Collapsing monoids consisting of permutations and constants

Miklós Dormán

Abstract. In this paper we determine all collapsing transformation monoids that contain at least one unary constant operation and whose nonconstant operations are permutations. Furthermore, we describe a subclass of transformation monoids that consist of at least three unary constant operations and some permutations for which the corresponding monoidal intervals are 2-element chains.

Introduction

Let A be a finite set with at least two elements. It is well known that for an arbitrary transformation monoid M on the set A the clones whose set of unary operations coincides with M form an interval in the lattice of all clones on A (see Á. Szendrei [9], Chapter 3). An interval of this form is called a monoidal interval. On the set A there are only finitely many transformation monoids, hence the monoidal intervals partition the lattice of clones into finitely many blocks. Since the lattice of clones on A has cardinality $2^{\aleph_0}$ if $|A| \geqslant 3$, one expects that "in most cases" a monoidal interval contains uncountably many clones. However, it turns out that for many transformation monoids the corresponding monoidal intervals are finite. So, studying these intervals may lead us to a better understanding of some parts of the lattice of clones.

In this paper we study the monoidal intervals corresponding to transformation monoids that consist of some permutations and at least one unary constant operation. The most important result for transformation monoids of this kind is the theorem of P. P. Pálfy [5], which states the following: if M is a transformation monoid on a finite set whose cardinality is greater than 3, and M consists of all unary constant operations and some permutations, then the monoidal interval corresponding to M has at most two elements. Furthermore, this interval has exactly two elements if and only if M coincides with the set of all unary polynomial operations of a vector space.

Our main result is a complete description of collapsing transformation monoids that consist of at least one unary constant operation and some permutations (Theorem 1). For a family of transformation monoids that consist of permutations

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and more than two unary constant operations we will show that the correspoding monoidal intervals are 2-element chains (Theorem 3). Furthermore, we will prove that the monoidal interval corresponding to a transformation monoid that contains exactly one unary constant operation and whose nonconstant operations are permutations is infinite (Theorem 2).

Preliminaries

Throughout this paper $\mathbb{N}$ will denote the set of positive natural numbers, and we will assume that A is a finite set. The set of all finitary operations on A will be denoted by $\mathcal{O}_A$. A set $\mathcal{C}$ of finitary operations on A is said to be a **clone** if it contains every projection and it is closed under composition. For a set F of finitary operations on A there is a least clone containing F which will be called the **clone generated by** F and will be denoted by $\langle F \rangle$. The set of all clones on A is a lattice with respect to set theoretic inclusion. This lattice will be denoted by $\mathcal{L}_A$.

Let $\mathcal{C}$ be a clone on A. For a positive integer n, the set of all n-ary operations of the clone $\mathcal{C}$ will be denoted by $\mathcal{C}^{(n)}$. It is easy to see that $\mathcal{C}^{(1)}$ is a transformation monoid. The monoid $\mathcal{C}^{(1)}$ will be called the **unary part of the clone** $\mathcal{C}$.

Let m and n be positive integers. We say that an n-ary operation $f \in \mathcal{O}_A$ **preserves** an m-ary relation $\varrho \subseteq A^m$ if ϱ is a subalgebra of $(A; f)^m$. The set of all operations on A that preserve a relation ϱ will be denoted by $\operatorname{Pol}(\varrho)$. It is easy to see that $\operatorname{Pol}(\varrho)$ is a clone.

Let M be an arbitrary transformation monoid on A. The **stabilizer** of the monoid M is the set

$$\operatorname{Sta}(M) = \big\{ f(x_1, \dots, x_n) \in \mathcal{O}_A \,\big|\, n \in \mathbb{N} \text{ and } f(m_1(x), \dots, m_n(x)) \in M \text{ for all } m_1, \dots, m_n \in M \big\}.$$

We note that the stabilizer of M is a clone on A. Furthermore, the unary part of a clone $\mathcal{C}$ is M if and only if $\langle M \rangle \subseteq \mathcal{C} \subseteq \operatorname{Sta}(M)$. Therefore the clones whose unary part is M form an interval in the lattice of all clones on A. The least and the greatest elements of this interval are the clone $\langle M \rangle$ of essentially unary operations generated by M and the stabilizer $\operatorname{Sta}(M)$ of M, respectively. This interval will be denoted by $\operatorname{Int}(M)$. An interval of this form is called a **monoidal interval**. If the interval $\operatorname{Int}(M)$ has only one element, then the transformation monoid M is called **collapsing**. In this case the only element of $\operatorname{Int}(M)$ is $\langle M \rangle$.

Main Results

The set of all unary constant operations and the set of all permutations on A will be denoted by $C(A)$ and $S(A)$, respectively. For arbitrary element $v \in A$ we will use the notation c_v for the unary constant operation with value v. Throughout the paper, the monoid M is supposed to be contained in $C(A) \cup S(A)$, moreover we will assume that M contains at least one but not all unary constant operations. We note that for collapsing monoids that contain all the unary constant operations and some permutations a complete desription is provided by Pálfy [5], as we mentioned in the introduction. Hence we will also assume that M does not contain all unary constant operations. Let V be the set of all elements $v \in A$ such that $c_v \in M$, and set $W = A \setminus V$. By the assumptions on the monoid M, we have that $\emptyset \subsetneq V, W \subsetneq A$. Define P to be the set of all permutations contained in M. The facts that A is finite and M is closed under composition ensure that P is a permutation group on A and

$$\alpha(V) = V, \qquad \alpha(W) = W \tag{1}$$

hold for all $\alpha \in P$. These equalities allow us to restrict P to V and W, and obtain the permutation groups

$$P_V = \{\alpha|_V : \alpha \in P\} \subseteq S(V),$$
$$P_W = \{\alpha|_W : \alpha \in P\} \subseteq S(W).$$

Furthermore, let $\mathfrak{i}_V$ be the restriction map $\mathfrak{i}_V \colon P \to P|_V,\ \alpha \mapsto \alpha|_V$. If the map $\mathfrak{i}_V$ is injective, then for every transformation $m \in M$ the unique extension of the map $m|_V$ to A is m. Hence, if the map $\mathfrak{i}_V$ is injective, the map

$$\mathfrak{j} \colon P_V \to P_W,\ \alpha|_V \mapsto \alpha|_W.$$

is well-defined.

