Transformation monoids with finite monoidal intervals

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Abstract. In this paper we investigate transformation monoids that are built up from inverse transformation monoids constructed from finite lattices by adding all the unary constant transformations. We give a complete description for the corresponding monoidal intervals in the clone lattice.

Introduction

Let A be a finite set with at least three elements, and let M be an arbitrary transformation monoid on A. It is well known that the clones whose set of unary operations coincides with M form an interval in the lattice of all clones on A (see Á. Szendrei [7], Chapter 3). An interval of this form is called a monoidal interval. If A is finite, then there are only finitely many transformation monoids on A. Hence the monoidal intervals partition the clone lattice into finitely many blocks. Since the clone lattice has cardinality $2^{\aleph_0}$ if $|A| \geqslant 3$, one might expect that 'for most M' the monoidal interval $\mathrm{Int}(M)$ contains uncountably many clones. Nevertheless, it turns out that for many interesting transformation monoids the corresponding monoidal intervals are countable. So, studying these intervals may lead to a better understanding of some parts of the clone lattice.

The problem of classifying transformation monoids according to the cardinalities of the corresponding monoidal intervals was posed by Á. Szendrei in [7]. A large family of monoids M with finite monoidal intervals is provided by Pálfy's theorem in [4]: if M consists of all constants and some permutations, then the corresponding monoidal interval contains at most two elements; moreover, this interval has a single element unless M coincides with the monoid of all unary polynomial operations of a finite vector space.

Although, a complete classification of transformation monoids according to the sizes of the corresponding monoidal intervals seems a very hard problem at present, for certain classes of monoids we can solve this problem (cf. Pálfy's theorem that was mentioned in the preceding paragraph). In this paper we will consider transformation monoids that consist of an inverse transformation monoid constructed from a finite lattice and all the unary constant operations. We will get a description

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Date: January 19, 2015.

1991 *Mathematics Subject Classification*: 08A40, 20M18, 20M20.

Key words and phrases: Inverse monoid, monoidal interval.

This research is supported by Hungarian National Foundation for Scientific Research grant no. K83219 and by the TAMOP-4.2.1/B-09/1/KONV-2010-0005 project.

that is similar to Pálfy's theorem. Namely, we will prove that the monoidal interval corresponding to such a transformation monoid is finite (cf. Theorems 3.1 and 3.2); moreover, this interval has a single element unless the lattice we start with has only one atom (cf. Theorem 2.4).

1. Preliminaries

Let X, Y, Y', and Z be sets for which $Y \subseteq Y'$ holds. By the composition of the maps $t\colon X \to Y$ and $t'\colon Y' \to Z$ we will mean the map $X \to Z,\ x \mapsto t'(t(x))$, denoted by $t' \circ t$. For arbitrary subset W of X the restriction of the map t to the set W is the map $t{\restriction_W}\colon W \to Y,\ x \mapsto t(x)$.

For a finite set A we will denote the full transformation semigroup, and the set of unary constant operations on A by T_A, and C_A, respectively. For an arbitrary element a of A we will use the notation c_a for the unary constant operation on A with value a, and a tuple whose all components are a will be denoted by $\hat{a}$.

For the set of positive integers we will use the notation $\mathbb{N}$, and we will refer to them as natural numbers.

Let A be a set and n be a positive integer. An **ℓ-ary operation on** A is a function $f\colon A^\ell \to A$. An operation is called **finitary** if it is ℓ-ary for a natural number ℓ. The set of all finitary operations on A will be denoted by $\mathcal{O}_A$. An ℓ-ary operation $f \in \mathcal{O}_A$ is said to **depend on its i-th variable** ($1 \leqslant i \leqslant \ell$) if there are elements $a_1, \dots, a_{i-1}, a_i, a_i', a_{i+1}, \dots, a_\ell$ of A such that

$$f(a_1, \dots, a_{i-1}, a_i, a_{i+1}, \dots, a_\ell) \neq f(a_1, \dots, a_{i-1}, a_i', a_{i+1}, \dots, a_\ell).$$

We call the operation f **essentially k-ary** ($k \in \mathbb{N},\ k \geqslant 2$) if it depends on exactly k of its variables. If f depends on at most one of its variables, we call f **essentially unary**. The **superposition** of an ℓ-ary operation $f \in \mathcal{O}_A$ by k-ary operations $g_1, \dots, g_\ell \in \mathcal{O}_A$ is the k-ary operation $f(g_1, \dots, g_\ell) \in \mathcal{O}_A$ defined by the rule

$$f(g_1, \dots, g_\ell)(x_1, \dots, x_k) = f\big(g_1(x_1, \dots, x_k), \dots, g_\ell(x_1, \dots, x_k)\big).$$

A set $\mathcal{C}$ of finitary operations on a set A is said to be a **clone** if it contains all the projections and is closed under superposition of operations. It is obvious that $\mathcal{O}_A$ and the set $\mathcal{P}_A$ of all projections on A are clones. Since the intersection of an arbitrary family of clones on A is also a clone, the set of all clones on A constitutes a complete lattice with respect to the set-theoretic inclusion. Furthermore, we can define the **clone generated by a subset** F of $\mathcal{O}_A$ as the intersection of all clones that contain F. This clone will be denoted by $\langle F \rangle$. For a natural number ℓ, the set of all ℓ-ary operations of a clone $\mathcal{C}$ will be denoted by $\mathcal{C}^{(\ell)}$.

Let M be a transformation monoid on A, and let $\operatorname{Int}(M)$ denote the collection of all clones $\mathcal{C}$ on A such that the set of unary operations of $\mathcal{C}$ is M. The clone $\langle M \rangle$ of essentially unary operations generated by M is a member of $\operatorname{Int}(M)$, in fact, it is the least member of $\operatorname{Int}(M)$, so $\operatorname{Int}(M)$ is non-empty. Furthermore, it is clear that every clone $\mathcal{C}$ in $\mathrm{Int}(M)$ is contained in the set

$$\mathrm{Sta}(M) = \big\{ f(x_1, \dots, x_\ell) \in \mathcal{O}_A \mid \ell \in \mathbb{N}, \text{ and } \\ f(t_1, \dots, t_\ell) \in M \text{ for all } t_1, \dots, t_\ell \in M \big\},$$

which is called the **stabilizer** of the monoid M. It is easy to verify that $\mathrm{Sta}(M)$ is a clone on A, therefore $\mathrm{Sta}(M)$ is the largest member of $\mathrm{Int}(M)$. So, we see that a clone $\mathcal{C}$ on A belongs to $\mathrm{Int}(M)$ if and only if $\langle M\rangle \subseteq \mathcal{C} \subseteq \mathrm{Sta}(M)$. Thus $\mathrm{Int}(M)$ is the interval $[\langle M\rangle, \mathrm{Sta}(M)]$ in the lattice of all clones on A. Such an interval is called a **monoidal interval**.

The monoid M will be called **collapsing** if the monoidal interval corresponding to it contains only one element, that is, there is no essentially at least binary operation in the stabilizer of M. In fact, it is enough to examine the binary operations in $\mathrm{Sta}(M)$ as the following result of Grabowski [2] states.

Theorem 1.1 (Grabowski [2])**.** *Let M be a transformation monoid on a finite set A. Then M is collapsing if and only if the stabilizer of M does not contain essentially binary operations.*

Let $\mathbf{L} = (L; \vee, \wedge)$ be a finite lattice. The least and greatest elements of $\mathbf{L}$ will be denoted by $0_{\mathbf{L}}$ and $1_{\mathbf{L}}$, respectively. If the lattice is clear from the context then we omit the subscript, and simply write 0 and 1, respectively. The set of atoms of $\mathbf{L}$ will be denoted by $\mathcal{A}(\mathbf{L})$, and we put $\mathcal{A}_0(\mathbf{L}) = \mathcal{A}(\mathbf{L}) \cup \{0\}$. If there is no danger of confusion, we simply write $\mathcal{A}$ and $\mathcal{A}_0$, respectively. Two elements b and d of $\mathbf{L}$ will be called **similar** iff the principal ideals $(b]$ and $(d]$ are isomorphic as lattices. We write $b \sim d$ to denote that b is similar to d. The relation $\sim$ is an equivalence relation on L. If the $\sim$-class containing b has only one element then b will be called **isolated**. For every element $b \in L$ we define a unary operation φ_b by the rule $\varphi_b(x) = x \wedge b$ $(x \in L)$. In particular, $\varphi_0 = c_0$ is the unary constant operation with range $\{0\}$. For similar elements $b, d \in L$ the symbol $\mathrm{Iso}(b, d)$ will denote the set of all lattice isomorphism between the principal ideals $(b]$ and $(d]$.

Define the set $\mathrm{IS}(\mathbf{L})$ of transformations on L in the following way:

$$\mathrm{IS}(\mathbf{L}) = \{\beta_{b,d} \circ \varphi_b \mid b, d \in L,\ b \sim d, \text{ and } \beta_{b,d} \in \mathrm{Iso}(b, d)\}.$$

Then $\mathrm{IS}(\mathbf{L})$ is an inverse submonoid of the full transformation semigroup on L (cf. Saito–Katsura [6], Lemma 3.1).

Let $N = \mathrm{IS}(\mathbf{L})$ be the inverse monoid determined by the lattice $\mathbf{L}$. The monoidal interval corresponding to N was examined in [1]. In Propositions 1.2 and 1.3 we recall some properties of the transformations in N and of the operations in $\mathrm{Sta}(N)$.

Proposition 1.2 (cf. Proposition 2.1 in [1])**.** *Let t be an arbitrary transformation from N. Then*

(a) *t is monotone;*
(b) *there is a unique element $b \sim t(1)$ of L such that $t = \beta_{b,t(1)} \circ \varphi_b$ for some isomorphism $\beta_{b,t(1)} \in \mathrm{Iso}(b, t(1))$; furthermore, for any $l \in L$ we have $t(l) = t(1)$ if and only if $b \leqslant l$;*

(c) *$t(\mathcal{A}_0) \subseteq \mathcal{A}_0$, and $t(0) = 0$, moreover, for arbitrary atom $a \in \mathcal{A}$ we have that $0 < t(a)$ if and only if $a \leqslant b$;*
(d) *if $t(a) = 0$ for every atom a of $\mathbf{L}$ then $t = \varphi_0$.*

Proposition 1.3 (cf. Lemma 2.2 in [1])**.** *Suppose $\mathbf{L}$ has at least two atoms. If f is a binary operation in the stabilizer of N then*

(a) *$f(\mathcal{A}_0 \times \mathcal{A}_0) \subseteq \mathcal{A}_0$ and $f(0,0) = 0$;*
(b) *$f\restriction_{\mathcal{A}_0}$ is an essentially unary operation;*
(c) *if $f\restriction_{\mathcal{A}_0}$ does not depend on its first variable [second variable] then $f(l,0) = 0$ [$f(0,l) = 0$] for all $l \in L$.*

2. Monoidal intervals with a single element

Throughout this section, $\mathbf{L}$ will be a finite lattice with underlying set L, and M will be a transformation monoid on L that is obtained from the monoid $\mathrm{IS}(\mathbf{L})$ associated to $\mathbf{L}$ (see Section 1) by adding some constant transformations, that is, $M = \mathrm{IS}(\mathbf{L}) \cup C'$ for some subset C' of C_L. Our goal is to present two sufficient conditions for such a monoid to be collapsing (Corollary 2.2 and Theorem 2.4). Both proofs rely on Proposition 2.1 below.

Before stating Proposition 2.1, we discuss some basic properties of the monoids $M = \mathrm{IS}(\mathbf{L}) \cup C'$ ($C' \subseteq C_L$). The definition of $\mathrm{IS}(\mathbf{L})$ shows that $c_0 = \varphi_0 \in \mathrm{IS}(\mathbf{L})$ and $t(0) = 0$ holds for every transformation $t \in \mathrm{IS}(\mathbf{L})$ (cf. Proposition 1.2 (c)), hence it follows from the definition of M that $t = c_l$ holds with an element $l \in L \setminus \{0\}$ if $t \in M \setminus \mathrm{IS}(\mathbf{L})$. The following consequences of these observations will be used later on without further references:

- for $t \in M$ we have $t(0) = 0$ if and only if $t \in \mathrm{IS}(\mathbf{L})$;
- if $t \in M$ and $t(0) \neq 0$ then $t = c_{t(0)}$.

Proposition 2.1. *Let $N = \mathrm{IS}(\mathbf{L})$ for a finite lattice $\mathbf{L}$ with underlying set L. If M is a transformation monoid of the form $M = \mathrm{IS}(\mathbf{L}) \cup C'$ with $C' \subseteq C_L$, then $\mathrm{Sta}(M)^{(2)} \setminus \langle M \rangle \subseteq \mathrm{Sta}(N)^{(2)}$.*

Proof. Suppose that M satisfies the assumption of the proposition. We will show that every operation in $\mathrm{Sta}(M)^{(2)} \setminus \mathrm{Sta}(N)^{(2)}$ is constant, and so, it belongs to $\left\langle \mathrm{Sta}(M)^{(1)} \right\rangle = \langle M \rangle$, which implies the desired conclusion.

Let $f \in \mathrm{Sta}(M)^{(2)} \setminus \mathrm{Sta}(N)^{(2)}$ be an arbitrary operation. Then $f \notin \mathrm{Sta}(N)^{(2)}$, and so, for some $n_1, n_2 \in N$ the unary operation $t = f(n_1, n_2)$ does not belong to N. However, $n_1, n_2 \in M$ and $f \in \mathrm{Sta}(M)$ provide that $t \in M$. Hence, $t \in M \setminus N = C' \setminus \{c_0\}$, that is, $t = c_l$ for an element $l \in L \setminus \{0\}$. We will show that f is the binary constant operation with value l. To prove this, choose arbitrary elements b and d in L, and set $m = f(\varphi_b, \varphi_d)$. Then $m \in M$, because $\varphi_b, \varphi_d \in M$ and $f \in \mathrm{Sta}(M)$. Furthermore, the equalities $n_1(0) = 0$ and $n_2(0) = 0$ also hold, because $n_1, n_2 \in N$. Hence, using the definitions of m and t, we get that

$$m(0) = f(\varphi_b(0), \varphi_d(0)) = f(0,0) = f(n_1(0), n_2(0)) = t(0) = l \neq 0.$$

Thus that $m = c_l$, and hence $f(b,d) = f(\varphi_b(1), \varphi_d(1)) = m(1) = l$. Therefore, f is the constant operation with value l, as claimed. $\square$

Combining Proposition 2.1 with Theorem 1.1 we get the following corollary.

