Collapsing inverse monoids

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Abstract. In this paper we investigate a class of inverse transformation monoids constructed from finite lattices, and we describe a necessary and sufficient condition for such a transformation monoid to be collapsing.

1. Introduction

Let A be a finite set with at least three elements. It is well known that for an arbitrary transformation monoid M on the set A the clones whose set of unary operations coincides with M form an interval in the lattice of all clones on A (see Á. Szendrei [11], Chapter 3). An interval of this form is called a monoidal interval. On the set A there are only finitely many transformation monoids, hence the monoidal intervals partition the lattice of clones into finitely many blocks. Since the lattice of clones on A has cardinality $2^{\aleph_0}$ if $|A| \geqslant 3$, one expects that "in most cases" a monoidal interval contains uncountably many clones. However, it turns out that for many transformation monoids the corresponding monoidal intervals are finite. So, studying these intervals may lead us to a better understanding of some parts of the lattice of clones.

The problem of classifying transformation monoids according to the cardinalities of the corresponding monoidal intervals was posed by Á. Szendrei in [11]. A large family of monoids M with finite monoidal intervals is provided by Pálfy's theorem in [7]: if M consists of all constants and some permutations, then the corresponding monoidal interval contains at most two elements; moreover, this interval has a single element unless M coincides with the monoid of all unary polynomial operations of a finite vector space. The full transformation semigroup on A is an example of a monoid M such that the monoidal interval is finite with more than two elements; in fact, in this case this interval is an $(|A| + 1)$-element chain (cf. Burle [1]). The monoidal interval corresponding to the one-element transformation monoid has cardinality $2^{\aleph_0}$ (cf. Marčenkov [6]). The first explicit construction of a transformation monoid M with a countably infinite monoidal interval is due to Krokhin in [5].

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A complete classification of transformation monoids according to the sizes of the corresponding monoidal intervals seems out of reach at present. Most attention has been given to finding the collapsing monoids, that is, the monoids that have one-element monoidal intervals. As we mentioned in the preceding paragraph, almost all transformation monoids that arise from a permutation group by adding all constants are collapsing. For permutation groups without constants the results known so far indicate that 'large' permutation groups, e.g. all primitive permutation groups, are collapsing (cf. Pálfy–Szendrei [8] and Kearnes–Szendrei [3]). This motivated us in extending the investigation of collapsing monoids to 'large' inverse monoids.

In this paper we investigate the monoidal intervals corresponding to a class of inverse transformation monoids constructed from finite lattices. These inverse monoids arise from finite lattices by applying the construction introduced by Saito–Katsura [10] to describe maximal inverse transformation monoids. In Section 3 we describe a necessary and sufficient condition for an inverse monoid constructed from a finite lattice to be collapsing. In Section 4 we present some examples of maximal inverse monoids for which the corresponding monoidal intervals are large.

2. The inverse monoid $\mathrm{IS}(\mathbf{L})$

Throughout this paper we will assume that the base set A is finite. The set of all finitary operations on A will be denoted by $\mathcal{O}_A$. Let $\mathcal{C}$ be a clone on A. For a positive integer n, the set of all n-ary operations of the clone $\mathcal{C}$ will be denoted by $\mathcal{C}^{(n)}$. It is easy to see that the unary operations in $\mathcal{C}$ form a transformation monoid. This monoid will be called the **unary part of the clone**. For a set F of finitary operations on A there is a least clone containing F which will be called the **clone generated by** F and will be denoted by $\langle F\rangle$.

Let m and n be positive integers. We say that an n-ary operation $f \in \mathcal{O}_A$ **preserves** an m-ary relation $\rho \subseteq A^m$ if ρ is a subalgebra of $(A; f)^m$. The set of all operations which preserve a relation ρ will be denoted by $\mathrm{Pol}(\rho)$. It is easy to see that $\mathrm{Pol}(\rho) \subseteq \mathcal{O}_A$ is a clone.