Our first theorem characterizes all collapsing monoids that consist of permutations and at least one unary constant operation. This extends the results obtained by A. Fearnley and I. Rosenberg in [1].

Theorem 1. *Let A be a finite set with at least two elements, and let M be a transformation monoid on A that consists of at least one unary constant operation and some permutations. Then M is collapsing if and only if*

(i) $|V| \geqslant 2$,
(ii) P_W *is transitive,*
(iii) $\mathfrak{i}_V$ *is injective, and*
(iv) *one of the following conditions holds:*
  (a) *the monoid $M|_V$ is collapsing,*
  (b) *the map $\mathfrak{j}$ is not injective,*
  (c) *the permutation group P_W is not regular.*

Thus, a transformation monoid M that consists of at least one unary constant operation and some permutations is not collapsing, i.e., staisfies $|\mathrm{Int}(M)| \geqslant 2$, iff one of (i)–(iv) fails for M. In some of these cases, namely if (i) fails or if (i)–(iii) hold but (iv) fails, we know more about the monoidal interval $\mathrm{Int}(M)$. Theorem 2 below treats the case when (i) fails, and Theorem 3 the case when (i)–(iii) hold but (iv) fails.

Theorem 2. *Let M be a monoid such that it contains only one unary constant operation and some permutations. Then the monoidal interval* $\mathrm{Int}(M)$ *is infinite.*

Theorem 3. *Let A be a finite set with at least two elements, and let M be a transformation monoid on A that consists of at least two unary constant operations* *and some permutations. If conditions (i)–(iii) of Theorem 1 hold but condition (iv) of Theorem 1 fails for M then* $\operatorname{Int}(M)$ *is isomorphic to* $\operatorname{Int}(M|_V)$*. Hence,*

- *if* $|V| = 2$ *and* $M|_V = \{\mathrm{id}_V, c_0|_V, c_1|_V\}$*, then* $|W| = 1$*,* $M = \{\mathrm{id}_A, c_0, c_1\}$*, and* $\operatorname{Int}(M)$ *is isomorphic to the direct square of the 2-element chain;*
- *if* $|V| = 2$ *and* $M|_V$ *is the full transformation semigroup on* V*, then* $|W| = 2$*,* $M = \{\mathrm{id}_A, c_0, c_1, (0\,1)(2\,3)\}$*, and* $\operatorname{Int}(M)$ *is a 3-element chain;*
- *if* $|V| \geqslant 3$*, then* $\operatorname{Int}(M)$ *is a 2-element chain.*

Proof of Theorem 2

In this section we prove Theorem 2.

Proof of Theorem 2. We may assume without loss of generality that $0 \in A$ and $c_0 \in M$. For every natural number $n \geqslant 4$ define the n-ary operation f_n as follows:

$$f_n(x_1, \ldots, x_n) = \begin{cases} 0 & \text{if } |\{i : x_i = 0\}| \geqslant 2, \\ x_{\min\{j : x_j \neq 0\}} & \text{otherwise.} \end{cases}$$

We will prove that the operations f_n $(n \geqslant 4)$ are in $\operatorname{Sta}(M)$ and the clones $\mathcal{C}_n = \langle \{f_n\} \cup M \rangle$ $(n \geqslant 4)$ are pairwise distinct.

Consider arbitrary transformations $m_1, \ldots, m_n \in M$, and let $m = f_n(m_1, \ldots, m_n)$. To prove that $m \in M$ suppose first that $|\{i : m_i = c_0\}| \geqslant 2$. Then there are indices $1 \leqslant j < k \leqslant n$ such that $m_j = m_k = c_0$. Let a be an arbitrary element of A, and set $\mathbf{a} = (m_1(a), \ldots, m_n(a))$. Then $f_n(\mathbf{a}) = 0$ since the equalities $m_j(a) = 0$ and $m_k(a) = 0$ ensure that more than one component of $\mathbf{a}$ is 0. Hence $m(a) = f_n(\mathbf{a}) = 0$ for every element $a \in A$, proving that $m = c_0 \in M$. It remains to consider the case when $|\{i : m_i = c_0\}| \leqslant 1$. In this case either $m_i \neq c_0$ for all i $(1 \leqslant i \leqslant n)$ or there is exactly one $i \in \{1, \ldots, n\}$ such that $m_i = c_0$. Hence by (1), $|\{i : m_i(a) = 0\}| = n \geqslant 2$ if $a = 0$, and $|\{i : m_i(a) = 0\}| \leqslant 1$ if $a \neq 0$. Then by the definition of f_n we get that for arbitrary element $a \in A$

$$m(a) = f_n(\mathbf{a}) = \begin{cases} 0 & \text{if } a = 0, \\ m_2(a) & \text{if } a \neq 0 \text{ and } m_1 = c_0, \\ m_1(a) & \text{otherwise.} \end{cases}$$

Hence the unary operation

$$f_n(m_1, \ldots, m_n) = m = \begin{cases} m_2 & \text{if } m_1 = c_0, \\ m_1 & \text{otherwise} \end{cases}$$

is in M, proving that the operations f_n $(n \geqslant 4)$ are in $\operatorname{Sta}(M)$.