Corollary 2.2. *Let* $\mathbf{L}$ *be a finite lattice with underlying set* L*. If* $\mathrm{IS}(\mathbf{L})$ *is collapsing then so is every transformation monoid* M *on* L *that has the form* $\mathrm{IS}(\mathbf{L}) \cup C'$ *for some* $C' \subseteq C_L$*.*

The finite lattices for which the transformation monoid $\mathrm{IS}(\mathbf{L})$ is collapsing is characterized in [1, Theorem 3.1]. The characterization shows, among others, that for $\mathrm{IS}(\mathbf{L})$ to be collapsing it is necessary that $\mathbf{L}$ has at least two atoms [1, Theorem 3.1], and sufficient that $\mathbf{L}$ is an atomistic lattice with at least three elements [1, Corollary 3.12].

Example 2.3. Let $\mathbf{L}$ be the 4-element lattice that can be seen in Figure 1. Let β_{a_1,a_2} be the (unique) isomorphism between $(a_1]$ and $(a_2]$ with inverse β_{a_2,a_1}, furthermore, let $\beta_{1,1}$ be the (unique) automorphism of the lattice $\mathbf{L}$ with $\beta_{1,1}(a_1) = a_2$.

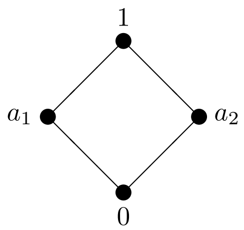

Figure 1: The lattice $\mathbf{L}$.

It is easy to see that

$$\mathrm{IS}(\mathbf{L}) = \{\varphi_0, \varphi_{a_1}, \beta_{a_1,a_2} \circ \varphi_{a_1}, \varphi_{a_2}, \beta_{a_2,a_1} \circ \varphi_{a_2}, \varphi_1, \beta_{1,1} \circ \varphi_1\}\,.$$

By Theorem 3.1 in [1], the monoid $\mathrm{IS}(\mathbf{L})$ is collapsing. Thus, Corollary 2.2 implies that the monoids $\mathrm{IS}(\mathbf{L}) \cup \{c_0, c_{a_1}, c_{a_2}\}$ and $\mathrm{IS}(\mathbf{L}) \cup C_L$ are also collapsing.

From now on, we will assume that M is the transformation monoid $\mathrm{IS}(\mathbf{L}) \cup C_L$ for a finite lattice $\mathbf{L} = (L; \wedge, \vee)$.

Theorem 2.4. *If* $\mathbf{L}$ *contains at least two atoms then the monoid* $M = \mathrm{IS}(\mathbf{L}) \cup C_L$ *is collapsing.*

Proof. Let N denote the transformation monoid $\mathrm{IS}(\mathbf{L}) \leqslant T_L$. By Grabowski's result in [2], it is enough to prove that the stabilizer of M contains no essentially binary operations. Let $f \in \mathrm{Sta}(M)$ be an essentially binary operation. Then Proposition 2.1 implies that f belongs to $\mathrm{Sta}(N)$. Therefore, $f{\restriction}_{\mathcal{A}_0}$ is an essentially unary operation by Proposition 1.3 (b). We may assume, without loss of generality, that $f{\restriction}_{\mathcal{A}_0}$ does not depend on its second variable. Then by Proposition 1.3 (c) we get that $f(0, l) = 0$ holds for all $l \in L$, that is, $f(c_0, \mathrm{id}_L) = c_0$.

Suppose that $f(\mathrm{id}_L, c_0) = c_0$. Let b and d be arbitrary elements of $L \setminus \{0\}$, and set $t = f(\varphi_b, \varphi_d)$. Then for all atoms $a \in \mathcal{A}$ we get that

$$t(a) = f(\varphi_b(a), \varphi_d(a)) = f(a \wedge b, a \wedge d) = f(a \wedge b, 0) = 0,$$

where the third equality holds, because $a \wedge b, a \wedge d \in \mathcal{A}_0$ and $f{\restriction}_{\mathcal{A}_0}$ does not depend on its second variable. Therefore, the operation t belongs to N, and so, $t = \varphi_0 = c_0$

by Proposition 1.2 (d). Thus,

$$f(b,d) = f(\varphi_b(1), \varphi_d(1)) = t(1) = 0.$$

Hence, f is an essentially unary operation contradicting our assumption on f. Therefore, the unary operation $f(\mathrm{id}_L, c_0)$ is in $N \setminus \{c_0\}$. From the facts that $f(c_a, \mathrm{id}_L) \in M$ holds for every element $a \in \mathcal{A}$ and $f\restriction_{\mathcal{A}_0}$ does not depend on its second variable we get that $f(c_a, \mathrm{id}_L) \in C_L$ for all atoms a in $\mathbf{L}$.

Since f depends on its second variable there is an element $l \in L$ such that $f(c_l, \mathrm{id}_L) \in M$ is not a unary constant operation. Then $f(c_l, \mathrm{id}_L) \in N \setminus \{c_0\}$, and there is an atom $a_0 \in \mathcal{A}$ for which $f(c_l, \mathrm{id}_L)(a_0) \neq 0$. Define the following unary operations:

$$\begin{aligned} n &= f(c_l, \mathrm{id}_L), \\ m_1 &= f(\mathrm{id}_L, c_0), \\ m_2 &= f(\mathrm{id}_L, c_{a_0}). \end{aligned}$$

Then $n, m_1, m_2 \in M$, because $f \in \mathrm{Sta}(M)$. In fact, we just proved that $n \in N$. We also have that $m_1, m_2 \in N$ since $m_1(0) = m_2(0) = 0$, and so, there are similar elements $b_i, d_i \in L$ and $\beta_{b_i,d_i} \in \mathrm{Iso}(b_i, d_i)$ such that $m_i = \beta_{b_i,d_i} \circ \varphi_{b_i}$ $(i \in \{1,2\})$. Furthermore,

$$0 \neq f(c_l, \mathrm{id}_L)(a_0) = f(l, a_0) = m_2(l) = \beta_{b_2,d_2}(l \wedge b_2)$$

implies that $l \wedge b_2 > 0$. Then there is an atom $a_1 \in \mathcal{A}$ such that $a_1 \leqslant l \wedge b_2 \leqslant l, b_2$. As $m_1(l) = f(l, 0) = n(0) = 0$ and by Proposition 1.2 (a) we get that

$$m_1(a_1) \leqslant m_1(l) = 0,$$

that is, $m_1(a_1) = 0$. Recalling that $f\restriction_{\mathcal{A}_0}$ does not depend on its second variable and using that for the atom a_1 the inequality $a_1 \leqslant b_2$ holds, we obtain the following series of equalities:

$$0 = m_1(a_1) = f(a_1, 0) = f(a_1, a_0) = m_2(a_1) = \beta_{b_2,d_2}(a_1 \wedge b_2) = \beta_{b_2,d_2}(a_1) \in \mathcal{A},$$

which is a contradiction. The proof of the theorem is complete. □

3. Finite monoidal intervals with more than one elements

Let $\mathbf{L} = (L; \wedge, \vee)$ be a finite lattice that contains exactly one atom. Then there is a largest element $z \in L \setminus \{0\}$ such that

- $[0, z]$ is a chain,
- $L = [0, z] \cup [z, 1]$.

Let $\mathbf{T}$ and $\mathbf{H}$ be the sublattices of $\mathbf{L}$ with universes $T = [0, z]$ and $H = [z, 1]$, respectively (see Figure 2). We note that the set H has either exactly one element or more than three elements, and the former case occurs if and only if $z = 1$. Let N and M be the transformation monoids $\mathrm{IS}(\mathbf{L})$ and $\mathrm{IS}(\mathbf{L}) \cup C_L$, respectively.

Furthermore, let M_T and M_H be the transformation monoids $\mathrm{IS}(\mathbf{T}) \cup C_T$ on T and $\mathrm{IS}(\mathbf{H}) \cup C_H$ on H, respectively. It is easy to see that

$$M_T = \{m{\restriction}_T \mid m \in N \text{ or } m = c_l \ (l \in T)\}\,,$$
$$M_H = \{m{\restriction}_H \mid m \in N,\ m(z) = z, \text{ or } m = c_l \ (l \in H)\}\,.$$

The unique upper cover of 0 and the unique lower cover of z will be denoted by a and s, respectively.

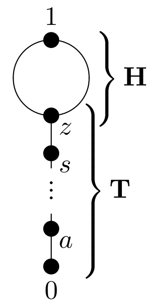

Figure 2: The structure of lattice $\mathbf{L}$.

Define the binary operations $\sqcap_l$ $(l \in L,\ l \neq 0)$ and $\sqcup$ on L in the following way:

$$b \sqcap_l d = \varphi_l(b \wedge d) \quad (b, d \in L),$$
$$b \sqcup d = \varphi_a(b \vee d) \quad (b, d \in L).$$

We remark that the operation $\sqcap_1$ coincides with $\wedge$.

The main results of the article are the following two theorems. The first one deals with the general case leading the statement back to second one, which deals with the case when the lattice is a chain.

Theorem 3.1. *Let* $\mathbf{L}$ *be a finite lattice that contains exactly one atom. Then the monoidal interval* $\mathrm{Int}(\mathrm{IS}(\mathbf{L}) \cup C_L)$ *is isomorphic to* $\mathrm{Int}(\mathrm{IS}(\mathbf{T}) \cup C_T)$*, where* $\mathbf{T}$ *is the sublattice of* $\mathbf{L}$ *with universe* $[0, z]$ *with* z *the largest element in* $L \setminus \{0\}$ *such that* $[0, z]$ *is a chain and* $L = [0, z] \cup [z, 1]$*. Hence, it is isomorphic to the lattice that can be seen in Figure 3.*

Theorem 3.2. *Let* $\mathbf{L}$ *be a finite chain with at least two elements in which* $0_{\mathbf{L}} \prec a$ *and* $s \prec z = 1_{\mathbf{L}}$ *hold, furthermore, let* M *be the transformation monoid* $\mathrm{IS}(\mathbf{L}) \cup C_L$*. Then the monoidal interval* $\mathrm{Int}(M)$ *consists of the clones*

$$\langle M \rangle\,,\ \langle M \cup \{\sqcup\} \rangle\,,\ \langle M \cup \{\sqcap_l\} \rangle\,,\ \textit{and}\ \langle M \cup \{\sqcap_l, \sqcup\} \rangle \quad (l \in L,\ l \neq 0).$$

These clones are pairwise distinct, and the monoidal interval $\mathrm{Int}(M)$ *is the lattice in Figure 3.*

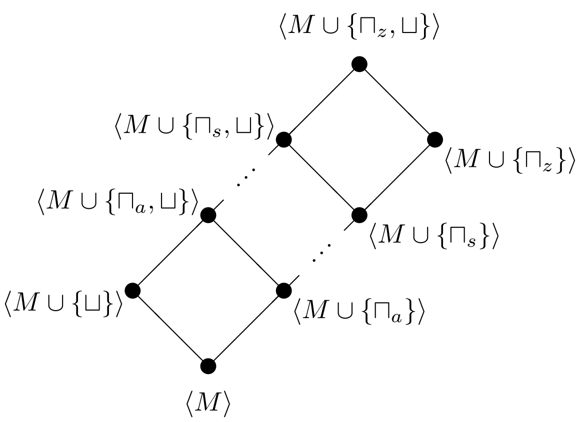

Figure 3: *The monoidal interval* $\mathrm{Int}(M)$.

We continue the section with some basic properties of the operations in $\mathrm{Sta}(M)$.

Proposition 3.3. *Every operation in* $\mathrm{Sta}(M)$ *is monotone with respect to the lattice order.*

Proof. Let f be an ℓ-ary operation in $\mathrm{Sta}(M)$, and let $(b_1, \dots, b_\ell)$, $(d_1, \dots, d_\ell)$ be ℓ-tuples in L^ℓ such that $b_i \leqslant d_i$ holds for every i ($1 \leqslant i \leqslant \ell$). From the monotonicity of the transformations

$$f(\mathrm{id}_L, c_{b_2}, \dots, c_{b_\ell}), \dots, f(c_{d_1}, \dots, c_{d_{i-1}} \mathrm{id}_L, c_{b_{i+1}}, \dots, c_{b_\ell}), \dots, f(c_{d_1}, \dots, c_{d_{\ell-1}}, \mathrm{id}_L)$$

in M we obtain that

$$\begin{aligned} f(b_1, b_2, \dots, b_\ell) &\leqslant f(d_1, b_2, \dots, b_\ell) \\ &\vdots \\ f(d_1, \dots, d_{i-1}, b_i, b_{i+1}, \dots, b_\ell) &\leqslant f(d_1, \dots, d_{i-1}, d_i, b_{i+1}, \dots, b_\ell) \\ &\vdots \\ f(d_1, \dots, d_{\ell-1}, b_\ell) &\leqslant f(d_1, d_2, \dots, d_\ell), \end{aligned}$$

hence, $f(b_1, b_2, \dots, b_\ell) \leqslant f(d_1, d_2, \dots, d_\ell)$ holds by the transitivity of $\leqslant$. $\square$

Let u be an arbitrary element in $T \setminus \{0\}$, and set $U = [0, u]$ and $V = [u, 1]$. Let M_U be the transformation monoid

$$\{m{\restriction_U} \mid m \in N \text{ or } m = c_l \ (l \in U)\}$$

on U.