Let M be an arbitrary transformation monoid on A. The **stabilizer** of the monoid M is the set

$$\begin{aligned}\mathrm{Sta}(M) = \big\{ f(x_1, \dots, x_n) \in \mathcal{O}_A \,\big|\, & n \in \mathbb{N}, \text{ and} \\ & f(m_1(x), \dots, m_n(x)) \in M \text{ for all } m_1, \dots, m_n \in M \big\}.\end{aligned}$$

We note that the stabilizer of M is a clone on A, in fact, $\mathrm{Sta}(M)$ is the clone $\mathrm{Pol}(\varrho_M)$, where

$$\varrho_M = \big\{ (m(a_1), \dots, m(a_k)) \,\big|\, m \in M \big\},$$

$k = |A|$, and $(a_1, \dots, a_k)$ is a fixed k-tuple with pairwise different components. Furthermore, the unary part of a clone $\mathcal{C}$ is M if and only if $\langle M\rangle \subseteq \mathcal{C} \subseteq \mathrm{Pol}(\varrho_M) = \mathrm{Sta}(M)$ (cf. Pálfy–Szendrei [8], Proposition 1). Therefore the clones whose unary part is M form an interval in the lattice of all clones on A. The least and the greatest elements of this interval are the clone $\langle M\rangle$ of essentially unary operations generated by M and the clone $\mathrm{Sta}(M)$, respectively. This interval will be denoted by $\mathrm{Int}(M)$. An interval of this form is called a **monoidal interval**. Hence, the monoidal interval $\mathrm{Int}(M)$ is the interval $[\langle M\rangle, \mathrm{Sta}(M)]$ in the lattice of clones on A. If the interval $\mathrm{Int}(M)$ has only one element, then the transformation monoid M is called **collapsing**. In this case the only element of $\mathrm{Int}(M)$ is $\langle M\rangle$.

Let $\mathbf{L} = (L; \vee, \wedge)$ be a finite lattice. The least and greatest elements of $\mathbf{L}$ will be denoted by $0_{\mathbf{L}}$ and $1_{\mathbf{L}}$, respectively. If the lattice is clear from the context then we omit the subscript, and simply write 0 and 1, respectively. The set of atoms and the set of join-irreducible elements of $\mathbf{L}$ will be denoted by $\mathcal{A}(\mathbf{L})$ and $\mathcal{J}(\mathbf{L})$, respectively, and we put $\mathcal{A}_0(\mathbf{L}) = \mathcal{A}(\mathbf{L}) \cup \{0\}$. If there is no danger of confusion, we simply write $\mathcal{A}$, $\mathcal{A}_0$ and $\mathcal{J}$, respectively. Two elements a and b of $\mathbf{L}$ will be called **similar** iff the principal ideals $(a]$ and $(b]$ are isomorphic. We write $a \sim b$ to denote that a is similar to b. The relation $\sim$ is an equivalence relation on L. If the $\sim$-class containing a has only one element then a will be called **isolated**. For every element $a \in L$ we define a unary operation φ_a by the rule $\varphi_a(x) = x \wedge a$ ($x \in L$). In particular, φ_0 is constant with range $\{0\}$. For similar elements $a, b \in L$ the symbol $\beta_{a,b}$ will denote an isomorphism between the principal ideals $(a]$ and $(b]$. Define a set $\mathrm{IS}(\mathbf{L})$ of transformations on L in the following way:

$$\mathrm{IS}(\mathbf{L}) = \{\beta_{v,w} \circ \varphi_v \mid v, w \in L,\ v \sim w, \text{ and } \beta_{v,w}\colon (v] \to (w] \text{ is an isomorphism}\}.$$

Then $\mathrm{IS}(\mathbf{L})$ is an inverse submonoid of the full transformation semigroup on L (cf. Saito–Katsura [10], Lemma 3.1). We note that, with the help of Proposition 2.1 (b), one can easily verify that the set $\mathrm{IS}(\mathbf{L})$ is closed under composition.

Let $M = \mathrm{IS}(\mathbf{L})$ be the inverse monoid determined by the lattice $\mathbf{L}$.

Proposition 2.1. *Let m be an arbitrary transformation from M. Then*

(a) *m is monotone;*
(b) *there is a unique element $v \sim m(1)$ of L such that $m = \beta_{v,m(1)} \circ \varphi_v$ for some isomorphism $\beta_{v,m(1)}\colon (v] \to (m(1)]$; furthermore, for any $u \in L$, $m(u) = m(1)$ if and only if $v \leqslant u$;*
(c) *$m(\mathcal{A}_0) \subseteq \mathcal{A}_0$, and $m(0) = 0$, moreover for arbitrary atom d we have that $0 < m(d)$ if and only if $d \leqslant v$;*
(d) *if $m(d) = 0$ for every atom d of $\mathbf{L}$ then $m = \varphi_0$.*

Proof. (a) As $m \in M$, there are elements $v, w \in L$ and an isomorphism $\beta_{v,w}\colon (v] \to (w]$ such that $m = \beta_{v,w} \circ \varphi_v$. Since both φ_v and $\beta_{v,w}$ are monotone, the operation m is monotone, as well.