Now we prove that the clones $\mathcal{C}_n$ $(n \geqslant 4)$ are pairwise distinct. The binary relation $\varrho = \{(0,0)\} \cup (W \times W)$ is a congruence relation of the algebras $\mathbf{A}_n = (A; \mathcal{C}_n)$ $(n \geqslant 4)$. Identify the sets $\{0\}$ and W with 0 and 1, respectively. The meet and join operations with respect to the partial order $0 < 1$ will be denoted by $\wedge$ and $\vee$, respectively. Then the clone of term operations of the quotient algebra $\mathrm{Clo}\,(\mathbf{A}_n/\varrho)$ is

$$\mathrm{Clo}\,(\mathbf{A}_n/\varrho) = \{f/\varrho : f \in \mathcal{C}_n\} = \langle \{f_n/\varrho\} \cup \{m/\varrho : m \in M\}\rangle = \langle f_n/\varrho, c_0/\varrho\rangle\,,$$

since the clone $\mathcal{C}_n$ is generated by the set $\{f_n\} \cup \{m : m \in M\}$ and we have that $\{m/\varrho : m \in M\} = \{\mathrm{id}_A/\varrho, c_0/\varrho\}$. The operation c_0/ϱ is the unary constant operation on $\{0, 1\}$ with value 0, and the operation f_n/ϱ is the following:

$$(f_n/\varrho)(x_1, \ldots, x_n) = \bigvee_{k=1}^{n} (x_1 \wedge \cdots \wedge x_{k-1} \wedge x_{k+1} \wedge \cdots \wedge x_n).$$

Using Post's results in [7], we get that the clones $\mathrm{Clo}\,(\mathbf{A}_n/\varrho) = \langle f_n/\varrho, c_0/\varrho\rangle$ $(n \geqslant 4)$ are pairwise distinct. Hence the clones $\mathcal{C}_n$ $(n \geqslant 4)$ are pairwise distinct as well. This completes the proof of the theorem. □

Proof of Theorems 1 and 3

Lemma 4. *If the permutation group P_W is intransitive, then the monoid M is not collapsing.*

Proof. Assume that the permutation group P_W is intransitive. Then for all elements $w \in W$ we have that $\{\alpha(w) : \alpha \in P\} \subsetneq W$. Consider an arbitrary element $a \in A$. Then

$$\{m(a) : m \in M\} = V \cup \{\alpha(a) : \alpha \in P\} \subsetneq V \cup W = A.$$

Hence, M is not weakly transitive, and by a result of Ihringer–Pöschel [3], the monoid M is not collapsing. □

Lemma 5. *Every operation in* $\mathrm{Sta}(M)$ *can be restricted to* V.

Proof. Let f be an arbitrary n-ary operation in $\mathrm{Sta}(M)$, and choose arbitrary elements $v_1, \ldots, v_n$ in V. The unary operation $m = f(c_{v_1}, \ldots, c_{v_n})$ is constant with value $f(v_1, \ldots, v_n)$, and m belongs to M, since $c_{v_1}, \ldots, c_{v_n} \in M$ and $f \in \mathrm{Sta}(M)$. Thus it follows from the definition of V that $f(v_1, \ldots, v_n) \in V$. □

We will denote the set $\{f|_V : f \in \mathrm{Sta}(M)\}$ of restrictions of operations in $\mathrm{Sta}(M)$ by $\mathrm{Sta}(M)|_V$.

Lemma 6. *Suppose that*

(i) $|V| \geqslant 2$, *and*
(ii) *the permutation group P_W is transitive.*

If the map $\mathfrak{i}_V$ is not injective then M is not collapsing.

Proof. Suppose that the map $\mathfrak{i}_V$ is not injective. We will prove that the monoid M is not collapsing by exhibiting an essentially binary operation in the stabilizer of M. Since $\mathfrak{i}_V$ is not injective, there are permutations $\alpha, \beta \in P$ such that $\alpha \neq \beta$

but $\alpha|_V = \beta|_V$. Choose distinct elements v and v' from V, and define the binary operation f as follows:

$$f(x,y) = \begin{cases} \alpha(x) & \text{if } x \in V, \text{ or } x \in W \text{ and } y \neq v', \\ \beta(x) & \text{if } x \in W \text{ and } y = v'. \end{cases}$$

It follows from this definition that $f(x,v) = \alpha(x)$ for all $x \in A$. Since α is a permutation, there are distinct elements $a_1,\ a_2 \in A$ such that $f(a_1,v) \neq f(a_2,v)$. Furthermore, there is an element $w \in W$ such that $\alpha(w) \neq \beta(w)$, and so $f(w,v) = \alpha(w) \neq \beta(w) = f(w,v')$. These equalities prove that the operation f is essentially binary.