Proposition 3.4. *Let* $f \in \mathrm{Sta}(M)$ *be an* ℓ*-ary operation* ($\ell \in \mathbb{N}$)*. If* f *is not a constant operation then*

(a) $f(\hat{0}) = 0$,

(b) $f(U^\ell) \subseteq U$,
(c) $f\restriction_U \in \mathrm{Sta}(M_U)$.

Proof. (a) Suppose that $f(\hat{0}) = b \neq 0$. We will prove that f is the constant operation with value b. Let $b_1, \dots, b_\ell$ be arbitrary elements of L, and set $t = f(\varphi_{b_1}, \dots, \varphi_{b_\ell})$. Then

$$t(0) = f\big(\varphi_{b_1}(0), \dots, \varphi_{b_\ell}(0)\big) = f(\hat{0}) = b \neq 0$$

implies that $t = c_b$, hence,

$$f(b_1, \dots, b_\ell) = f\big(\varphi_{b_1}(1), \dots, \varphi_{b_\ell}(1)\big) = t(1) = b.$$

Therefore, the operation f is a constant operation with value b.

(b) Assume f to be a nonconstant operation. Then $f(\hat{0}) = 0$ holds by (a). Let $b_1, \dots, b_\ell$ be arbitrary elements of $U = [0, u]$, and set $t = f(\varphi_{b_1}, \dots, \varphi_{b_\ell})$. Then $t \in N$ since

$$t(0) = f(\varphi_{b_1}(0), \dots, \varphi_{b_\ell}(0)) = f(\hat{0}) = 0,$$

and so, there are simlar elements $b, d \in L$ and $\beta_{b,d} \in \mathrm{Iso}(b, d)$ such that $t = \beta_{b,d} \circ \varphi_b$. Since $b_1, \dots, b_\ell \leqslant u$ we get that

$$t(u) = f(b_1 \wedge u, \dots, b_\ell \wedge u) = f(b_1, \dots, b_\ell) = t(1),$$

hence by Proposition 1.2 (b), $b \leqslant u$ holds. Then $d \leqslant u$ follows from $d \sim b$ and the fact that u is an isolated element, and we get that

$$f(b_1, \dots, b_\ell) = t(1) = d \leqslant u,$$

that is, $f(b_1, \dots, b_\ell) \in U$.

(c) Let $t_1, \dots, t_\ell$ be arbitrary elements of M_U. Then there are transformations $t'_1, \dots, t'_\ell \in M$ such that $t'_i\restriction_U = t_i$ holds for every i $(1 \leqslant i \leqslant \ell)$. Moreover, $f(t'_1, \dots, t'_\ell) \notin \{c_l \mid l \in V,\ l \neq u\}$ follows from (b). Then

$$\begin{aligned} f\restriction_U(t_1, \dots, t_\ell) &= f\restriction_U(t'_1\restriction_U, \dots, t'_\ell\restriction_U) \\ &= f(t'_1, \dots, t'_\ell)\restriction_U \in M_U \end{aligned}$$

implies that $f\restriction_U \in \mathrm{Sta}(M_U)$.

This completes the proof of Proposition 3.4. □

We consider the following subsets of $\mathrm{Sta}(M)$ that have a rôle in the rest of the section. For the subset $U = [0, u]$ of L let $\mathrm{Sta}_{L,U}(M)$ be the set of all operations from $\mathrm{Sta}(M)$ whose ranges are contained in U, and let $\mathrm{Sta}^{[U]}(M) = \langle M \rangle \cup \mathrm{Sta}_{L,U}(M)$. It is clear that $\mathrm{Sta}^{[U]}(M)$ is a clone on L.

For an arbitrary ℓ-ary operation $g \in \mathrm{Sta}(M_U)$ $(\ell \in \mathbb{N})$ let $\mathfrak{f}_g$ be the operation

$$\mathfrak{f}_g \colon L^\ell \to L,\ \mathfrak{f}_g(b_1, \dots, b_\ell) = g(b_1 \wedge u, \dots, b_\ell \wedge u),$$

and for arbitrary operation $f \in \mathrm{Sta}(M)$ let $\mathfrak{g}_f$ be the operation $(\varphi_u \circ f)\restriction_U$. Define maps $\mathfrak{f}$ and $\mathfrak{g}$ as follows

$$\begin{aligned} &\mathfrak{f} \colon \mathrm{Sta}(M_U) \to \mathcal{O}_L,\ g \mapsto \mathfrak{f}_g, \\ &\mathfrak{g} \colon \mathrm{Sta}(M) \to \mathcal{O}_U,\ f \mapsto \mathfrak{g}_f. \end{aligned}$$

Proposition 3.5. *The map $\mathfrak{f}$ preserves superposition, has range in* $\mathrm{Sta}_{L,U}(M)$*, and satisfies $\mathfrak{f}_g\restriction_U = g$ for all $g \in \mathrm{Sta}(M_U)$.*

To prove the proposition we need some more properties of the connection between U and the transformations in M. It is straightforward to check that the following statements are true.

Lemma 3.6. (a) *If $m \in M$ and $m(U) \subseteq U$ then $m(v) \wedge u = m(u) \wedge u$ holds for every element $v \in V$ and $m\restriction_U = \mathfrak{f}_{m\restriction_U}$.*
(b) *If $m \in M$ and $m(U) \not\subseteq U$ then $m \in \{c_v \mid v \in V,\ v \neq u\}$.*
(c) *If m is a transformation on L such that $m\restriction_U \in M_U$ and $m(v) = m(u)$ holds for every element $v \in V$ then $m \in M$.*

Proof of Proposition 3.5. To prove that $\mathfrak{f}$ preserves superposition, choose operations $g_0, g_1, \dots, g_\ell$ be in $\mathrm{Sta}(M_U)$, where g_0 is ℓ-ary and $g_1, \dots, g_\ell$ are k-ary ($\ell, k \in \mathbb{N}$). We note that for arbitrary k-tuples $\boldsymbol{b} = (b_1, \dots, b_k) \in L^k$ we have that

$$\begin{aligned}
\mathfrak{f}_{g_0(g_1,\dots,g_\ell)}(\boldsymbol{b}) &= \big(g_0(g_1, \dots, g_\ell)\big)(b_1 \wedge u, \dots, b_k \wedge u)\\
&= g_0\big(g_1(b_1 \wedge u, \dots, b_k \wedge u), \dots, g_\ell(b_1 \wedge u, \dots, b_k \wedge u)\big)\\
&= g_0\big(g_1(b_1 \wedge u, \dots, b_k \wedge u) \wedge u, \dots, g_\ell(b_1 \wedge u, \dots, b_k \wedge u) \wedge u\big)\\
&= \mathfrak{f}_{g_0}(\mathfrak{f}_{g_1}, \dots, \mathfrak{f}_{g_\ell})(\boldsymbol{b}),
\end{aligned}$$

where the third equality is true, because the ranges of $g_0, g_1, \dots, g_\ell$ are contained in $U = [0, u]$, that is,

$$\mathfrak{f}_{g_0}(\mathfrak{f}_{g_1}, \dots, \mathfrak{f}_{g_\ell}) = \mathfrak{f}_{g_0(g_1,\dots,g_\ell)},$$

which proves the required property of $\mathfrak{f}$.

Let g be an arbitrary ℓ-ary operation in $\mathrm{Sta}(M_U)$ ($\ell \in \mathbb{N}$). It is obvious that $\mathfrak{f}_g\restriction_U = g$, furthermore, $\mathfrak{f}_g(L^\ell) \subseteq g(U^\ell) \subseteq U$, that is, the range of $\mathfrak{f}_g$ is contained in U. Let $m_1, \dots, m_\ell$ be arbitrary transformations in M, and set $m = \mathfrak{f}_g(m_1, \dots, m_\ell)$. For every j ($1 \leqslant j \leqslant \ell$) let m'_j be the transformation m_j if $m_j(U) \subseteq U$ and c_u if $m_j(U) \not\subseteq U$, that is, m_j is a constant operation with value in $V \setminus \{u\}$ by Lemma 3.6 (b). As

$$m'_j(b) \wedge u = \begin{cases} m_j(b) \wedge u & \text{if } m_j(U) \subseteq U,\\ c_u(b) \wedge u = u = m_j(b) \wedge u & \text{if } m_j(U) \not\subseteq U \end{cases}$$

hold for arbitrary elements $j \in \{1, \dots, \ell\}$ and $b \in L$, we obtain that

$$\begin{aligned}
m(b) &= \mathfrak{f}_g\big(m_1(b), \dots, m_\ell(b)\big)\\
&= g\big(m_1(b) \wedge u, \dots, m_\ell(b) \wedge u\big)\\
&= g\big(m'_1(b) \wedge u, \dots, m'_\ell(b) \wedge u\big)\\
&= \mathfrak{f}_g\big(m'_1(b), \dots, m'_\ell(b)\big),
\end{aligned} \tag{1}$$

that is, $m = \mathfrak{f}_g(m'_1, \ldots, m'_\ell)$. Then

$$\begin{aligned} m\restriction_U &= \mathfrak{f}_g(m'_1, \ldots, m'_\ell)\restriction_U = \mathfrak{f}_g(m'_1\restriction_U, \ldots, m'_\ell\restriction_U) \\ &= \mathfrak{f}_g(\mathfrak{f}_{m'_1\restriction_U}, \ldots, \mathfrak{f}_{m'_\ell\restriction_U}) \overset{\text{by (a)}}{=} \mathfrak{f}_{g(m'_1\restriction_U, \ldots, m'_\ell\restriction_U)} \\ &= g(m'_1\restriction_U, \ldots, m'_\ell\restriction_U)\restriction_U, \end{aligned} \tag{2}$$

and for every $v \in V$

$$\begin{aligned} m(v) &= g(m'_1(v) \wedge u, \ldots, m'_\ell(v) \wedge u) \\ &= g(m'_1(u) \wedge u, \ldots, m'_\ell(u) \wedge u) \\ &= m(u), \end{aligned} \tag{3}$$

where in the first and third equalities we used (1), while the second equality follows from Lemma 3.6 (a). Then (2) and (3), via Lemma 3.6 (c), ensure that m is in M, hence $\mathfrak{f}_g$ is in $\mathrm{Sta}(M)$. Since the range of $\mathfrak{f}_g$ is contained in U, we get that the range of $\mathfrak{f}$ is a subset of $\mathrm{Sta}_{L,U}(M)$.

The proof of Proposition 3.5 is complete. □

We will consider $\mathfrak{f}$ as a map $\mathrm{Sta}(M_U) \to \mathrm{Sta}(M)$ or $\mathrm{Sta}(M_U) \to \mathrm{Sta}^{[U]}(M)$ or $\mathrm{Sta}(M_U) \to \mathrm{Sta}_{L,U}(M)$, as the context requires.

Proposition 3.7. *The map $\mathfrak{g}\colon \mathrm{Sta}(M) \to \mathcal{O}_U$, $f \mapsto \mathfrak{g}_f$ preserves superposition, has range in $\mathrm{Sta}(M_U)$, and satisfies the following conditions: $\mathfrak{g}_{\varphi_u \circ f} = \mathfrak{g}_f$ for every $f \in \mathrm{Sta}(M)$, and $\mathfrak{g}_f = f\restriction_U$ for every $f \in \mathrm{Sta}_{L,U}(M)$.*

Proof. First, we prove the second statement of the proposition. To prove it, let $f \in \mathrm{Sta}(M)$ be an arbitrary ℓ-ary operation ($\ell \in \mathbb{N}$) and choose arbitrary transformations $t_1, \ldots, t_\ell \in M_U$. Then there exist transformations $t'_1, \ldots, t'_\ell \in M$ such that $t'_i\restriction_U = t_i$ ($1 \leqslant i \leqslant \ell$). Therefore,

$$\begin{aligned} \mathfrak{g}_f(t_1, \ldots, t_\ell) &= (\varphi_u \circ f)\restriction_U(t'_1\restriction_U, \ldots, t'_\ell\restriction_U) \\ &= \big((\varphi_u \circ f)(t'_1, \ldots, t'_\ell)\big)\restriction_U \\ &= \big(\varphi_u \circ f(t'_1, \ldots, t'_\ell)\big)\restriction_U \in M_U, \end{aligned}$$

since $f(t'_1, \ldots, t'_\ell) \in M$. This proves that $\mathfrak{g}_f \in \mathrm{Sta}(M_U)$, hence, the range of $\mathfrak{g}$ is contained in $\mathrm{Sta}(M_U)$.

For the truth of the first part of the third statement, we note that

$$\mathfrak{g}_{\varphi_u \circ f} = (\varphi_u \circ (\varphi_u \circ f))\restriction_U = (\varphi_u \circ f)\restriction_U = \mathfrak{g}_f$$

for every operation $f \in \mathrm{Sta}(M)$. The second part of the third statement is obvious.

