f\rangle_M$.

Suppose that $\langle g\rangle_{M_{A_2}} = \mathcal{L}$. Then $+ \in \langle g\rangle_{M_{A_2}}$, and so, $+_{\vee''}$ is a member of $\langle f\rangle_M$.

Let $\mathcal{C}$ be a clone in $\mathrm{Int}(M)$. Set

$$\mathcal{C}_{A_2} = \left\{ f|_{A_2} : f\in\mathcal{C},\ f({A_2}^{\mathrm{arity}(f)}) \subseteq A_2 \right\},$$
$$\mathcal{C}_{A_1} = \{(\varepsilon_0\circ f)|_{A_1} : f\in\mathcal{C}\}.$$

Then $\mathcal{C}_{A_2}$ and $\mathcal{C}_{A_1}$ are clones, moreover, $\mathcal{C}_{A_2} \in \mathrm{Sta}(M_{A_2})$ and $\mathcal{C}_{A_1} \in \mathrm{Sta}(M_{A_1})$, where $M_{A_2} = \{m|_{A_2} : m\in M,\ m(A_2)\subseteq A_2\} = T_{A_2}$ and $M_{A_1} = M_{\varepsilon_0} = C_{A_1} \cup \{\mathrm{id}_{A_1}\}$.

Proposition 8. *If* $\mathcal{C}_{A_1} = \langle M_{A_1}\rangle$ *then* $\mathcal{C} = \langle M\rangle$.

Proof. Let $f\in\mathrm{Sta}(M)$ be an n-ary operation that depends on its variables. If $n \geqslant 2$ then by a theorem of A. Salomaa (see Lemma 4.1. in Hobby–McKenzie [5]) there are distinct indexes i and j ($1\leqslant i<j\leqslant n$) and elements a_k ($1\leqslant k\leqslant n,\ k\neq i,j$) such that the binary operation

$$f'\colon A^2\to A,\ (x,y)\mapsto f(a_1,\ldots,a_{i-1},x,a_{i+1},\ldots,a_{j-1},y,a_{j+1},\ldots,a_n)$$

depends on both of its variables. Then f' belongs to $\langle f\rangle_M$ and $(\varepsilon_0\circ f')|_{A_1}$ depends on its variables.

The statement of the proposition follows. ☐

Proposition 9. *Let* f *be a surjective operation in* $\mathrm{Sta}(M)\setminus\langle 0_{\wedge''}, 0_{\vee''}\rangle_M$*, and set* $\mathcal{C} = \langle f\rangle_M$*. Then* $\rightrightarrows_{\vee''}\in\mathcal{C}$*, furthermore* $\langle\rightrightarrows_{\vee''}\rangle_M$ *is the unique upper cover of* $\langle 0_{\vee''}\rangle_M$ *and* $\langle 0_{\wedge''}, \rightrightarrows_{\vee''}\rangle_M$ *is the unique upper cover of* $\langle 0_{\wedge''}, 0_{\vee''}\rangle_M$*.*

Proof. We may assume that f depends on all of its variables. By Proposition 2 (c), $f(A_2^n)\subseteq A_2$ and $g_f = f|_{A_2}$ is not a constant operation.

If g_f depends on at least two of its variables then, by Salomaa's theorem, there is a binary operation in $\langle g_f\rangle_{\Gamma_{A_2}}$ that depends on its variables, hence, there is a binary surjective operation in $\langle f\rangle_{\Gamma_A}$ that depends on its variables. Therefore, in this case the statement follows.

Suppose that g_f is an essentially unary operation. We may assume that it depends on its first variable. Then the $(n-1)$-ary operation

$$f[2,1]\colon A^{n-1}\to A,\ (x_2,\dots,x_n)\mapsto f(2,x_2,\dots,x_n)$$

is a constant operation with value 2 by Proposition 6. As f depends on its variables, there is an n-tuple $\mathbf{a}\in A^n$ such that $\mathbf{a}_{[1]}\in A_2$ and $f(\mathbf{a})=2$. Define the binary operation w on A as follows:

$$w(a,b)=f(a,\mu_2(b),\dots,\mu_n(b))\ (a,b\in A),$$

where for every index i $(2\leqslant i\leqslant n)$

$$\mu_i=\begin{cases}\gamma_{\mathbf{a}_{[i]}}, & \text{if } \mathbf{a}_{[i]}\in A_2,\\ \mathrm{id}_A, & \text{if } \mathbf{a}_{[i]}=2.\end{cases}$$

Then $w(2,b)=2$ $(b\in A)$ and $w(\mathbf{a}_{[1]},2)=f(\mathbf{a})=2$, which imply that $w=\rightrightarrows_{\vee''}$.

This completes the proof. □

Proposition 10. *Let f be a surjective operation in* $\mathrm{Sta}(M)\setminus\langle 0_{\wedge''},\rightrightarrows_{\vee''}\rangle_M$*, and set $\mathcal{C}=\langle f\rangle_M$. Then $+_{\vee''}\in\mathcal{C}$, furthermore $\langle +_{\vee''}\rangle_M$ is the unique upper cover of $\langle\rightrightarrows_{\vee''}\rangle_M$ and $\langle 0_{\wedge''},+_{\vee''}\rangle_M$ is the unique upper cover of $\langle 0_{\wedge''},\rightrightarrows_{\vee''}\rangle_M$.*

Proof. Suppose that $\mathcal{C}_{A_2}=\langle M_{A_2}\rangle$. We may assume that $f(\hat{0})=0$. Then $g_f=f|_{A_2}$ belongs to $\mathcal{C}_{A_2}$, and so, it is an essentially unary operation that is not constant since f is surjective. Reordering the variables of f if necessary, we may assume that $g_f(\mathbf{x})=\mathbf{x}_{[1]}$ holds for every n-tuple $\mathbf{x}\in A^n$.