To prove that f is in the stabilizer of M, choose arbitrary transformations $m_1, m_2 \in M$. Assume first that m_1 is a permutation. Then by (1), $m_1(a) \in V$ for every element $a \in V$, and $m_1(b) \in W$ for every element $b \in W$. Since $\alpha|_V = \beta|_V$, we have that $\alpha(m_1(a)) = \beta(m_1(a))$, hence

$$f(m_1(a), m_2(a)) = \alpha(m_1(a)) = \beta(m_1(a)). \tag{2}$$

If m_2 is the unary constant operation with value v', then for every element $b \in W$ we have that $(m_1(b), m_2(b)) \in W \times \{v'\}$, and so, $f(m_1(b), m_2(b)) = \beta(m_1(b))$ holds by the definition of f. Therefore $f(m_1, m_2)$ is the unary operation $\beta \circ m_1 \in M$ by (2). If $m_2 \neq c_{v'}$, then $m_2(b) \neq v'$ holds for every element $b \in W$. Hence, by the definition of f, for all elements $b \in W$ we have that $f(m_1(b), m_2(b)) = \alpha(m_1(b))$. Therefore $f(m_1, m_2) = \alpha \circ m_1 \in M$ holds by (2).

If m_1 is not a permutation, then $m_1 = c_a$ for some element $a \in V$. Then $m_1(x) = a \in V$ for every element $x \in A$. Hence $f(m_1(x), m_2(x)) = \alpha(x)$ for every element $x \in A$, that is, $f(m_1, m_2) = c_{\alpha(a)}$ is in M.

Hence f is an essentially binary operation in the stabilizer of M, which proves that the monoid M is not collapsing. $\square$

The next three lemmas are concerned with the case when (i)–(iii) hold for M.

Lemma 7. *Suppose that*

(i) $|V| \geqslant 2$,
(ii) *the permutation group* P_W *is transitive, and*
(iii) *the map* $\mathfrak{i}_V$ *is injective.*

If $\mathrm{Sta}(M)|_V$ *contains only essentially unary operations, then* M *is collapsing.*

Proof. Let f be an arbitrary n-ary operation in $\mathrm{Sta}(M)$. By Lemma 5, the operation f can be restricted to V, moreover by the assumption, the restriction $f|_V$ of f to V is an essentially unary operation. Hence there is an index $i \in \{1, \dots, n\}$ and there is a unary operation $m \in M$ for which $f(v_1, \dots, v_n) = m(v_i)$ holds for every n-tuple $(v_1, \dots, v_n) \in V^n$. Our aim is to prove that f is essentially unary. To prove this, fix an element $w_0 \in W$, and let $(a_1, \dots, a_n)$ be an arbitrary n-tuple in A^n. Then there are transformations $t_1, \dots, t_n \in M$ such that $t_i(w_0) = a_i$ $(1 \leqslant i \leqslant n)$. Set $t = f(t_1, \dots, t_n)$. Since f is in $\mathrm{Sta}(M)$, the unary operation t is in M. Furthermore,

$$t(w_0) = f(t_1, \dots, t_n)(w_0) = f(t_1(w_0), \dots, t_n(w_0)) = f(a_1, \dots, a_n), \tag{3}$$

and for every element $a \in V$ we have that

$$t(a) = f(t_1, \dots, t_n)(a) = f(t_1(a), \dots, t_n(a)) = m(t_i(a)). \tag{4}$$

The unary operation $m \in M$ is either a permutation or a unary constant operation. If m is a unary constant operation, then there is an elements $v \in V$ such that $m = c_v$. Then (4) implies that $t(a) = v$ for all elements $a \in V$. However, since $|V| \geqslant 2$ this latter fact shows that $t = c_v$, and so by (3), $f(a_1, \dots, a_n) = t(w_0) = v = m(a_i)$. Otherwise, if m is a permutation, then $t|_V = (mt_i)|_V$ by (4), and the injectivity of $\mathfrak{i}_V$ implies that $t = mt_i$. Hence $f(a_1, \dots, a_n) = t(w_0) = (mt_i)(w_0) = m(t_i(w_0)) = m(a_i)$. Therefore, in both cases we get that for arbitrary n-tuple $(a_1, \dots, a_n) \in A^n$ the equality

$$f(a_1, \dots, a_n) = m(a_i)$$

holds, proving that f is essentially unary. Hence M is collapsing, since the stabilizer of M contains only essentially unary operations. This completes the proof. □

If the monoid $M|_V$ is collapsing, then the monoid M is also collapsing, by Lemma 7. Henceforth we will investigate the monoids M for which $M|_V$ is not collapsing.

Lemma 8. *Suppose that*

(i) $|V| \geqslant 2$,
(ii) *the permutation group P_W is transitive,*
(iii) *the map $\mathfrak{i}_V$ is injective, and*
(iv) *the monoid $M|_V$ is not collapsing.*

If the map $\mathfrak{j}\colon P_V \to P_W,\ \alpha|_V \mapsto \alpha|_W$ is not injective or the permutation group P_W is not regular, then M is collapsing.

Proof. To prove the statement, suppose that either the map $\mathfrak{j}$ is not injective or the permutation group P_W is not regular. Then there are permutations $\alpha, \beta \in P$ such that $\alpha|_V \neq \beta|_V$ and $\alpha(w^*) = \beta(w^*)$ for some element $w^* \in W$. By Proposition 7, it is enough to prove that $\mathrm{Sta}(M)|_V$ contains only essentially unary operations.