To verify that $\mathfrak{g}$ preserves superposition, let the operations $f_0, f_1, \ldots, f_\ell$ be in $\mathrm{Sta}(M)$, where f_0 is ℓ-ary and $f_1, \ldots, f_\ell$ are k-ary ($\ell, k \in \mathbb{N}$). Since

$$\mathfrak{g}_{f_0(f_1, \ldots, f_\ell)} = \mathfrak{g}_{(\varphi_u \circ f_0)(f_1, \ldots, f_\ell)} \quad \text{and} \quad \mathfrak{g}_{f_0}(\mathfrak{g}_{f_1}, \ldots, \mathfrak{g}_{f_\ell}) = \mathfrak{g}_{\varphi_u \circ f_0}(\mathfrak{g}_{f_1}, \ldots, \mathfrak{g}_{f_\ell}),$$

follows from the equality $\mathfrak{g}_{\varphi_u \circ f} = \mathfrak{g}_f$, we may assume that the range of f_0 is contained in U. Let $\boldsymbol{b} = (b_1, \ldots, b_k)$ be an arbitrary k-tuple in U^k, and set $t = f_0(\varphi_{f_1(\boldsymbol{b})}, \ldots, \varphi_{f_\ell(\boldsymbol{b})})$. Then $t \in M$ and $t(1) = f_0(f_1(\boldsymbol{b}), \ldots, f_\ell(\boldsymbol{b})) \in U$, together with Lemma 3.6 (a), implies that $t(v) \wedge u = t(u) \wedge u$ for every element $v \in V$. Thus, we obtain the following chain of equalities:

$$\begin{aligned}
\mathfrak{g}_{f_0(f_1,\ldots,f_\ell)}(\boldsymbol{b}) &= \big(f_0(f_1,\ldots,f_\ell)\big)(\boldsymbol{b}) \wedge u \\
&= f_0\big(f_1(\boldsymbol{b}),\ldots,f_\ell(\boldsymbol{b})\big) \wedge u \\
&= t(1) \wedge u \\
&= t(u) \wedge u \\
&= f_0\big(u \wedge f_1(\boldsymbol{b}),\ldots,u \wedge f_\ell(\boldsymbol{b})\big) \wedge u \\
&= \mathfrak{g}_{f_0}\big(\mathfrak{g}_{f_1}(\boldsymbol{b}),\ldots,\mathfrak{g}_{f_\ell}(\boldsymbol{b})\big) \\
&= \big(\mathfrak{g}_{f_0}(\mathfrak{g}_{f_1},\ldots,\mathfrak{g}_{f_\ell})\big)(\boldsymbol{b}),
\end{aligned}$$

for all elements $\boldsymbol{b} \in U^\ell$, which proves that $\mathfrak{g}_{f_0}(\mathfrak{g}_{f_1},\ldots,\mathfrak{g}_{f_\ell}) = \mathfrak{g}_{f_0(f_1,\ldots,f_\ell)}$, that is, $\mathfrak{g}$ is a superposition-preserving map.

This concludes the proof of Proposition 3.7. $\square$

We will consider $\mathfrak{g}$ as a map $\mathrm{Sta}(M) \to \mathrm{Sta}(M_U)$. The restriction of $\mathfrak{g}$ to $\mathrm{Sta}_{L,U}(M)$ will be denoted by $\hat{\mathfrak{g}}$.

We note that the sequence of maps $(\mathfrak{g}_0, \mathfrak{g}_1, \ldots)\colon \mathrm{Sta}(M) \to \mathrm{Sta}(M_U)$, where

$$\mathfrak{g}_i\colon \mathrm{Sta}(M)^{(i)} \to \mathrm{Sta}(M_U)^{(i)},\ f \mapsto \mathfrak{g}_f \quad (i \in \mathbb{N}_0),$$

is a homomorphism between the clones $\mathrm{Sta}(M)$ and $\mathrm{Sta}(M_U)$ (as multisorted algebras) since $\mathfrak{g}$ preserves superposition and projections. We will refer to this fact that the map $\mathfrak{g}$ is a clone homomorphism.

Proposition 3.8. *The superposition-preserving maps*

$$\mathfrak{f}\colon \mathrm{Sta}(M_U) \to \mathrm{Sta}_{L,U}(M) \quad \textit{and} \quad \hat{\mathfrak{g}}\colon \mathrm{Sta}_{L,U}(M) \to \mathrm{Sta}(M_U)$$

are mutually inverse bijections.

Proof. Let $f \in \mathrm{Sta}_{L,U}(M)$ be an ℓ-ary operation ($\ell \in \mathbb{N}$). It is straightforward to check that equality $\mathfrak{f}_{\mathfrak{g}_f} = f$ holds if f is a constant operation. If f is not constant then $f(\hat{0}) = 0$, by Proposition 3.4 (a). Let $\boldsymbol{b} = (b_1,\ldots,b_\ell)$ be an arbitrary ℓ-tuple in L^ℓ, and set $t = f(\varphi_{b_1},\ldots,\varphi_{b_\ell})$. Then $t \in M$, furthermore, $t(0) = f(\hat{0}) = 0$ and $t(1) = f(\boldsymbol{b})$ imply that $t = \varphi_{f(\boldsymbol{b})}$. Since $f(\boldsymbol{b}) \leqslant u$ we obtain that $t(1) = t(u)$ holds. From the latter equality we get that

$$\begin{aligned}
f(\boldsymbol{b}) &= t(1) \\
&= t(u) \\
&= f(u \wedge b_1,\ldots,u \wedge b_\ell) \\
&= f(u \wedge b_1,\ldots,u \wedge b_\ell) \wedge u \\
&= \mathfrak{g}_f(b_1 \wedge u,\ldots,b_\ell \wedge u) \\
&= \mathfrak{f}_{\mathfrak{g}_f}(\boldsymbol{b}),
\end{aligned}$$

that is,

$$f = \mathfrak{f}_{\mathfrak{g}_f} = \mathfrak{f}_{\hat{\mathfrak{g}}_f}. \tag{4}$$

Let g be an arbitrary ℓ-ary operation in $\mathrm{Sta}(M_U)$ ($\ell \in \mathbb{N}$). By Proposition 3.5, the range of $\mathfrak{f}_g$ is contained in U. Hence, $\mathfrak{g}_{\mathfrak{f}_g} = (\varphi_u \circ \mathfrak{f}_g)\restriction_U = \mathfrak{f}_g\restriction_U$. Then for every $\boldsymbol{b} = (b_1, \dots, b_\ell) \in U^\ell$ we get that

$$g(\boldsymbol{b}) = g(b_1 \wedge u, \dots, b_\ell \wedge u) = \mathfrak{f}_g(\boldsymbol{b}) = \mathfrak{f}_g\restriction_U(\boldsymbol{b}) = \mathfrak{g}_{\mathfrak{f}_g}(\boldsymbol{b}),$$

that is,

$$g = \mathfrak{g}_{\mathfrak{f}_g} = \hat{\mathfrak{g}}_{\mathfrak{f}_g}. \tag{5}$$

Then (4) and (5) prove the statement of the proposition. □

For an arbitrary clone $\mathcal{D} \in \mathrm{Int}(M_U)$ let $\mathfrak{F}_{\mathcal{D}}$ be the clone $\langle M \cup \{\mathfrak{f}_g \mid g \in \mathcal{D}\}\rangle \subseteq \mathcal{O}_L$, and for an arbitrary clone $\mathcal{C} \in \mathrm{Int}(M)$ let $\mathfrak{G}_{\mathcal{C}}$ be the clone $\mathfrak{g}(\mathcal{C}) = \{\mathfrak{g}_f \mid f \in \mathcal{C}\}$; it is a clone because $\mathfrak{g}$ is a clone homomorphism.

By Proposition 3.5, $\mathfrak{F}_{\mathcal{D}} \subseteq \mathrm{Sta}^{[U]}(M)$, while the inclusion $M \subseteq \mathfrak{F}_{\mathcal{D}}$ is obvious. That is, $\mathfrak{F}_{\mathcal{D}} \in \mathrm{Int}(M)$. Furthermore, for every operation $f \in \mathfrak{F}_{\mathcal{D}} \setminus \langle M\rangle$ the range of f is contained in U.

By Proposition 3.7, $\mathfrak{G}_{\mathcal{C}} \subseteq \mathrm{Sta}(M_U)$, so $\mathfrak{G}_{\mathcal{C}}$ belongs to $\mathrm{Int}(M_U)$.

In the next proposition we summarize these results.

Proposition 3.9. (a) *If $\mathcal{D} \in \mathrm{Int}(M_U)$ then $\mathfrak{F}_{\mathcal{D}} \in \mathrm{Int}(M)$, moreover, the ranges of operations in $\mathfrak{F}_{\mathcal{D}} \setminus \langle M\rangle$ are contained in U.*

(b) *If $\mathcal{C} \in \mathrm{Int}(M)$ then $\mathfrak{G}_{\mathcal{C}} \in \mathrm{Int}(M_U)$.*

Now, we are ready to define maps $\mathfrak{F}$ and $\mathfrak{G}$ in the following way:

$$\mathfrak{F}\colon \mathrm{Int}(M_U) \to [\langle M\rangle, \mathrm{Sta}^{[U]}(M)],\ \mathcal{D} \mapsto \mathfrak{F}_{\mathcal{D}},$$
$$\mathfrak{G}\colon [\langle M\rangle, \mathrm{Sta}^{[U]}(M)] \to \mathrm{Int}(M_U),\ \mathcal{C} \mapsto \mathfrak{G}_{\mathcal{C}}.$$

By Proposition 3.9, these maps are well-defined. The monotonicity of $\mathfrak{F}$ and $\mathfrak{G}$ with respect to set-theoretic inclusion is an immediate consequence of their definitions. Our aim is to prove that maps $\mathfrak{F}$ and $\mathfrak{G}$ are mutually inverse isomorphisms between the lattices $\mathrm{Int}(M_U)$ and $[\langle M\rangle, \mathrm{Sta}^{[U]}(M)]$.

Theorem 3.10. *Let $\mathbf{L}$ be a finite lattice that contains exactly one atom. If $u \in L$ is an element such that $L = [0_{\mathbf{L}}, u] \cup [u, 1_{\mathbf{L}}]$ and $U = [0_{\mathbf{L}}, u]$ is a chain. Then the lattices $\mathrm{Int}(M_U)$ and $[\langle M\rangle, \mathrm{Sta}^{[U]}(M)]$ are isomorphic. In particular, the maps $\mathfrak{F}$ and $\mathfrak{G}$ are mutually inverse lattice isomorphisms.*

Proof. As $\mathfrak{F}$ and $\mathfrak{G}$ are monotone maps with respect to set-theoretic inclusion, if we prove that they are mutually inverse bijections, the assertion follows.

Let $\mathcal{D}$ be an arbitrary clone in $\mathrm{Int}(M_U)$. Then $\mathfrak{G}_{\mathfrak{F}_{\mathcal{D}}} = \{\mathfrak{g}_f \mid f \in \mathfrak{F}_{\mathcal{D}}\}$. For any operation g in $\mathcal{D}$ we have that $g = \mathfrak{g}_{\mathfrak{f}_g}$, by Proposition 3.8, hence, $g \in \mathfrak{G}_{\mathfrak{F}_{\mathcal{D}}}$ since $\mathfrak{f}_g \in \mathfrak{F}_{\mathcal{D}}$. This implies that $\mathcal{D} \subseteq \mathfrak{G}_{\mathfrak{F}_{\mathcal{D}}}$. The reverse inclusion follows from the fact that the set of generators of the clone $\mathfrak{F}_{\mathcal{D}}$, namely, the set $M \cup \{\mathfrak{f}_g \mid g \in \mathcal{D}\}$ is mapped into $\mathcal{D}$ by the clone homomorphism $\mathfrak{g}$, by Proposition 3.8. Then by the preceding argument,

$$\mathfrak{G}_{\mathfrak{F}_{\mathcal{D}}} = \mathcal{D}. \tag{6}$$

Let $\mathcal{C}$ be an arbitrary clone in $[\langle M\rangle, \mathrm{Sta}^{[U]}(M)]$. Then the equality $\mathfrak{f}_{\mathfrak{g}_f} = f$ holds for all $f \in \mathcal{C} \setminus \langle M\rangle$, by Proposition 3.8. Thus, $\mathcal{C} \subseteq \mathfrak{F}_{\mathfrak{G}_{\mathcal{C}}}$. To prove the reverse inclusion it is enough to verify that

$$\{\mathfrak{f}_g \mid g \in \mathfrak{G}_{\mathcal{C}} \setminus \langle M_U\rangle\} \subseteq \mathcal{C}.$$

Let g be an arbitrary element of $\mathfrak{G}_{\mathcal{C}} \setminus \langle M_U\rangle$. Then $g = \mathfrak{g}_f = (\varphi_u \circ f)\restriction_U$ for a suitable operation $f \in \mathcal{C} \setminus \langle M\rangle$, hence,

$$\mathfrak{f}_g = \mathfrak{f}_{\mathfrak{g}_f} = f$$

holds by Proposition 3.8, since by our assumption on $\mathcal{C}$, the range of f is contained in U. This proves that

$$\mathcal{C} = \mathfrak{F}_{\mathfrak{G}_{\mathcal{C}}}. \tag{7}$$

To finish the proof, we note that equalities (6) and (7) imply that the compositions $\mathfrak{F} \circ \mathfrak{G}$ and $\mathfrak{G} \circ \mathfrak{F}$ are the identity automorphisms of the lattices $\mathrm{Sta}(M_U)$ and $[\langle M\rangle, \mathrm{Sta}^{[U]}(M)]$, respectively. Hence, they are bijective maps that are mutually inverses lattice isomorphisms. From this, the assertion of the theorem follows. □

We will use Theorem 3.10 to prove both of the Theorems 3.1 and 3.2.

Before we start the proofs of our main theorems, we recall some definitions from the beginning of the this section. The lattice $\mathbf{L} = (L; \wedge, \vee)$ is finite with at least two elements that contains exactly one atom a. Furthermore, z is the largest element in $\mathbf{L} \setminus \{0\}$ such that $[0, z]$ is a chain and $L = [0, z] \cup [z, 1]$. The element s is the unique under cover of z. The sublattices of $\mathbf{L}$ with universes $T = [0, z]$ and $H = [z, 1]$ are $\mathbf{T}$ and $\mathbf{H}$, respectively. The transformation monoids N and M are $\mathrm{IS}(\mathbf{L})$ and $\mathrm{IS}(\mathbf{L}) \cup C_L$, respectively.