The clone generated by the operation $h_f=f|_{A_1}$ over M_{A_1} is either $\langle\vee''\rangle_{M_{A_1}}$ or $\langle\wedge'',\vee''\rangle_{M_{A_1}}$ since $\rightrightarrows_{\vee''}\in\mathcal{C}$, furthermore, in the latter case $0_{\wedge''}\in\mathcal{C}$ also holds. Thus, there is an operation $\tilde{f}$ in $\langle 0_{\wedge''},\rightrightarrows_{\vee''}\rangle_M$ such that $g_{\tilde{f}}=g_f$ and $h_{\tilde{f}}=h_f$, which proves that $f=\tilde{f}$. This contradicts to our assumption on f.

Then $\langle M_{A_2}\rangle\subsetneq\mathcal{C}_{A_2}$, and so, $+\in\mathcal{C}_{A_2}$. Since $+_{\vee''}\in\mathrm{Sta}(M)$ is the unique operation whose restriction to A_2 is $+$, we get that $+_{\vee''}\in\mathcal{C}$. □

Proposition 11. *Let f be a surjective operation in* $\mathrm{Sta}(M)\setminus\langle 0_{\wedge''},+_{\vee''}\rangle_M$*. Then $\vee_{\vee''}\in\mathcal{C}$,*

Proof. Set $\mathcal{C}=\langle f\rangle_M$. We may assume that $f(\hat{0})=0$. Then $+\notin\mathcal{C}_{A_2}$, and so, $\mathcal{C}_B=\langle\vee\rangle_{M_{A_2}}$. Since $\vee_{\vee''}\in\mathrm{Sta}(M)$ is the unique operation whose restriction to A_2 is $\vee$, we get that $\vee_{\vee''}\in\mathcal{C}$. □

Proof of Theorem 1. Combining Propositions 7–11, we get that $\mathrm{Int}(M)$ consists of the clones

$$\begin{array}{lllll}\langle M\rangle, & \langle 0_{\vee''}\rangle_M, & \langle\rightrightarrows_{\vee''}\rangle_M, & \langle +_{\vee''}\rangle_M, & \langle\vee_{\vee''}\rangle_M,\\ \langle 0_{\wedge''}\rangle_M, & \langle 0_{\wedge''},0_{\vee''}\rangle_M, & \langle 0_{\wedge''},\rightrightarrows_{\vee''}\rangle_M, & \langle 0_{\wedge''},+_{\vee''}\rangle_M, & \langle 0_{\wedge''},\vee_{\vee''}\rangle,\end{array}$$

and the monoidal interval can be seen in Figure 1 on page 9. □

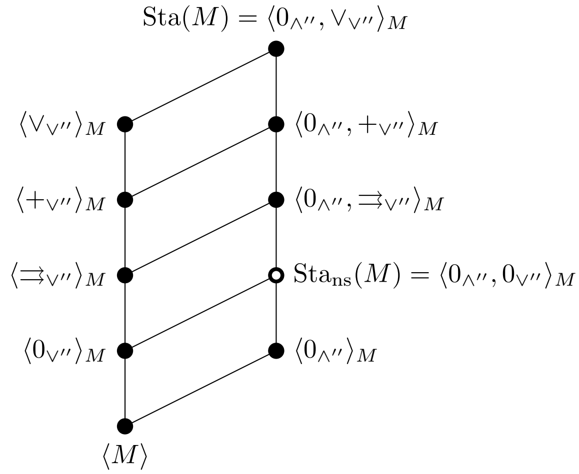

FIGURE 1. The interval $\mathrm{Int}(M)$

REFERENCES

[1] Dormán, M.: Collapsing inverse monoids. Algebra Universalis **56**, 241–261 (2007)
[2] Dormán, M., Makay, G., Maróti, M., Vajda R.: Monoidal intervals on three- and four-element sets. Acta Sci. Math. (Szeged) **81**, 399–424 (2015)
[3] Dormán, M.: Transformation monoids with finite monoidal intervals. Algebra Universalis **77**, 1–27 (2017)
[4] Grabowski, J.-U.: Binary operations suffice to test collapsing of monoidal intervals. Algebra Universalis **38**, 92–95 (1997)
[5] Hobby, D. and McKenzie, R.: The Structure of finite algebras. AMS Contemporary Mathematics 76 (1996)
[6] Krokhin, A.: Monoidal intervals in lattices of clones. Algebra Logika **34**, 288–310 (1993) (Russian, english translation: Algebra Logic **34**, 155–168 (1993))
[7] Pálfy, P. P.: Unary polynomials in algebras I. Algebra Universalis **18**, 262–273 (1984)
[8] Post, E. L.: The two-valued iterative systems of mathematical logic. Ann. Math. Studies, no. 5, Princeton University Press, Princeton, N.J., (1941)
[9] Salomaa, A.: On essential variables of functions, especially in the algebra of logic, Ann. Acad. Sci. Fenn. Ser. AI, No. 339, (1963)
[10] Szendrei, Á.: Clones in universal algebra. Séminare de mathématiques supérieures, vol. 99. Les presses de l'Université de Montréal, Montréal (1986)

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