Suppose that $\mathrm{Sta}(M)|_V$ contains an operation that is not essentially unary. Thus $\mathrm{Sta}(M)|_V$ is a clone on V with unary part $M|_V$ that is different from the essentially unary clone $\langle M|_V \rangle$. If $|V| \geqslant 3$, this implies by the result of Pálfy [5] that the clone $\mathrm{Sta}(M)|_V$ is the set of all polynomial operations of some finite vector space $(V; +, \lambda\cdot\ (\lambda \in K))$ over a finite field K. If $|V| = 2$, say $V = \{0,1\}$, then the assumtion that $\mathfrak{j}$ is not injective or P_W is not regular implies that $|P_V| \neq 1$. Therefore $M|_V$ is the full transformation semigroup on V, and the monoidal interval $\mathrm{Int}(M|_V)$ is the 3-element chain $\langle M|_V \rangle \subsetneq \langle +, c_1 \rangle \subsetneq \mathcal{O}_V$, where $+$ is addition modulo 2. Hence, either $\mathrm{Sta}(M)|_V = \langle +, c_1 \rangle$ or $\mathrm{Sta}(M)|_V = \mathcal{O}_V$. Thus we get that, in all cases, the monoid $M|_V$ is the set of all unary polynomial operations of some finite vector space $(V; +, \lambda\cdot\ (\lambda \in K))$ over a finite field K, moreover, the binary operation $x - y$ is in $\mathrm{Sta}(M)|_V$. Then there are elements $\lambda_1, \lambda_2 \in K$ and $v_1, v_2 \in V$ such that $\alpha|_V = \lambda_1 \cdot x + v_1$ and $\beta|_V = \lambda_2 \cdot x + v_2$, and there is a binary operation $f \in \mathrm{Sta}(M)$

for which $f|_V(x,y) = x - y$. Define the unary transformations t_1 and t_2 to be the transformations $f(\alpha,\alpha)$ and $f(\alpha,\beta)$, respectively. Then for all $v \in V$ we have that

$$\begin{aligned} t_1|_V(v) &= f|_V(\alpha|_V, \beta|_V)(v) \\ &= \alpha(v) - \alpha(v) \\ &= 0, \\ t_2|_V(v) &= f|_V(\alpha|_V, \beta|_V)(v) \\ &= \alpha|_V(v) - \beta|_V(v) \\ &= (\lambda_1 - \lambda_2) \cdot v + (v_1 - v_2). \end{aligned}$$

Hence, $t_1 = c_0$. Suppose that $\lambda_1 \neq \lambda_2$. Then $t_2|_V$ is a permutation, hence t_2 must be a permutation. Since $\alpha(w^*) = \beta(w^*)$ and $w^* \in W$, we get that

$$0 = t_1(w^*) = f(\alpha,\alpha)(w^*) = f(\alpha(w^*),\alpha(w^*)) = f(\alpha(w^*),\beta(w^*)) = t_2(w^*) \in W.$$

This is a contradiction, since $0 \in V$. Thus $\lambda_1 = \lambda_2$. This implies that $v_1 \neq v_2$, since $\alpha|_V \neq \beta|_V$, and that $t_2|_V$ is constant with value $v_1 - v_2$. Hence t_2 is the unary constant operation $c_{v_1 - v_2}$, and we have

$$\begin{aligned} 0 &= t_1(w^*) = f(\alpha,\alpha)(w^*) = f(\alpha(w^*),\alpha(w^*)) \\ &= f(\alpha(w^*),\beta(w^*)) = f(\alpha,\beta)(w^*) = t_2(w^*) = v_1 - v_2. \end{aligned}$$

This contradiction proves that $\mathrm{Sta}(M)|_V$ contains only essentially unary operations. It follows from Lemma 7 that the monoid M is collapsing, and this completes the proof of Lemma 8. □

Lemma 9. *Suppose that*

(i) $|V| \geqslant 2$,

(ii) *the permutation group P_W is transitive,*

(iii) *the map $\mathfrak{i}_V$ is injective, and*

If the map $\mathfrak{j}\colon P_V \to P_W,\ \alpha|_V \mapsto \alpha|_W$ is injective and the permutation group P_W is regular, then the intervals $\mathrm{Int}(M)$ *and* $\mathrm{Int}(M|_V)$ *are isomorphic.*

Proof. We note that the map $\mathfrak{j}$ is well-defined, since $\mathfrak{i}_V$ is injective by the assumption.

Claim 10. *For arbitrary elements $a \in A$ and $w \in W$ there is a unique transformation $m_{w,a}$ in M such that $m_{w,a}(w) = a$. Moreover, if $a \in V$, then $m_{w,a}$ is the unary constant operation c_a, and if $a \in W$, then $m_{w,a}$ is a permutation.*

If $a \in V$, then $m_{w,a}$ must be the unary operation c_a, since for all permutations $m \in P$ we have that $m(w) \in W$ by (1). If $a \in W$ then (1) shows that $m_{w,a}$ must be a permutation, and the regularity of P_W ensures the existence and uniqueness of such permutation. This completes the proof of Claim 10.

For an arbitrary n-ary operation g in $\mathrm{Sta}(M|_V)$ we will define an n-ary operation $\widehat{g}$ on A^n in the following way. Choose and fix an element w_0 in W, and consider arbitrary elements $a_1, \ldots, a_n$ of A. Let $m \in M$ be the unique extension of $g(m_{w_0,a_1}|_V, \ldots, m_{w_0,a_n}|_V) \in M|_V$, and let the value of $\widehat{g}$ on the n-tuple $(a_1, \ldots, a_n)$ be $m(w_0)$.