When the lattice L is a chain

In this part, we will assume that $z = 1$ holds in $\mathbf{L}$. Then $\mathbf{L}$ is a chain, H is a 1-element set that contains only 1 and $\mathbf{T} = \mathbf{L}$. Let $\mathbf{S}$ be the sublattice of $\mathbf{L}$ with universe $S = [0, s]$, and let M_S be the transformation monoid $\mathrm{IS}(\mathbf{S}) \cup C_S$ on S. It is straightforward to check that

$$M = \{\varphi_l \mid l \in L\} \cup C_L,$$
$$M_S = \{\varphi_l\restriction_S \mid l \in S\} \cup C_S.$$

Assume that the chain $\mathbf{L}$ has at least three elements, that is, assume that

$$a \leqslant s.$$

In a series of propositions we will examine the operations in $\mathrm{Sta}(M)$. For any positive integer ℓ let $h_\ell^{\mathbf{L}}$ be the mapping

$$h_\ell^{\mathbf{L}} \colon L^\ell \to P(\mathbb{N}),\ (b_1, \dots, b_\ell) \mapsto \{i \in \mathbb{N} \mid 1 \leqslant i \leqslant \ell,\ b_i = 1\},$$

and for the ℓ-ary operation $f \in \mathrm{Sta}(M)$ let $W_f^{\mathbf{L}}$ and $w_f^{\mathbf{L}}$ be defined as follows:

$$W_f^{\mathbf{L}} = \left\{ \boldsymbol{b} \in L^\ell \mid f(\boldsymbol{b}) = 1 \right\},$$

$$w_f^{\mathbf{L}} = \begin{cases} \min \left\{ |h_\ell^{\mathbf{L}}(\boldsymbol{b})| \mid \boldsymbol{b} \in W_f^{\mathbf{L}} \right\} & \text{if } W_f^{\mathbf{L}} \neq \emptyset, \\ \infty & \text{if } W_f^{\mathbf{L}} = \emptyset. \end{cases}$$

It follows from the definition of $w_f^{\mathbf{L}}$ and in (8) by Proposition 3.4 (a), that

$$w_f^{\mathbf{L}} = 0 \text{ if and only if } f \text{ is a constant operation with value } 1, \tag{8}$$

$$w_f^{\mathbf{L}} = \ell \text{ if and only if } W_f^{\mathbf{L}} = \{\hat{1}\}, \tag{9}$$

$$w_f^{\mathbf{L}} = \infty \text{ if and only if } f(L^\ell) \subseteq S. \tag{10}$$

If the lattice $\mathbf{L}$ is clear from the context then we will omit the superscript, that is, we will write h_ℓ, W_f, and w_f instead of $h_\ell^{\mathbf{L}}$, $W_f^{\mathbf{L}}$, and $w_f^{\mathbf{L}}$, respectively.

For a subset I of $\{1, \dots, \ell\}$ and for an element $\boldsymbol{b} = (b_1, \dots, b_\ell) \in L^\ell$ let $\boldsymbol{b}_I$ denote the ℓ-tuple $(b'_1, \dots, b'_\ell)$ for which

$$b'_i = \begin{cases} b_i & \text{if } i \in I, \\ 0 & \text{otherwise} \end{cases}$$

holds for every i $(1 \leqslant i \leqslant \ell)$.

Proposition 3.11. *Let f be an arbitrary ℓ-ary operation in* $\mathrm{Sta}(M)$ $(\ell \in \mathbb{N})$.

(i) *If J is an arbitrary subset of $\{1, \dots, \ell\}$, then for every element $\boldsymbol{b} \in L^\ell$ if $f(\boldsymbol{b}_J) \neq 0$ then $f(\boldsymbol{b}) = f(\boldsymbol{b}_J)$.*

(ii) *If $\boldsymbol{d} \in L^\ell$ is an ℓ-tuple for which $f(\boldsymbol{d}) = 1$ holds and $I = h_\ell(\boldsymbol{d})$, then $f(\boldsymbol{d}_I) = f(\hat{1}) = 1$.*

Proof. Let $\boldsymbol{b} = (b_1, \dots, b_\ell)$ be an arbitrary ℓ-tuple in L^ℓ. For every $i \in \{1, \dots, \ell\}$ let t_i be the transformation c_{b_i} if $i \in J$ and φ_{b_i} otherwise, and set $t = f(t_1, \dots, t_\ell) \in M$. Clearly $t(0) = f(\boldsymbol{b}_J)$ and $t(1) = f(\boldsymbol{b})$. Hence, if $f(\boldsymbol{b}_J) \neq 0$ then $t = c_{f(\boldsymbol{b}_J)}$, and so, $f(\boldsymbol{b}) = t(1) = f(\boldsymbol{b}_J)$. This proves (i).

To prove (ii), let $\boldsymbol{d} = (d_1, \dots, d_\ell) \in L^\ell$ be an ℓ-tuple for which $f(\boldsymbol{d}) = 1$ holds and let $I = h_\ell(\boldsymbol{d})$. By the monotonicity of f, we obtain that $1 = f(\boldsymbol{d}) \leqslant f(\hat{1}) \leqslant 1$, that is, $f(\hat{1}) = 1$. Applying the same construction as in the proof of (i) in the case when $\boldsymbol{b} = \boldsymbol{d}$ and $J = I$, we get that $t(0) = f(\boldsymbol{d}_I)$ and $t(1) = f(\boldsymbol{d}) = 1$. Moreover,

$$t_i(s) = \begin{cases} c_{d_i}(s) = d_i & \text{if } i \in I, \\ \varphi_{d_i}(s) = d_i \wedge s = d_i & \text{if } i \notin I \end{cases}$$

holds for every i $(1 \leqslant i \leqslant \ell)$, since if $i \notin I$ then $d_i \leqslant s$. Hence, $t(s) = f(\boldsymbol{d}) = 1$, which implies that $t = c_1$. Therefore, $f(\boldsymbol{d}_I) = t(0) = 1$. The proof of the proposition is complete. □

Proposition 3.12. *Let f be an arbitrary ℓ-ary operation in* $\mathrm{Sta}(M)$ $(\ell \in \mathbb{N})$. *Suppose that $0 < w_f < \ell$ holds for the operation f, and let $\boldsymbol{d} = (d_1, \dots, d_\ell)$ be an element in W_f such that $|h_\ell(\boldsymbol{d})| = w_f$. Then for arbitrary element $\boldsymbol{b} \in L^\ell$ we have that $\boldsymbol{b} \in W_f$ if and only if $h_\ell(\boldsymbol{d}) \subseteq h_\ell(\boldsymbol{b})$.*

Proof. Suppose that $\boldsymbol{b} = (b_1, \dots, b_\ell) \in W_f$. Set $I = h_\ell(\boldsymbol{d})$ and $J = h_\ell(\boldsymbol{b})$. By Lemma 3.11 (ii), we may assume that $d_i = 0$ if $i \in \{1, \dots, \ell\} \setminus I$ and $b_j = 0$ if $j \in \{1, \dots, \ell\} \setminus J$. Let g be the binary operation $f(g_1, \dots, g_\ell)$ on L, where for $i \in \{1, \dots, \ell\}$ the binary operation g_i is defined to be projection onto the first variable if $i \in I \setminus J$, projection onto the second variable if $i \in J \setminus I$, constant 1 if $i \in I \cap J$, and constant 0 in all the other cases. Then $g \in \mathrm{Sta}(M)$, because $f, g_1, \dots, g_\ell \in \mathrm{Sta}(M)$. Furthermore, $g(1,0) = f(\boldsymbol{d}) = 1$ and $g(0,1) = f(\boldsymbol{b}) = 1$ imply that $g(c_1, \mathrm{id}_L) = g(\mathrm{id}_L, c_1) = c_1$, hence, $g(0,0) = 1$. And so, the ℓ-tuple $(g_1(0,0), \dots, g_\ell(0,0)) \in W_f$ and $h_\ell(g_1(0,0), \dots, g_\ell(0,0)) = I \cap J \subseteq I$. By the minimality of $w_f = |I|$, we get that $I \cap J = I$, that is, $h_\ell(\boldsymbol{d}) = I \subseteq J = h_\ell(\boldsymbol{b})$.

Suppose that $h_\ell(\boldsymbol{d}) \subseteq h_\ell(\boldsymbol{b})$. Then $\boldsymbol{d} = \boldsymbol{d}_I \leqslant \boldsymbol{b}$, hence, the monotonicity of f implies that $1 = f(\boldsymbol{d}) \leqslant f(\boldsymbol{b})$, that is, $\boldsymbol{b} \in W_f$.

This completes the proof of the proposition. □

By the preceding proposition, for every operation $f \in \mathrm{Sta}(M)$, say f is ℓ-ary, there is a unique set in $\{h_\ell(b) \mid b \in W_f\}$ of size w_f.

In the next statement we will describe the essentially binary operations in the stabilizer of M.

Proposition 3.13. *If f is an essentially binary operation in* $\mathrm{Sta}(M)$*, then f coincides with either $\sqcup$ or $\sqcap_l$ for some $l \in L \setminus \{0\}$.*

Proof. Let f be an essentially binary operation in $\mathrm{Sta}(M)$. Then $f(0,0) = 0$ and $f(S \times S) \subseteq S$ hold by Proposition 3.4 (a) and (b), respectively.

The proof splits according to the value of w_f. We note that $w_f = 0$ is impossible because this equality would imply that f is the constant operation with value 1.

In the following we will use some simple facts concerning the unary operations in M. Let m be an arbitrary element in M. Since all the elements of L are isolated, we get that for arbitrary elements $b, d \in L$ the relation $b \sim d$ holds if and only if $b = d$, furthermore, $\mathrm{Iso}(b,d) = \{\mathrm{id}_{(b]}\}$. Then by Proposition 3.4 (b) we have that

($\dagger$) either $m = c_{m(1)}$ or $m = \varphi_{m(1)}$,

in particular,

($\dagger_c$) if $m(1) = 0$ then $m = c_0$, and if $m(0) > 0$ then $m = c_{m(0)}$,
($\dagger_\varphi$) if $m(0) = 0$ then $m = \varphi_{m(1)}$.

Case 1: $w_f = \infty$, that is the range of f is contained in S. Let b^* denote the element $f(1,1) \in L \setminus \{0,1\}$; $b^* \neq 0$, because f is a monotone and essentially binary operation, and $b^* \neq 1$, because of (10). Then the unary operations $f(c_1, \mathrm{id}_L)$ and $f(\mathrm{id}_L, c_1)$ are in $\{c_{b^*}, \varphi_{b^*}\}$ by ($\dagger$), since $f(c_1, \mathrm{id}_L)(1) = f(\mathrm{id}_L, c_1)(1) = f(1,1) = b^*$.

If $f(c_1, \mathrm{id}_L) = c_{b^*}$ then $f(\mathrm{id}_L, c_0) = \varphi_{b^*}$ by ($\dagger_\varphi$), since

$$f(\mathrm{id}_L, c_0)(0) = f(0,0) = 0,$$
$$f(\mathrm{id}_L, c_0)(1) = f(1,0) = f(c_1, \mathrm{id}_L)(0) = b^*,$$

and so,

$$f(c_l, \mathrm{id}_L) = c_{\varphi_{b^*}(l)} \tag{11}$$

for every element $l \in L \setminus \{0\}$ by $(\dagger_c)$, since

$$f(c_l, \mathrm{id}_L)(0) = f(l, 0) = f(\mathrm{id}_L, c_0)(l) = \varphi_{b^*}(l) = l \wedge b^* > 0.$$

If $b^* > a$, where a is the unique atom in the chain $\mathbf{L}$, then for all $l \in L$ we have that

$$f(\mathrm{id}_L, c_l)(a) = f(a, l) = f(c_a, \mathrm{id}_L)(l) \overset{\text{by (11)}}{=} \varphi_{b^*}(a) = a \wedge b^* = a,$$

$$f(\mathrm{id}_L, c_l)(b^*) = f(b^*, l) = f(c_{b^*}, \mathrm{id}_L)(l) \overset{\text{by (11)}}{=} \varphi_{b^*}(b^*) = b^* > a,$$

which show that $f(\mathrm{id}_L, c_l)$ is not a unary constant operation, hence $f(0, l) = f(\mathrm{id}_L, c_l)(0) = 0$ holds for all $l \in L$. Thus, it follows that

$$f(c_0, \mathrm{id}_L) = c_0. \tag{12}$$

Hence, combining (11) and (12), we get that f does not depend on its second variable, which contradicts the assumption on f.

If $b^* = a$ then for all elements $l, l' \in L$ with $l \geqslant a$ we get that

$$f(l, l') = f(c_l, \mathrm{id}_L)(l') \overset{\text{by (11)}}{=} c_{\varphi_a(l)}(l') = \varphi_a(l) = a \wedge l = a. \tag{13}$$

Since $f(c_0, \mathrm{id}_L)(0) = f(0, 0) = 0$ and $f(c_0, \mathrm{id}_L)(1) = f(0, 1) \leqslant f(1, 1) = a$, we see that

$$f(c_0, \mathrm{id}_L) \in \{c_0, \varphi_a\}$$

by $(\dagger_\varphi)$. The equality $f(c_0, \mathrm{id}_L) = c_0$ can be excluded, because in this case (11) and (12) would imply that f is not an essentially binary operation. Hence, $f(c_0, \mathrm{id}_L) = \varphi_a$, and this with (13) show that $f = \sqcup$.

As the assumption $f(\mathrm{id}_L, c_1) = c_{b^*}$ leads to similar results, we may suppose that the equalities

$$f(c_1, \mathrm{id}_L) = f(\mathrm{id}_L, c_1) = \varphi_{b^*}$$

hold for f. Applying these equalities we get that

$$f(c_0, \mathrm{id}_L)(1) = f(0, 1) = f(\mathrm{id}_L, c_1)(0) = \varphi_{b^*}(0) = 0,$$

$$f(\mathrm{id}_L, c_0)(1) = f(1, 0) = f(c_1, \mathrm{id}_L)(0) = \varphi_{b^*}(0) = 0,$$

which imply that

$$f(c_0, \mathrm{id}_L) = f(\mathrm{id}_L, c_0) = c_0,$$

by $(\dagger_c)$. By the above, we have for all elements $l \in L$ that

$$f(c_l, \mathrm{id}_L)(0) = f(l, 0) = f(\mathrm{id}_L, c_0)(l) = c_0(l) = 0,$$

$$f(c_l, \mathrm{id}_L)(1) = f(l, 1) = f(\mathrm{id}_L, c_1)(l) = \varphi_{b^*}(l),$$

hence,

$$f(c_l, \mathrm{id}_L) = \varphi_{\varphi_{b^*}(l)}$$

for every element $l \in L$, by $(\dagger_\varphi)$. Then for arbitrary elements $l, l' \in L$ we obtain that

$$f(l, l') = f(c_l, \mathrm{id}_L)(l') = \varphi_{\varphi_{b^*}(l)}(l') = l' \wedge (l \wedge b^*) = (l' \wedge l) \wedge b^* = l \sqcap_{b^*} l',$$

that is, $f = \sqcap_{b^*} = \sqcap_{f(1,1)}$.