(b) For the element w in part (a) we get that $w = \beta_{v,w}(v) = \beta_{v,w}(1 \wedge v) = m(1)$. Since $\beta_{v,w}$ is an isomorphism, the elements v and w are similar, and so $v \sim m(1)$. Furthermore, for arbitrary element u of L we have that

$$\begin{aligned} m(u) = w \quad &\iff \quad \beta_{v,w}(u \wedge v) = w \\ &\iff \quad u \wedge v = v \\ &\iff \quad v \leqslant u. \end{aligned}$$

This proves that the element v must be the least element of the set

$$\{u \in L \mid m(u) = m(1)\},$$

and hence it is uniquely determined by m.

(c) It is straightforward to check that $m(0)=0$. Let $d \in L$ be an arbitrary atom. If $d \nleqslant v$ then $d \wedge v = 0$ implies that $m(d) = \beta_{v,w}(d \wedge v) = \beta_{v,w}(0) = 0$. If $d \leqslant v$ then $m(d) = \beta_{v,w}(d \wedge v) = \beta_{v,w}(d)$. Since $0 \prec d$ and $\beta_{v,w}$ is an isomorphism, we get that $0 \prec \beta_{v,w}(d)$. Hence, $m(d) = \beta_{v,w}(d)$ is an atom, as well. Thus, the inclusion $m(\mathcal{A}_0) \subseteq \mathcal{A}_0$ and all other claims in (c) are proved.

(d) Assume that $m(d) = 0$ holds for every atom d of $\mathbf{L}$. If the inequality $0 < v$ were true then there would be an atom $d \leqslant v$. Then by part (c), $0 < m(d)$ would hold, which contradicts the assumption. Hence, $v = 0$. Then for arbitrary element $x \in L$ we get that $m(x) = \beta_{0,w}(x \wedge 0) = \beta_{0,w}(0) = 0$. Therefore, $m = \varphi_0$. □

Lemma 2.2. *Suppose* $\mathbf{L}$ *has at least two atoms. If* f *is a binary operation in the stabilizer of* M *then*

(a) $f(\mathcal{A}_0 \times \mathcal{A}_0) \subseteq \mathcal{A}_0$ *and* $f(0,0) = 0$*;*
(b) $f|_{\mathcal{A}_0}$ *is an essentially unary operation;*
(c) *if* $f|_{\mathcal{A}_0}$ *does not depend on its first variable [second variable] then* $f(x,0) = 0$ *[*$f(0,x) = 0$*] for all* $x \in L$*.*

Proof. Throughout this proof we will repeatedly use the fact that for any two atoms k, l of $\mathbf{L}$ there is a unique isomorphism $\beta_{k,l} \colon (k] \to (l]$, and hence the transformations φ_k and $\beta_{k,l} \circ \varphi_k$ belong to M. Now choose and fix two distinct atoms d_0 and d_1 of $\mathbf{L}$, and let d and d' be arbitrary atoms of $\mathbf{L}$.

(a) Since $f \in \mathrm{Sta}(M)$, the transformations $t = f(\varphi_d, \beta_{d,d'} \circ \varphi_d)$, $r = f(\varphi_d, \varphi_0)$, and $l = f(\varphi_0, \varphi_d)$ belong to M. Thus we get from Proposition 2.1 (c) that $t(d) = f(d,d')$, $r(d) = f(d,0)$, and $l(d) = f(0,d)$ belong to $\mathcal{A}_0$, and $0 = t(0) = f(0,0)$. This proves the first statement.