Claim 11. *The value of $\widehat{g}$ on the n-tuple $(a_1, \ldots, a_n)$ does not depend on the choice of w_0.*

Let w_0' be an arbitrary element of W, and let $m' \in M$ be the unary operation for which

$$m'|_V = g(m_{w_0',a_1}|_V, \ldots, m_{w_0',a_n}|_V).$$

Our goal is to prove that $m'(w_0') = m(w_0)$. By Claim 10 we get that

$$m_{w_0,a_i} = m_{w_0',a_i} m_{w_0,w_0'} \quad (1 \leqslant i \leqslant n),$$

and so, for every element $v \in V$ we get that

$$\begin{aligned}
m|_V(v) &= g(m_{w_0,a_1}|_V, \ldots, m_{w_0,a_n}|_V)(v) \\
&= g(m_{w_0,a_1}|_V(v), \ldots, m_{w_0,a_n}|_V(v)) \\
&= g(m_{w_0,a_1}(v), \ldots, m_{w_0,a_1}(v)) \\
&= g((m_{w_0',a_1} m_{w_0,w_0'})(v), \ldots, (m_{w_0',a_n} m_{w_0,w_0'})(v)) \\
&= g(m_{w_0',a_1}(m_{w_0,w_0'}(v)), \ldots, m_{w_0',a_n}(m_{w_0,w_0'}(v))) \\
&= g(m_{w_0',a_1}|_V(m_{w_0,w_0'}(v)), \ldots, m_{w_0',a_n}|_V(m_{w_0,w_0'}(v))) \\
&= g(m_{w_0',a_1}|_V, \ldots, m_{w_0',a_n}|_V)(m_{w_0,w_0'}(v)) \\
&= m'(m_{w_0,w_0'}(v)) \\
&= (m' m_{w_0,w_0'})(v) \\
&= (m' m_{w_0,w_0'})|_V(v).
\end{aligned}$$

Hence $m|_V = (m' m_{w_0,w_0'})|_V$, and the injectivity of $\mathfrak{i}_V$ implies that $m = m' m_{w_0,w_0'}$. Therefore

$$m'(w_0') = m'(m_{w_0,w_0'}(w_0)) = (m' m_{w_0,w_0'})(w_0) = m(w_0).$$

This completes the proof of Claim 11.

Claim 12. *For arbitrary unary operations $t_1, \ldots, t_n \in M$ and for arbitrary element $w \in W$ we have that*

$$\widehat{g}(t_1(w), \ldots, t_n(w)) = t(w),$$

where $t \in M$ is the unique extension of the unary operation $g(t_1|_V, \ldots, t_n|_V)$.

By Claim 10 we get that

$$t|_V = g(t_1|_V, \ldots, t_n|_V) = g(m_{w,t_1(w)}|_V, \ldots, m_{w,t_n(w)}|_V),$$

and so, $\widehat{g}(t_1(w), \ldots, t_n(w)) = t(w)$ by Claim 11 and by the definition of $\widehat{g}$. This proofs Claim 12.

Claim 13. *The operation $\widehat{g}$ is the unique extension of g in the stabilizer of M.*

First we show that $\widehat{g}$ is an extension of g. Consider arbitrary elements $v_1, \ldots, v_n$ of V. By definition, $\widehat{g}(v_1, \ldots, v_n) = m(w_0)$ where m is the unique extension of $g(m_{w_0,v_1}|_V, \ldots, m_{w_0,v_n}|_V)$. By Claim 10,

$$g(m_{w_0,v_1}|_V, \ldots, m_{w_0,v_n}|_V) = g(c_{v_1}|_V, \ldots, c_{v_n}|_V) = c_{g(v_1,\ldots,v_n)}|_V,$$

and so, (i) and the injectivity of $\mathfrak{i}_V$ imply that $m = c_{g(v_1,\ldots,v_n)}$. Hence

$$\widehat{g}(v_1,\ldots,v_n) = m(w_0) = g(v_1,\ldots,v_n).$$

This proves that $\widehat{g}$ is an extension of g.

Next we prove that $\widehat{g}$ is in the stabilizer of M. Consider arbitrary elements $t_1,\ldots,t_n$ of M, and set $t = \widehat{g}(t_1,\ldots,t_n)$. Our aim is to prove that t belongs to M. By the preceding paragraph, the restriction of t to V is the unary operation

$$t|_V = \widehat{g}(t_1,\ldots,t_n)|_V = \widehat{g}|_V(t_1|_V,\ldots,t_n|_V) = g(t_1|_V,\ldots,t_n|_V) \in M|_V.$$

Let $\widehat{t} \in M$ be the unique extension of $g(t_1|_V,\ldots,t_n|_V) \in M|_V$ to A. Then $\widehat{t}|_V = t|_V$, and for arbitrary element $w \in W$ we have that

$$t(w) = \widehat{g}(t_1,\ldots,t_n)(w) = \widehat{g}(t_1(w),\ldots,t_n(w)) = \widehat{t}(w),$$

where the first equality follows from the definition of $\widehat{g}$ and the last equality from Claim 12. Since $t|_V = \widehat{t}|_V$, this proves that $t = \widehat{t} \in M$. Hence the operation $\widehat{g}$ is in $\mathrm{Sta}(M)$.