Case 2: $w_f = 1$. Let $\boldsymbol{d} \in W_f$ be an element such that $|h_2(\boldsymbol{d})| = 1$. We may assume, without loss of generality, that $h_2(\boldsymbol{d}) = \{1\}$. Then $f(1,0) = 1$, and so, $f(c_1, \mathrm{id}_L)(0) = f(1,0) = 1$ and $f(\mathrm{id}_L, c_0)(1) = f(1,0) = 1$ imply that $f(c_1, \mathrm{id}_L) = c_1$ and $f(\mathrm{id}_L, c_0) = \varphi_1 = \mathrm{id}_L$ by $(\dagger_c)$ and $(\dagger_\varphi)$, respectively. Hence,

$$f(c_l, \mathrm{id}_L) = c_l \tag{14}$$

for every element $l \in L \setminus \{0\}$ by $(\dagger_c)$, since $f(c_l, \mathrm{id}_L)(0) = f(l,0) = f(\mathrm{id}_L, c_0)(l) = l > 0$. Using this, we get for arbitrary element $l' \in L \setminus \{0\}$

$$f(\mathrm{id}_L, c_l)(l') = f(l', l) = f(c_{l'}, \mathrm{id}_L)(l) = c_{l'}(l) = l',$$

which yields that $f(\mathrm{id}_L, c_l) = \mathrm{id}_L$ holds for all l $(l \in L)$, hence,

$$f(0, l'') = f(\mathrm{id}_L, c_{l''})(0) = \mathrm{id}_L(0) = 0$$

hold for all elements $l'' \in L$. This implies that $f(c_0, \mathrm{id}_L) = c_0$, which with (14) forces f not to depend on its second variable. This contradiction shows that there is no essentially binary operation f with $w_f = 1$.

Case 3: $w_f = 2$. Then $f(c_1, \mathrm{id}_L)(1) = f(1,1) = 1$ and $f(c_1, \mathrm{id}_L)(0) = f(1,0) \neq 1$ imply that $f(c_1, \mathrm{id}_L) = \varphi_1 = \mathrm{id}_L$, by $(\dagger)$. In a similar way, we get that $f(\mathrm{id}_L, c_1) = \mathrm{id}_L$. Therefore, $f(c_0, \mathrm{id}_L) = c_0$ and $f(\mathrm{id}_L, c_0) = c_0$ hold by $(\dagger_c)$, since

$$f(c_0, \mathrm{id}_L)(1) = f(0,1) = f(\mathrm{id}_L, c_1)(0) = 0,$$
$$f(\mathrm{id}_L, c_0)(1) = f(1,0) = f(c_1, \mathrm{id}_L)(0) = 0.$$

Hence, for every element $l \in L$ we have that

$$f(c_l, \mathrm{id}_L)(0) = f(l,0) = f(\mathrm{id}_L, c_0)(l) = c_0(l) = 0,$$
$$f(c_l, \mathrm{id}_L)(1) = f(l,1) = f(\mathrm{id}_L, c_1)(l) = \mathrm{id}_L(l) = l,$$

which yields that $f(c_l, \mathrm{id}_L) = \varphi_l$ $(l \in L)$, by $(\dagger_\varphi)$. Therefore,

$$f(l, l') = f(c_l, \mathrm{id}_L)(l') = \varphi_l(l') = l \wedge l' = l \sqcap_1 l'$$

holds for arbitrary elements $l, l' \in L$, that is, $f = \sqcap_1$.

This concludes the proof of Proposition 3.13. □

Proposition 3.14. *If f is an ℓ-ary operation in* $\mathrm{Sta}(M) \setminus \langle M \rangle$ *with $w_f = \ell$ then $f(b_1, \dots, b_\ell) = b_1 \sqcap_1 \cdots \sqcap_1 b_\ell$ for all $b_1, \dots, b_\ell \in L$.*

Proof. Let us remark that if f is an operation in $\mathrm{Sta}(M) \setminus \langle M \rangle$, then the arity of f is at least 2. We will proceed by induction on the arity of f. Proposition 3.13 ensures that the assertion is true for $\ell = 2$. Assume the statement is true for operations in $\mathrm{Sta}(M) \setminus \langle M \rangle$ with arity less than ℓ, and let $f \in \mathrm{Sta}(M) \setminus \langle M \rangle$ be an ℓ-ary operation $(\ell \geqslant 3)$ such that $w_f = \ell$. Define the operations $f_1, \dots, f_\ell$ in the following way

$$f_i \colon L^{\ell-1} \to L,\ f_i(b_1, \dots, b_{i-1}, b_{i+1}, \dots, b_\ell) = f(b_1, \dots, b_{i-1}, 1, b_{i+1}, \dots, b_\ell)$$

$(1 \leqslant i \leqslant \ell)$. It is obvious that for every i $(1 \leqslant i \leqslant \ell)$

$$f_i(b_1, \dots, b_{i-1}, b_{i+1}, \dots, b_\ell) = 1 \iff b_1 = \cdots = b_{i-1} = b_{i+1} = \cdots = b_\ell = 1$$

holds, which ensures that $w_{f_i} = \ell - 1$ and f_i depends on all of its variables. The operations $f_1, \dots, f_\ell$ belong to $\mathrm{Sta}(M)$ since for every i $(1 \leqslant i \leqslant \ell)$

$$f_i = f(\pi_1^{(\ell-1)}, \dots, \pi_{i-1}^{(\ell-1)}, c_1 \circ \pi_1^{(\ell-1)}, \pi_i^{(\ell-1)}, \dots, \pi_{\ell-1}^{(\ell-1)}) \in \mathrm{Sta}(M),$$

where $\pi_j^{(\ell-1)}\colon L^{\ell-1} \to L$ is the $(\ell-1)$-ary j^{th} projection $(1 \leqslant j \leqslant \ell - 1)$. Hence, $f_1, \dots, f_\ell \in \mathrm{Sta}(M) \setminus \langle M \rangle$ and we can apply the inductive hypothesis to the operations $f_1, \dots, f_\ell$:

$$f_i(b_1, \dots, b_{i-1}, b_{i+1}, \dots, b_\ell) = b_1 \sqcap_1 \cdots \sqcap_1 b_{i-1} \sqcap_1 b_{i+1} \sqcap_1 \cdots \sqcap_1 b_\ell \tag{15}$$

holds for every $(\ell-1)$-tuple $(b_1, \dots, b_{i-1}, b_{i+1}, \dots, b_\ell) \in L^{\ell-1}$ and for every i $(1 \leqslant i \leqslant \ell)$. Let $\boldsymbol{b} = (b_1, \dots, b_\ell)$ be an arbitrary element in L^ℓ, and set $b_0 = b_1 \sqcap_1 \cdots \sqcap_1 b_\ell$. Our aim is to prove that $f(\boldsymbol{b}) = b_0$. Let $j \in \{1, \dots, \ell\}$ be an index such that $b_j = b_1 \vee \cdots \vee b_\ell$ holds in $\mathbf{L}$, and set $t = f(c_{b_1}, \dots, c_{b_{j-1}}, \mathrm{id}_L, c_{b_{j+1}}, \dots, c_{b_\ell}) \in M$. Then

$$\begin{aligned} t(1) &= f(b_1, \dots, b_{j-1}, 1, b_{j+1}, \dots, b_\ell) \\ &= f_j(b_1, \dots, b_{j-1}, b_{j+1}, \dots, b_\ell) \\ &= b_1 \sqcap_1 \cdots \sqcap_1 b_{j-1} \sqcap_1 b_{j+1} \sqcap_1 \cdots \sqcap_1 b_\ell \\ &= b_1 \sqcap_1 \cdots \sqcap_1 b_\ell \\ &= b_0 \end{aligned}$$

implies that t is equal to either c_{b_0} or φ_{b_0}. However,

$$t(b_j) = \begin{cases} b_0 & \text{if } t = c_{b_0}, \\ b_j \wedge b_0 = b_0 & \text{if } t = \varphi_{b_0}, \end{cases}$$

that is, in both cases we get that $t(b_j) = b_0$, hence, $f(\boldsymbol{b}) = t(b_j) = b_0$.

With this the statement of the proposition is proved. □

Proposition 3.15. *If f is an ℓ-ary operation in* $\mathrm{Sta}(M) \setminus \langle M \rangle$ *such that* $0 < w_f \leqslant \ell$ *then* $\langle f \rangle = \langle \sqcap_1 \rangle$.

Proof. Let f be an ℓ-ary operation in $\mathrm{Sta}(M) \setminus \langle M \rangle$ such that $0 < w_f \leqslant \ell$ $(\ell \in \mathbb{N})$. Let $\boldsymbol{d} \in W_f$ be an element for which $|h_\ell(\boldsymbol{d})| = w_f$ holds, and set $I = h_\ell(\boldsymbol{d})$. We may suppose, without loss of generality, that $I = \{1, \dots, w_f\}$. Let g be the operation

$$g\colon L^{w_f} \to L,\ (l_1, \dots, l_{w_f}) \mapsto f(l_1, \dots, l_{w_f}, 0, \dots, 0).$$

Then $w_g = w_f$, by Lemma 3.11 (ii), hence by Lemma 3.14, we have that

$$g(b_1, \dots, b_{w_f}) = b_1 \sqcap_1 \cdots \sqcap_1 b_{w_f}$$

for all $b_1, \dots, b_{w_f} \in L$. By Lemma 3.11 (i), for every element $\boldsymbol{b} = (b_1, \dots, b_\ell) \in L^\ell$ if $f(\boldsymbol{b}_I) \neq 0$ then

$$f(\boldsymbol{b}) = f(\boldsymbol{b}_I) = g(b_1, \dots, b_{w_f}) = b_1 \sqcap_1 \cdots \sqcap_1 b_{w_f}.$$

Now we will prove that the equality $f(\boldsymbol{b}) = f(\boldsymbol{b}_I)$ also holds when $f(\boldsymbol{b}_I) = 0$. Suppose that, on the contrary, there is an ℓ-tuple $\boldsymbol{b} \in L^\ell$ for which $f(\boldsymbol{b}_I) = 0$

and $f(\boldsymbol{b}) \neq 0$ hold. For every element $i \in \{1, \ldots, \ell\}$ let $t_i \in M$ be the following transformation:

$$t_i = \begin{cases} \mathrm{id}_L & \text{if } i \in I \text{ and } b_i = 0, \\ c_{b_i} & \text{otherwise,} \end{cases}$$

and set $t = f(t_1, \ldots, t_\ell)$. Then $t \in M$ and $t(0) = f(\boldsymbol{b}) \neq 0$, which implies that $t = c_{f(\boldsymbol{b})}$ by ($\dagger_c$). Let $b_0 = \sqcap_1 \{b_i \mid i \in I, b_i \neq 0\}$. Since a is the unique atom in the lattice $\mathbf{L}$, we see that $b_0 \geqslant a$. Moreover, if $d \in L \setminus \{0\}$ then $d \geqslant a$ and for every $i \in I$

$$t_i(d) = \begin{cases} \mathrm{id}_L(d) = d \geqslant a & \text{if } b_i = 0, \\ c_{b_i}(d) = b_i \geqslant a & \text{if } b_i \neq 0 \end{cases}$$

holds, and so,

$$\begin{aligned} f\Big(\big(t_1(d), \ldots, t_\ell(d)\big)_I\Big) &= g\big(t_1(d), \ldots, t_{w_f}(d)\big) \\ &= t_1(d) \sqcap_1 \cdots \sqcap_1 t_{w_f}(d) \\ &= d \sqcap_1 b_0 \geqslant a, \end{aligned}$$

where in the third equality we used that d occurs among the elements $t_i(d)$ $(i \in I)$ since $0 \in \{b_i \mid i \in I\}$, which follows from

$$b_1 \sqcap_1 \cdots \sqcap_1 b_{w_f} = g(b_1, \ldots, b_{w_f}) = f(\boldsymbol{b}_I) = 0.$$

Therefore by Lemma 3.11 (i),

$$t(d) = f\big(t_1(d), \ldots, t_\ell(d)\big) = f\Big(\big(t_1(d), \ldots, t_\ell(d)\big)_I\Big) = d \sqcap_1 b_0 \geqslant a$$

for all $d \in L \setminus \{0\}$. Hence, as a consequence of the equalities $t = c_{f(\boldsymbol{b})}$ and

$$\begin{aligned} t(a) &= a \sqcap_1 b_0 = a, \\ t(1) &= 1 \sqcap_1 b_0 = b_0, \end{aligned}$$

we get that $f(\boldsymbol{b}) = b_0 = a$ hold. Furthermore, the equality of b_0 and a implies that the set $J = \{i \in I \mid b_i = a\}$ is not empty. Define transformations t_i' for $i = 1, \ldots, \ell$ in the following way:

$$t_i' = \begin{cases} \mathrm{id}_L & \text{if } i \in I \text{ and } b_i = 0, \\ c_1 & \text{if } i \in J, \\ c_{b_i} & \text{otherwise,} \end{cases}$$

and set $t' = f(t_1', \ldots, t_\ell')$. Then all the transformations $t_1', \ldots, t_\ell', t'$ are in M. In particular, for $i \in I$ we have that

$$t_i'(a) = \begin{cases} a & \text{if } i \in I \text{ and } b_i = 0, \\ 1 & \text{if } i \in J, \\ b_i & \text{if } i \in I \text{ and } b_i > a, \end{cases}$$

$$t_i'(1) = \begin{cases} 1 & \text{if } i \in I \text{ and } b_i = 0, \\ 1 & \text{if } i \in J, \\ b_i & \text{if } i \in I \text{ and } b_i > a, \end{cases}$$

hold. Using again that $0 \in \{b_i \mid i \in I\}$, we obtain that

$$f((t'_1(a), \dots, t'_\ell(a))_I) = g(t'_1(a), \dots, t'_{w_f}(a)) = t'_1(a) \sqcap_1 \cdots \sqcap_1 t'_{w_f}(a) = a,$$
$$f((t'_1(1), \dots, t'_\ell(1))_I) = g(t'_1(1), \dots, t'_{w_f}(1)) = t'_1(1) \sqcap_1 \cdots \sqcap_1 t'_{w_f}(1) > a.$$