(b) To prove the second statement, define two unary transformations m and n as follows:

$$m = f(\beta_{d_0,d} \circ \varphi_{d_0}, \beta_{d_1,d'} \circ \varphi_{d_1}), \qquad n = f(\beta_{d_0,d} \circ \varphi_{d_0}, \beta_{d_0,d'} \circ \varphi_{d_0}).$$

Again, $f \in \mathrm{Sta}(M)$ implies that $m, n \in M$. Furthermore, we have

$$m(x) = f(\beta_{d_0,d}(x \wedge d_0), \beta_{d_1,d'}(x \wedge d_1)) = \begin{cases} f(d,0) & \text{if } x = d_0, \\ f(0,d') & \text{if } x = d_1, \\ f(0,0) = 0 & \text{if } x \in \mathcal{A}_0 \setminus \{d_0, d_1\}, \end{cases} \tag{1}$$

and

$$n(x) = f(\beta_{d_0,d}(x \wedge d_0), \beta_{d_0,d'}(x \wedge d_0)) = \begin{cases} f(d,d') & \text{if } x = d_0, \\ f(0,0) = 0 & \text{if } x \in \mathcal{A}_0 \setminus \{d_0\}. \end{cases} \tag{2}$$

First we will prove that at least one of the elements $f(d,0)$ and $f(0,d')$ is 0. We proceed by contradiction. Suppose that $f(d,0), f(0,d') > 0$. By part (a), the element $f(d,d') = n(d_0)$ is in $\mathcal{A}_0$, and $m(d_0) = f(d,0) > 0$, $m(d_1) = f(0,d') > 0$. Since $m \in M$, there are similar elements $v, w \in L$ and an isomorphism $\beta_{v,w} \colon (v] \to (w]$ such that $m = \beta_{v,w} \circ \varphi_v$. Hence, by Proposition 2.1 (c), this implies that

$d_0, d_1 \leqslant v$, and

$$m(d_0) = \beta_{v,w}(d_0 \wedge v) = \beta_{v,w}(d_0) \neq \beta_{v,w}(d_1) = \beta_{v,w}(d_1 \wedge v) = m(d_1),$$

since d_0 and d_1 are distinct atoms. Thus $f(d, d') = m(1) \geqslant m(d_0) \vee m(d_1) \notin \mathcal{A}_0$. This contradicts part (a), and therefore proves that $f(d, 0)$ or $f(0, d')$ is 0.

If $f(d, 0) = f(0, d') = 0$ then by formula (1) the value of m is 0 for all atoms of $\mathbf{L}$, therefore $m = \varphi_0$ by Proposition 2.1 (d). Thus, $f(d, d') = m(1) = 0 = f(d, 0) = f(0, d')$.

Suppose that $f(d, 0) > 0$ or $f(0, d') > 0$ holds. Without loss of generality, we may suppose that $f(d, 0) > 0$. Then $f(0, d') = 0$, so by formula (1) we have that $m(d_0) = f(d, 0) > 0$, and $m(c) = 0$ for all atoms c distinct from d_0. Furthermore, $m(1) = f(d, d') = n(d_0) \in \mathcal{A}_0$, hence the monotonicity of m implies that $f(d, d') = f(d, 0)$.

Thus, we get that if for all atoms d, d' of $\mathbf{L}$ the equalities $f(d, 0) = 0$ and $f(0, d') = 0$ hold then $f|_{\mathcal{A}_0}$ is constantly 0. Otherwise, there is an atom d of $\mathbf{L}$ for which either $f(d, 0) > 0$ or $f(0, d) > 0$. If $f(d, 0) > 0$ then $f(0, d') = 0$ for all atoms d' of $\mathbf{L}$. Hence, by the preceeding argument $f(k, l) = f(k, 0)$ for all atoms $k, l \in \mathcal{A}$. Therefore, the operation $f|_{\mathcal{A}_0}$ is essentially unary. A similar argument shows that if $f(0, d) > 0$ then $f|_{\mathcal{A}_0}$ is also an essentially unary unary operation.

(c) Without loss of generality, we may assume that $f|_{\mathcal{A}_0}$ does not depend on its second variable. Then $f(0, d) = f(0, 0) = 0$ for all atoms d of $\mathbf{L}$, by part (a). For the operation $t = f(\varphi_0, \varphi_1) \in M$, this means that t is 0 for every atom of $\mathbf{L}$. Then by Proposition 2.1 (d), $t = \varphi_0$. Hence, $0 = t(x) = f(\varphi_0(x), \varphi_1(x)) = f(0, x)$ for all $x \in L$. □

3. When the monoid IS(L) is collapsing

This section is devoted to the proof of the following theorem which characterizes the collapsing monoids among the inverse monoids of the form $\mathrm{IS}(\mathbf{L})$ where $\mathbf{L}$ is a finite lattice.