Finally, we show that there are no other extensions of g in $\mathrm{Sta}(M)$. Assume that $\widetilde{g}$ is an extension of g in the stabilizer of M. Then for every n-tuple $(v_1,\ldots,v_n) \in V^n$ we have that $\widehat{g}(v_1,\ldots,v_n) = \widetilde{g}(v_1,\ldots,v_n) = g(v_1,\ldots,v_n)$. Consider arbitrary n-tuple $(a_1,\ldots,a_n) \in A^n$, and let $m \in M$ be the unique extension of $g(m_{w_0,a_1}|_V,\ldots,m_{w_0,a_n}|_V)$. Then

$$\begin{aligned} m|_V &= g(m_{w_0,a_1}|_V,\ldots,m_{w_0,a_n}|_V) \\ &= \widetilde{g}|_V(m_{w_0,a_1}|_V,\ldots,m_{w_0,a_n}|_V) \\ &= \widetilde{g}(m_{w_0,a_1},\ldots,m_{w_0,a_n})|_V. \end{aligned}$$

Since $\widetilde{g} \in \mathrm{Sta}(M)$, the unary operation $\widetilde{g}(m_{w_0,a_1},\ldots,m_{w_0,a_n})$ is in M, and the injectivity of $\mathfrak{i}_V$ implies that $\widetilde{g}(m_{w_0,a_1},\ldots,m_{w_0,a_n}) = m$. Furthermore, we get that

$$\begin{aligned} \widetilde{g}(a_1,\ldots,a_n) &= \widetilde{g}(m_{w_0,a_1}(w_0),\ldots,m_{w_0,a_n}(w_0)) \\ &= \widetilde{g}(m_{w_0,a_1},\ldots,m_{w_0,a_n})(w_0) \\ &= m(w_0) \\ &= \widehat{g}(a_1,\ldots,a_n). \end{aligned}$$

Thus $\widetilde{g}(a_1,\ldots,a_n) = \widehat{g}(a_1,\ldots,a_n)$ holds for arbitrary n-tuples $(a_1,\ldots,a_n) \in A^n$, and so $\widetilde{g} = \widehat{g}$. This proves Claim 13.

Claim 14. *For an arbitrary clone $\mathcal{C} \in \mathrm{Int}(M|_V)$ the set $\widehat{\mathcal{C}} = \{\widehat{g} : g \in \mathcal{C}\}$ is a clone, which belongs to the monoidal interval $\mathrm{Int}(M)$.*

First we note that for every operation $f \in \mathrm{Sta}(M)$ the equality

$$\widehat{f|_V} = f \tag{5}$$

holds by Claim 13. Let $f \in \mathrm{Sta}(M)$ be an arbitrary projection on the set A, say f is the n-ary i-th projection. Then $f|_V$ is the n-ary i-th projection on V, hence $f|_V \in \mathcal{C}$. Furthermore by (5), $f = \widehat{f|_V} \in \widehat{\mathcal{C}}$. This proves that the set $\widehat{\mathcal{C}}$ contains all the projections.

Let $f \in \widehat{\mathcal{C}}$ be an arbitrary k-ary operation, and let $f_1, \dots, f_k \in \widehat{\mathcal{C}}$ be arbitrary n-ary operations. Then there are operations $g \in \mathcal{C}^{(k)}$ and $g_1, \dots, g_k \in \mathcal{C}^{(n)}$ such that $f = \widehat{g}$ and $f_i = \widehat{g_i}$ $(1 \leqslant i \leqslant k)$. Our aim is to prove that the operation $\widehat{h}$ is in $\widehat{\mathcal{C}}$. To prove this consider arbitrary n-tuple $(a_1, \dots, a_n) \in A^n$, set $h = g(g_1, \dots, g_k)$, and let $t, t_1, \dots, t_k \in M$ be the unique unary transformations for which

$$\begin{aligned} t|_V &= h(m_{w_0,a_1}|_V, \dots, m_{w_0,a_n}|_V), \\ t_i|_V &= g_i(m_{w_0,a_1}|_V, \dots, m_{w_0,a_n}|_V) \quad (1 \leqslant i \leqslant k). \end{aligned}$$

Then by the definition of $\widehat{h}, \widehat{g_i}$ $(1 \leqslant i \leqslant k)$ we have that $\widehat{h}(a_1, \dots, a_n) = t(w_0)$ and $\widehat{g_i}(a_1, \dots, a_n) = t_i(w_0)$. Since

$$\begin{aligned} g(t_1|_V, \dots, t_k|_V) &= g(g_1(m_{w_0,a_1}|_V, \dots, m_{w_0,a_n}|_V), \dots, g_k(m_{w_0,a_1}|_V, \dots, m_{w_0,a_n}|_V)) \\ &= (g(g_1, \dots, g_k))(m_{w_0,a_1}|_V, \dots, m_{w_0,a_n}|_V) \\ &= h(m_{w_0,a_1}|_V, \dots, m_{w_0,a_n}|_V) \\ &= t|_V, \end{aligned}$$

we get that $\widehat{g}(t_1(w_0), \dots, t_p(w_0)) = t(w_0)$ by Claim 12, and so, by the definition of $\widehat{h}$,

$$\begin{aligned} \widehat{g}(\widehat{g_1}, \dots, \widehat{g_k})(a_1, \dots, a_n) &= \widehat{g}(\widehat{g_1}(a_1, \dots, a_n), \dots, \widehat{g_k}(a_1, \dots, a_n)) \\ &= \widehat{g}(t_1(w_0), \dots, t_k(w_0)) \\ &= t(w_0) \\ &= \widehat{h}(a_1, \dots, a_n). \end{aligned}$$

Therefore $\widehat{h} = \widehat{g}(\widehat{g_1}, \dots, \widehat{g_k})$, which proves that the operation $\widehat{h}$ is in $\widehat{\mathcal{C}}$ since $h = g(g_1, \dots, g_k) \in \mathcal{C}$. These show that the set $\widehat{C}$ is a clone. It is remaining to prove that the unary part of $\widehat{\mathcal{C}}$ is M. As $\widehat{\mathcal{C}}^{(1)} = \left\{\widehat{m} : m \in \mathcal{C}^{(1)}\right\} = \{\widehat{m} : m \in M|_V\}$, the injectivity of $\mathfrak{i}_V$ and equation (5) ensure that $\widehat{\mathcal{C}}^{(1)} = M$. This completes the proof of Claim 14.