Hence, by Lemma 3.11 (i), it follows that

$$t'(a) = f\big(t'_1(a), \dots, t'_\ell(a)\big) = f\Big(\big(t'_1(a), \dots, t'_\ell(a)\big)_I\Big) = a,$$
$$t'(1) = f(t'_1(1), \dots, t'_\ell(1)) = f\Big(\big(t'_1(1), \dots, t'_\ell(1)\big)_I\Big) > a,$$

which means that $t' \in M$ is not a unary constant operation, and so, $t'(0) = 0$ by (†). This is impossible by the assumption on the choice of $\boldsymbol{b}$ for the following reason: since $b_i \leqslant t'_i(0)$ holds for every i ($i \in \{1, \dots, \ell\}$), the monotonicity of f implies that

$$0 \neq f(\boldsymbol{b}) = f(b_1, \dots, b_\ell) \leqslant f(t'_1(0), \dots, t'_\ell(0)) = t'(0) = 0.$$

This contradiction shows that, $f(\boldsymbol{b}) = f(\boldsymbol{b}_I)$ holds for all ℓ-tuples $\boldsymbol{b} \in L^\ell$, that is,

$$f(\boldsymbol{b}) = f(\boldsymbol{b}_I) = g(b_1, \dots, b_{w_f}) = b_1 \sqcap_1 \cdots \sqcap_1 b_{w_f}.$$

Thus, the operation f is in $\langle \sqcap_1 \rangle$. Since $f \notin \langle M \rangle$, it depends on at least two of its variables, which ensures that $w_f \geqslant 2$. Hence, the inclusion $\sqcap_1 \in \langle f \rangle$ also holds. Therefore, $\langle f \rangle = \langle \sqcap_1 \rangle$. □

As a corollary of Proposition 3.15 we obtain the following statement.

Corollary 3.16. *If $f \in \mathrm{Sta}(M) \setminus \langle M \rangle$ is an operation that depends on all of its variables then w_f equals either ∞ or the arity of f.*

Proof of Theorem 3.1. By the results of E. L. Post in [5], the statement is true for $|L| = 2$.

Assume that the theorem is valid for all chains $\mathbf{L}'$ with $2 \leqslant |L'| < |L|$. We proceed to prove the statement for the chain $\mathbf{L}$.

We will apply the inductive hypothesis for the sublattice $\mathbf{L}' = \mathbf{S}$ of $\mathbf{L}$ that was introduced at the beginning of this subsection. The universe of $\mathbf{S}$ is $[0, s]$, where $s \prec 1_{\mathbf{L}}$. In particular, $\mathbf{S}$ is a chain with $|\mathbf{L}| - 1$ elements. The transformation monoid M_S is

$$\mathrm{IS}(\mathbf{S}) \cup C_S = \{\varphi_l\restriction_S \mid l \in S\} \cup C_S.$$

By the inductive hypothesis for the chain $\mathbf{S}$ we obtain that $\mathrm{Int}(M_S)$ is the lattice that can be seen in Figure 4,

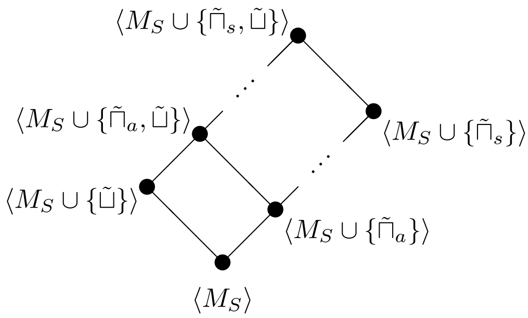

Figure 4: The monoidal interval $\mathrm{Int}(M_S)$.

where $\tilde{\sqcap}_l = \sqcap_l\restriction_S$ $(l \in S,\ l \neq 0)$ and $\tilde{\sqcup} = \sqcup\restriction_S$. By Theorem 3.10, the lattices $\mathrm{Int}(M_S)$ and $[\langle M\rangle, \mathrm{Sta}^{[S]}(M)]$ are isomorphic, the map $\mathfrak{F}\colon \mathrm{Int}(M_U) \to [\langle M\rangle, \mathrm{Sta}^{[S]}(M)]$ is a lattice isomorphism. Hence, $\mathrm{Sta}^{[S]}(M) = \mathfrak{F}_{\langle M_S \cup \{\tilde{\sqcap}_s, \tilde{\sqcup}\}\rangle}$.

Claim 3.17. *Let G be an arbitrary subset of $\mathrm{Sta}(M_S)$. Then*

$$\mathfrak{F}_{\langle G\rangle} = \langle M \cup \{\mathfrak{f}_g \mid g \in G\}\rangle .$$

Then $\langle G\rangle \subseteq \mathrm{Sta}(M_S)$ and $\mathfrak{F}_{\langle G\rangle} = \langle M \cup \{\mathfrak{f}_g \mid g \in \langle G\rangle\}\rangle$, which contains the clone $\langle M \cup \{\mathfrak{f}_g \mid g \in G\}\rangle$. Since $\mathfrak{f}$ is a superposition-preserving map, the set $\{\mathfrak{f}_g \mid g \in \langle G\rangle\}$ is contained in $\langle\{\mathfrak{f}_g \mid g \in G\}\rangle$, hence $M \cup \{\mathfrak{f}_g \mid g \in \langle G\rangle\} \subseteq \langle M \cup \{\mathfrak{f}_g \mid g \in G\}\rangle$, which proves the reverse inclusion $\mathfrak{F}_{\langle G\rangle} \subseteq \langle M \cup \{\mathfrak{f}_g \mid g \in G\}\rangle$. This completes the proof of the claim.

We recall that the operations $\tilde{\sqcap}_l$ $(l \in S,\ b \neq 0)$ and $\tilde{\sqcup}$ were defined by the rule $\tilde{\sqcap}_l = \sqcap_l\restriction_S$ and $\tilde{\sqcup} = \sqcup\restriction_S$. Using the previous claim and the facts that

$$\mathfrak{f}_{\tilde{\sqcap}_s} = \mathfrak{f}_{\sqcap_s\restriction_S} = \sqcap_s,$$
$$\mathfrak{f}_{\tilde{\sqcup}} = \mathfrak{f}_{\sqcup\restriction_S} = \sqcup_s,$$

we get that

$$\mathrm{Sta}^{[S]}(M) = \mathfrak{F}_{\langle M_S \cup \{\tilde{\sqcap}_s, \tilde{\sqcup}\}\rangle} = \langle M \cup \{\sqcap_s, \sqcup\}\rangle .$$

Let $\mathcal{C} \in \mathrm{Int}(M)$ be a clone such that $\mathcal{C} \not\subseteq \mathrm{Sta}^{[S]}(M)$. Choose an operation f from $\mathcal{C} \setminus \mathrm{Sta}^{[S]}(M)$. Then $0 < w_f < \infty$ by (8) and (10), and so, by Lemma 3.15 the operation $\sqcap_1$ belongs to $\mathcal{C}$. Hence, we get the following statement.

Claim 3.18. *Every clone in $\mathrm{Int}(M)$ that is not contained in $\mathrm{Sta}^{[S]}(M)$ contains the operation $\sqcap_1$.*

Claim 3.19. *If $\mathcal{C}$ is a clone in $\mathrm{Int}(M)$, which contains an operation f with $0 < w_f < \infty$ that depends on at least two of its variables, then $\mathcal{C} = \mathfrak{F}_{\mathfrak{G}_\mathcal{C}} \vee \langle\sqcap_1\rangle$. Hence, $\mathcal{C}$ is equal to either $\langle M \cup \{\sqcap_1, \sqcup\}\rangle$ or $\langle M \cup \{\sqcap_1\}\rangle$.*

Let $\mathcal{C}$ be a clone in $\mathrm{Int}(M)$, which contains an operation f with $0 < w_f < \infty$ that depends on at least two of its variables. Then f does not belong to $\langle M\rangle$. Thus, Lemma 3.15 implies that f is in $\langle \sqcap_1\rangle \setminus \langle M\rangle$. Therefore, the operation $\sqcap_1$ is in $\langle f\rangle \subseteq \mathcal{C}$.

Let g be an arbitrary operation in $\mathrm{Sta}(M) \setminus \langle M\rangle$, say g is ℓ-ary. If the range of g is contained is S then $g = \mathfrak{f}_{\mathfrak{g}_g} \in \mathfrak{F}_{\mathfrak{G}_{\mathcal{C}}}$, by Proposition 3.8. Otherwise, if $0 < w_g \leqslant \ell$ then $g \in \langle \sqcap_1\rangle$, by Lemma 3.15. Therefore, $\mathcal{C} \subseteq \mathfrak{F}_{\mathfrak{G}_{\mathcal{C}}} \vee \langle \sqcap_1\rangle$. Since $\sqcap_1 \in \mathcal{C}$ and $\mathfrak{F}_{\mathfrak{G}_{\mathcal{C}}} \subseteq \mathcal{C}$, we obtain that

$$\mathcal{C} = \mathfrak{F}_{\mathfrak{G}_{\mathcal{C}}} \vee \langle \sqcap_1\rangle\,. \tag{16}$$

The clone $\mathfrak{G}_{\mathcal{C}}$ is in $\mathrm{Int}(M_S)$ and it contains the operation $\mathfrak{g}_{\sqcap_1} = \tilde{\sqcap}_1$, hence by the inductive hypothesis, $\mathfrak{G}_{\mathcal{C}}$ is equal to either $\langle M_S \cup \{\tilde{\sqcap}_1, \tilde{\sqcup}\}\rangle$ or $\langle M_S \cup \{\tilde{\sqcap}_1\}\rangle$. Since $\mathfrak{f}_{\tilde{\sqcap}_1} = \sqcap_s$ and $\mathfrak{f}_{\tilde{\sqcup}} = \sqcup$, we have by Claim 3.17 that

$$\begin{aligned} \mathfrak{F}_{\langle M_S \cup \{\tilde{\sqcap}_1, \tilde{\sqcup}\}\rangle} &= \langle M \cup \{\sqcap_s, \sqcup\}\rangle\,, \\ \mathfrak{F}_{\langle M_S \cup \{\tilde{\sqcap}_1\}\rangle} &= \langle M \cup \{\sqcap_s\}\rangle\,. \end{aligned}$$

Hence, by (16), $\mathcal{C}$ coincides with one of the clones

$$\begin{aligned} \langle M \cup \{\sqcap_s, \sqcup\}\rangle \vee \langle \sqcap_1\rangle &= \langle M \cup \{\sqcap_1, \sqcup\}\rangle\,, \\ \langle M \cup \{\sqcap_s\}\rangle \vee \langle \sqcap_1\rangle &= \langle M \cup \{\sqcap_1\}\rangle\,, \end{aligned}$$

as required. This completes the proof of Claim 3.19. □

Now we return to the remaining case.

When the lattice L is not a chain

From now on we assume that $z \neq 1$. We recall that z is the largest element in $L \setminus \{0\}$ such that $[0, z]$ is a chain and $L = [0, z] \cup [z, 1]$, furthermore, $\mathbf{T}$ and $\mathbf{H}$ are the sublattices of $\mathbf{L}$ with universes $T = [0, z]$ and $H = [z, 1]$, respectively. Then M_T and M_H are the transformation monoids

$$\begin{aligned} &\mathrm{IS}(\mathbf{T}) \cup C_T = \{m{\restriction}_T \mid m \in N \text{ or } m = c_b\ (b \in T)\} \text{ and} \\ &\mathrm{IS}(\mathbf{H}) \cup C_H = \{m{\restriction}_H \mid m \in N,\ m(z) = z,\ \text{or } m = c_b\ (b \in H)\}\,, \end{aligned}$$

respectively, where $N = \mathrm{IS}(\mathbf{L})$. Since $z \neq 1$, the lattice $\mathbf{H}$ contains at least two atoms, and so, $|H| \geqslant 4$.

Our aim is to prove that the monoidal interval corresponding to $M = \mathrm{IS}(\mathbf{L})$ is isomorphic to $\mathrm{Int}(M_T)$. To prove this we will use Theorem 2.4 and Theorem 3.1. Before we state and give the proof of Theorem 3.2, in Proposition 3.20 and Proposition 3.21 we will examine the operations in $\mathrm{Sta}(M)$. Recall that, by Proposition 3.4 (b), if $f \in \mathrm{Sta}(M)$ is not a constant operation then $f(\hat{z}) \leqslant z$ holds.

Proposition 3.20. *Let f be an ℓ-ary operation from* $\mathrm{Sta}(M)$ *that is not a constant operation ($\ell \in \mathbb{N}$). If $f(\hat{z}) < z$ then for every ℓ-tuple $(b_1, \ldots, b_\ell) \in L^\ell$ we have that*

(a) $f(b_1, \ldots, b_\ell) < z$,
(b) $f(b_1, \ldots, b_\ell) = f(b_1 \wedge z, \ldots, b_\ell \wedge z)$.

Proof. Suppose that $f(\hat{z}) < z$ holds for f. First we prove that the inequality $f(\boldsymbol{b}) < z$ holds for every element $\boldsymbol{b} \in L^\ell$. Suppose for a contradiction that there is an element $\boldsymbol{b} = (b_1, \dots, b_\ell) \in L^\ell$ such that $f(\boldsymbol{b}) \geqslant z$.