Define the map Φ as follows:

$$\Phi \colon \mathrm{Int}(M|_V) \to \mathrm{Int}(M),\ \mathcal{C} \mapsto \widehat{\mathcal{C}}.$$

We will prove that Φ is an isomorphism between the lattices $\mathrm{Int}(M|_V)$ and $\mathrm{Int}(M)$. Let $\mathcal{C}$ and $\mathcal{D}$ be arbitrary clones in $\mathrm{Int}(M|_V)$. Suppose that $\mathcal{C} \neq \mathcal{D}$. Then we may assume, without loss of generality, that there is an operation g in $\mathcal{C} \setminus \mathcal{D}$. Then $\widehat{g} \in \Phi(\mathcal{C})$, and $\widehat{g} \notin \Phi(\mathcal{D})$ since otherwise $g = \widehat{g}|_V \in \mathcal{D}$ would hold. Hence $\widehat{g} \in \Phi(\mathcal{C}) \setminus \Phi(\mathcal{D})$, and so, $\Phi(\mathcal{C}) \neq \Phi(\mathcal{D})$. This proves that Φ is injective. Let $\mathcal{F}$ be an arbitrary clone in $\mathrm{Int}(M)$. Then $\mathcal{F}|_V = \{f|_V : f \in \mathcal{F}\}$ is a clone in $\mathrm{Int}(M|_V)$ for which

$$\Phi(\mathcal{F}|_V) = \left\{\widehat{f|_V} : f \in \mathcal{F}\right\} = \mathcal{F}$$

by (5). Hence, Φ is surjective, and so, it is bijective. Furthermore, the inverse map of Φ is

$$\Phi^{-1} \colon \mathrm{Int}(M) \to \mathrm{Int}(M|_V),\ \mathcal{F} \mapsto \mathcal{F}|_V.$$

Let $\mathcal{C}_1, \mathcal{C}_2 \in \mathrm{Int}(M|_V)$ and $\mathcal{F}_1, \mathcal{F}_2 \in \mathrm{Int}(M)$ be arbitrary clones such that $\mathcal{C}_1 \subseteq \mathcal{C}_2$ and $\mathcal{F}_1 \subseteq \mathcal{F}_2$. Then

$$\Phi(\mathcal{C}_1) = \{\widehat{g} : g \in \mathcal{C}_1\} \subseteq \{\widehat{g} : g \in \mathcal{C}_2\} = \Phi(\mathcal{C}_2)$$

and

$$\Phi^{-1}(\mathcal{F}_1) = \{f|_V : f \in \mathcal{F}_1\} \subseteq \{f|_V : f \in \mathcal{F}_2\} = \Phi^{-1}(\mathcal{F}_2)$$

prove that the map Φ is an isomorphism. This completes the proof of the statement and hence the proof of Lemma 9. □

Proof of Theorem 1. Suppose that the monoid M is collapsing. Then by Therorem 2, the number of unary constant operations in M is greater than 1. Hence $|V| \geqslant 2$, and (i) holds. Lemma 4 shows that the permutation group P_W must be transitive, while the injectivity of the map $\mathfrak{i}_V$ follows from Lemma 6. These prove that M has the properties (ii) and (iii). If neither one of the conditions (b) and (c) of (iv) holds for M, then the intervals $\mathrm{Int}(M|_V)$ and $\mathrm{Int}(M)$ are isomorphic by Lemma 9, and so, the monoid $M|_V$ is collapsing, since M is. Then condition (a) of (iv) holds for M. This shows that (iv) also holds for the monoid M.

Suppose that conditions (i)–(iv) hold for the monoid M. The assumption that (iv) holds for M means that either $M|_V$ is collapsing or $\mathfrak{j}$ is not injective or P_W is not regular. If $M|_V$ is collapsing, then $\mathrm{Sta}(M)|_V \subseteq \mathrm{Sta}(M|_V)$ contains only essentially unary operations, and therefore M is collapsing by Lemma 7. If $M|_V$ is not collapsing then (iv) implies that either $\mathfrak{j}$ is not injective or P_W is not regular. Therefore, M is collapsing by Lemma 8.

This completes the proof of Theorem 1. □

Proof of Theorem 3. If conditions (i)–(iii) of Theorem 1 hold but condition (iv) of Theorem 1 fails for M then $\mathrm{Int}(M)$ is isomorphic to $\mathrm{Int}(M|_V)$ by Lemma 9. If $|V| \geqslant 3$ then by the result of Pálfy [5], the monoidal interval $\mathrm{Int}(M|_V)$ is a 2-element chain, and so, $\mathrm{Int}(M)$ is a 2-element chain, as well. To finish the proof we note that if $|V| = 2$, say $V = \{0, 1\}$, then $M|_V$ is either the monoid $\{c_0|_V, c_1|_V, \mathrm{id}_V\}$ and $\mathrm{Int}(M|_V)$ is isomorphic to he direct square of the 2-element chain or $M|_V$ is the full transformation semigroup and $\mathrm{Int}(M|_V)$ is isomorphic to the 3-element chain (cf. Post [7] and Szendrei [9]). Therefore, in the former case $\mathrm{Int}(M)$ is isomorphic to he direct square of the 2-element chain, and in the latter case $\mathrm{Int}(M)$ is isomorphic to the 3-element chain. The proof of Theorem 3 is complete. □

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Bolyai Institute, Aradi vértanúk tere 1, H-6720 Szeged, Hungary
E-mail address: dorman@math.u-szeged.hu