It cannot be that $\mathbf{b} \in T^\ell$, since that would imply that $\boldsymbol{b} \leqslant \hat{z}$, from which it follows that $f(\boldsymbol{b}) \leqslant f(\hat{z}) < z$ holds, by the monotonicity of f. Hence, the ℓ-tuple $\boldsymbol{b}$ belongs to $L^\ell \setminus T^\ell$. Let t be the transformation $t = f(\mathrm{id}_L, \dots, \mathrm{id}_L) \in M$. The operation t cannot be a constant operation, since

$$t(z) = f(\hat{z}) < z \leqslant f(\boldsymbol{b}) \leqslant f(\hat{1}) = t(1)$$

holds, by the assumptions on f and $\boldsymbol{b}$, and by the monotonicity of f. Hence $t \in N \setminus \{c_0\}$, and so, there are similar elements $d, d' \in L$ and $\beta_{d,d'} \in \mathrm{Iso}(d, d')$ such that $t = \beta_{d,d'} \circ \varphi_d$. Then $d' = t(1) \geqslant z$ implies that $d \geqslant z$. However, from these we get that

$$f(\hat{z}) = t(z) = \beta_{d,d'}(z \wedge d) = \beta_{d,d'}(z) = z,$$

since z is an isolated element. This contradicts our assumption on f. Therefore, the inequality $f(\boldsymbol{b}) < z$ holds for every element $\boldsymbol{b} \in L^\ell$. This is what was to be proved in part (a).

To prove (b) let $\boldsymbol{b} = (b_1, \dots, b_\ell)$ be an arbitrary element of L^ℓ, and set $t = f(\varphi_{b_1}, \dots, \varphi_{b_\ell})$. Then $t(0) = f(\hat{0}) = 0$ follows from Proposition 3.4 (a), which implies that $t \in N$, and so, there are similar elements $d, d' \in L$ and $\beta_{d,d'} \in \mathrm{Iso}(d, d')$ such that $t = \beta_{d,d'} \circ \varphi_d$. Since $d \sim d' = t(1) = f(\boldsymbol{b}) < z$ and all the elements in T are isolated, we obtain that $d = d'$ and $\beta_{d,d'} = \mathrm{id}_{(d]}$. Hence $t = \varphi_{f(\boldsymbol{b})}$, and so,

$$f(\boldsymbol{b}) = t(1) = t(z) = f(z \wedge b_1, \dots, z \wedge b_\ell)$$

holds for arbitrary element $\boldsymbol{b} \in L^\ell$. This proves part (b).

With this we finished the proof of the Proposition 3.20. □

Proposition 3.21. *Let f be an ℓ-ary ($\ell \in \mathbb{N}$) operation from* $\mathrm{Sta}(M)$ *for which $f(\hat{z}) = z$ holds. Then*

(a) *$f(H^\ell) \subseteq H$ and $f{\restriction}_H \in \mathrm{Sta}(M_H)$,*

(b) *if f depends on at least two of its variables than for arbitrary elements $d_1, \dots, d_\ell \in L$ we have that*

$$f(d_1, \dots, d_\ell) = f(d_1 \wedge z, \dots, d_\ell \wedge z)$$

holds.

Proof. (a) Let $\boldsymbol{b}$ be an arbitrary element in H^ℓ. Then, by the monotonicity of f, $\hat{z} \leqslant \boldsymbol{b}$ implies that $z = f(\hat{z}) \leqslant f(\boldsymbol{b})$, that is, $f(\boldsymbol{b}) \in H$. This proves the first part of statement (a).

Let t be an arbitrary element of M_H. If $t(z) > z$ then t is constant since z is an isolated element, and $t' = c_{t(z)} \in M$ is an extension of t. Furthermore, if $t(z) = z$ then let t' be the following transformation of L:

$$t' \colon L \to L,\ t'(b) = \begin{cases} t(b) & \text{if } b \in H, \\ b & \text{if } b \in T. \end{cases}$$

Then $t' \in M$, and it is an extension of t.

Let $t_1, \dots, t_\ell$ be arbitrary elements in M_H, and let $t'_1, \dots, t'_\ell$ be their extensions to L, described in the preceding paragraph. Then

$$f{\restriction}_H(t_1, \dots, t_\ell) = f{\restriction}_H(t'_1{\restriction}_H, \dots, t'_\ell{\restriction}_H) = f(t'_1, \dots, t'_\ell){\restriction}_H,$$

$f(t'_1, \dots, t'_\ell) \in M$ and $f(t'_1, \dots, t'_\ell)(z) = f(\hat{z}) = z$ imply that $f(t'_1, \dots, t'_\ell){\restriction}_H \in M_H$. This completes the proof of part (a).

(b) We may assume without loss of generality that f depends on all of its variables. From part (a) we obtain that $f{\restriction}_H \in \mathrm{Sta}(M_H)$, and $f{\restriction}_H$ is an essentially unary operation, by Theorem 2.1, since $\mathbf{H}$ contains at least two atoms. Then there is an index $i_0 \in \{1, \dots, \ell\}$ and a transformation $m \in M_H$ such that

$$f{\restriction}_H(b_1, \dots, b_\ell) = m(b_{i_0}) \tag{17}$$

holds for every ℓ-tuple $(b_1, \dots, b_\ell)$ in H^ℓ. Moreover, it follows from $f(\hat{0}) = 0$ and $f(\hat{z}) = z$ that $m(z) = z$, which ensures that $m \in \mathrm{IS}(\mathbf{H})$.

Furthermore, $f{\restriction}_T$ belongs to $\mathrm{Sta}(M_T)$ by Proposition 3.4 (c). As $\mathbf{T}$ is a chain with largest element z and $f(\hat{z}) = z$ we get that $0 \leqslant w^{\mathbf{T}}_{f\restriction_T} < \infty$.

If $w^{\mathbf{T}}_{f\restriction_T} = 0$ then $f{\restriction}_T$ is the constant operation with value z, and by Proposition 3.4 (a), f is constant. This contradicts our assumption on f, hence, $0 < w^{\mathbf{T}}_{f\restriction_T} \leqslant \ell$. Then Corollary 3.16 implies that

$$f{\restriction}_T(b_1, \dots, b_\ell) = b_1 \sqcap_z \cdots \sqcap_z b_\ell = b_1 \wedge \cdots \wedge b_\ell$$

for all $b_1, \dots, b_\ell \in T$, that is,

$$f(b_1, \dots, b_\ell) = b_1 \wedge \cdots \wedge b_\ell \tag{18}$$

for all $b_1, \dots, b_\ell \in T$. Let $\boldsymbol{d} = (d_1, \dots, d_\ell)$ be an arbitrary element in L^ℓ, furthermore, set $d_0 = d_1 \wedge \cdots \wedge d_\ell$ and $t = f(\varphi_{d_1}, \dots, \varphi_{d_\ell})$. Since $t(0) = f(\hat{0}) = 0$, by Proposition 3.4 (a), the transformation t is in N. Thus, there are similar elements $d, d' \in L$ and $\beta_{d,d'} \in \mathrm{Iso}(d, d')$ such that $t = \beta_{d,d'} \circ \varphi_d$. The argument splits according to whether $d_0 < z$ or $z \leqslant d_0$ holds.

Case 1: $d_0 < z$. For any element $l \in T$ we get that $(l \wedge d_1, \dots, l \wedge d_\ell) \in T^\ell$, hence by (18), $t(l) = f(l \wedge d_1, \dots, l \wedge d_\ell) = (l \wedge d_1) \wedge \cdots \wedge (l \wedge d_\ell) = l \wedge d_0$, that is,

$$t(l) = l \wedge d_0 \tag{19}$$

for all $l \in T$. The assumption $z \leqslant d$ would imply that $t(l) = l$ holds for every element $l \in T$, since all the elements of T are isolated. However, $t(z) = z \wedge d_0 = d_0 < z$ holds in this case by (19). Thus, we get that $d < z$, and so, $d' = d$ and $t = \varphi_d$. Then $d = \varphi_d(d) = t(d) = d_0 \wedge d$ implies that $d \leqslant d_0$, moreover, $d_0 = t(d_0) \leqslant t(1) = d$ follows from the monotonicity of t. Hence, $d = d_0$ and we can produce the following series of equalities in which we use that $t = \varphi_d$ and $t(1) = d = d_0 = t(z)$:

$$f(d_1, \dots, d_\ell) = f\big(\varphi_{d_1}(1), \dots, \varphi_{d_\ell}(1)\big) = t(1) = t(z) = f(d_1 \wedge z, \dots, d_\ell \wedge z).$$

Case 2: $z \leqslant d_0$. Then $z \leqslant d_1, \dots, d_\ell$, and so, $d_1, \dots, d_\ell \in H$ and by (17), we get that

$$f{\restriction}_H(d_1, \dots, d_\ell) = f(d_1, \dots, d_\ell) = m(d_{i_0}) \in H.$$

We will show that $f(\boldsymbol{d}) = z$. Suppose for a contradiction that $z < f(\boldsymbol{d})$. Then by Proposition 1.2 (d), there is an atom $a^\sharp$ in $\mathbf{H}$ such that $m(a^\sharp) > z$. Let n be the unary operation $n = f(\mathrm{id}_L, \dots, \mathrm{id}_L, c_{a^\sharp}, \mathrm{id}_L, \dots, \mathrm{id}_L) \in M$, where $c_{a^\sharp}$ occurs in the i_0^{th} argument of f. Then $n(0) = f(0, \dots, 0, a^\sharp, 0, \dots, 0) = 0 \wedge \cdots \wedge 0 \wedge a^\sharp \wedge 0 \cdots \wedge 0 = 0$ by Case 1, and the following holds by (17):

$$n(z) = f(z, \dots, z, a^\sharp, z, \dots, z) = m(a^\sharp) > z,$$

which is a contradiction, since z is an isolated element in $\mathbf{L}$ and n is not a constant operation. Thus,

$$f(\boldsymbol{d}) = z = f(d_1 \wedge z, \dots, d_\ell \wedge z).$$

This concludes the proof of Proposition 3.21. □

Proof of Theorem 3.2. Let f be an arbitrary operation in $\mathrm{Sta}(M) \setminus \langle M \rangle$. Then f depends on at least two of its variables, and $f(\hat{z}) \leqslant z$ by Proposition 3.4 (b). The range of f is contained in T: if $f(\hat{z}) < z$ then it is a consequence of Proposition 3.20 (a), while if $f(\hat{z}) = z$ then it follows from Proposition 3.21 (b). Hence, the stabilizer $\mathrm{Sta}(M)$ of M coincides with $\mathrm{Sta}^{[T]}(M)$, and so,

$$[\langle M \rangle, \mathrm{Sta}(M)] = [\langle M \rangle, \mathrm{Sta}^{[T]}(M)].$$

By Theorem 3.10, the intervals $[\langle M \rangle, \mathrm{Sta}^{[T]}(M)]$ and $\mathrm{Int}(M_T)$ are isomorphic. As $\mathbf{T}$ is a chain they are isomorphic to the lattice that can be seen in Figure 3.

The statement of the theorem is proved. □

We close the article with some examples.

Example 3.22. *Let* $\mathbf{L}$ *be the* 2*-element lattice on the set* $L = \{0, 1\}$ *with* $0 < 1$. *Then* $\mathrm{IS}(\mathbf{L}) = \{c_0, \mathrm{id}_L\}$ *and* $\mathrm{Int}(\mathrm{IS}(\mathbf{L}))$ *has cardinality* $\aleph_0$*, while for the monoid* $\mathrm{IS}(\mathbf{L}) \cup C_L = \{\mathrm{id}_L, c_0, c_1\}$ *the corresponding monoidal interval is a four-element interval of the clone lattice on L (cf. Post* [5]*).*

Example 3.23. *Let* $\mathbf{L}$ *be the* 3*-element lattice on the set* $L = \{0, a, 1\}$ *with* $0 < a < 1$. *In the interval* $[\mathrm{IS}(\mathbf{L}), \mathrm{IS}(\mathbf{L}) \cup C_L]$ *of the submonoid lattice of* $T(L)$ *there are three monoids:* $\mathrm{IS}(\mathbf{L})$, $\mathrm{IS}(\mathbf{L}) \cup \{c_a\}$, *and* $\mathrm{IS}(\mathbf{L}) \cup C_L$. *The monoidal interval corresponding to* $\mathrm{IS}(\mathbf{L}) = \{c_0, \varphi_a, \mathrm{id}_L\}$ *has continuum many elements (cf. Dormán* [1]*, Theorem 4.2). The monoidal interval* $\mathrm{Int}(\mathrm{IS}(\mathbf{L}) \cup C_L)$ *has six elements (cf. Theorem 3.1). The cardinality of* $\mathrm{IS}(\mathbf{L}) \cup \{c_a\}$ *is unknown.*

Acknowledgements. The author wishes to express his thanks to the Referee for her/his helpful suggestions.

References

[1] Dormán, M., *Collapsing inverse monoids*, Algebra Universalis, **56** (2007), 241-261.
[2] Grabowski, J.-U., *Binary operations suffice to test collapsing of monoidal intervals*, Algebra Universalis, **38** (1997), 92–95.
[3] Krokhin, A., *Monoidal intervals in lattices of clones*, Algebra i Logika **34** (3) (1993), 288-310. (Russian, english translation: Algebra and Logic **34** (3) (1993), 155-168.)
[4] Pálfy, P. P., *Unary polynomials in algebras I*, Algebra Universalis, **18** (1984) 262–273.
[5] Post, E. L., *The two-valued iterative systems of mathematical logic*, Ann. Math. Studies, Princeton University Press, Princeton, N.J., **5** (1941).
[6] Saito, T. and Katsura, M., *Maximal inverse subsemigroups of the full transformation semigroup*, Semigroups with applications, 1992, World Sci. Publishing, River Edge, NJ, Oberwolfach, pp. 101-113.
[7] Szendrei, Á., *Clones in universal algebra*, Séminare de mathématiques supérieures vol. 99, Les presses de l'Université de Montréal, Montréal, 1986.

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