First, we need the following definition. Let a and b be arbitrary elements of L. We will say that the element b is **dwarfed** by a if for all elements $b' \in L$ such that $b' \sim b$ we have that $b' \leqslant a$. We will use the notation $b \ll a$ to denote that a dwarfs b. Now we are in a position to state the central result of this paper.

Theorem 3.1. *Let $\mathbf{L}$ be a finite lattice such that $|L| \geqslant 3$. Then the inverse monoid $M = \mathrm{IS}(\mathbf{L})$ is collapsing if and only if no element of $\mathcal{J} \setminus \mathcal{A}$ dwarfs a nonzero element of L.*

Proof. Suppose that there are elements $a \in \mathcal{J} \setminus \mathcal{A}$ and $b \in L \setminus \{0\}$ such that $b \ll a$. Then $b \leqslant a$ since we have that

$$b \leqslant \bigvee \{b' \in L \mid b' \sim b\} \leqslant a.$$

We will construct an essentially binary operation f that belongs to the stabilizer of M. Let $\bar{a}$ be the unique lower cover of a, and define f in the following way:

$$f(x,y)=\begin{cases} x\wedge a = x\wedge \bar{a} & \text{if } a\nleqslant x,\\ \bar{a}=x\wedge\bar{a} & \text{if } a\leqslant x,\ b\nleqslant y,\\ a & \text{if } a\leqslant x, b\leqslant y.\end{cases}$$

Since we have $f(0,1)=0$, $f(1,1)=a$ and $f(1,0)=\bar{a}$, therefore f is an essentially binary operation. To check that f belongs to the stabilizer of M, consider arbitrary elements $m_1=\beta_{u_1,v_1}\circ\varphi_{u_1}$ and $m_2=\beta_{u_2,v_2}\circ\varphi_{u_2}$ of M, and set $t=f(m_1,m_2)$.

If $a\nleqslant v_1$ or $b\nleqslant v_2$ then $a\nleqslant m_1(x)$ for every $x\in L$ or $b\nleqslant m_2(x)$ for every $x\in L$. Thus

$$t(x)=f(m_1(x),m_2(x))=m_1(x)\wedge\bar{a}=\varphi_{\bar{a}}(m_1(x)) \text{ for all } x\in L.$$

Hence $t=\varphi_{\bar{a}}\circ m_1\in M$.

Now assume that $a\leqslant v_1$ and $b\leqslant v_2$. Then $t(1)=f(v_1,v_2)=a$, and there exist elements $a'\leqslant u_1,\ b'\leqslant u_2$ such that $\beta_{u_1,v_1}(a')=a$ and $\beta_{u_2,v_2}(b')=b$. Next we prove that $b'\leqslant a'$.

Claim 3.2. *For any elements $c,d\in L$ the following statements are equivalent:*

(i) *The element d is dwarfed by c.*

(ii) *The inequality $d'\leqslant c'$ holds for all elements $c',d'\in \mathbf{L}$ for which $c'\sim c$ and $d'\sim d$.*

The implication (ii) $\Rightarrow$ (i) is an easy consequence of the definition.

To prove that (i) implies (ii) choose an arbitrary element $c'\in L$ such that $c'\sim c$, and let $\beta\colon [c)\to[c')$ be an isomorphism. Furthermore, let H_d denote the set $\{d'\in L\mid d'\sim d\}$. By (i), for arbitrary element $d'\in H_d$ we have that $d'\leqslant c$. Since β is an isomorphism, we get that

$$d\sim\beta(d')\leqslant\beta(c)=c',$$

and so $\beta(d')\in H_d$. Therefore the isomorphism β induces a permutation of H_d. Hence, $d'\leqslant c'$ holds for arbitrary elements $c',d'\in L$ for which $c'\sim c$ and $d'\sim d$, that is, $d\ll c$. This completes the proof of Claim 3.2.

Since $b\ll a$ and $a'\sim a,\ b'\sim b$, by Claim 3.2, we get that $b'\leqslant a'$. Let x be an arbitrary element of L. If $a'\nleqslant x$ then $a'\nleqslant x\wedge u_1$. Hence, $a\nleqslant\beta_{u_1,v_1}(x\wedge u_1)=m_1(x)$; therefore

$$t(x)=f(m_1(x),m_2(x))=m_1(x)\wedge a.$$

If $a'\leqslant x$ then $a'\leqslant x\wedge u_1$ and because of $b'\leqslant a'\leqslant x$ we have $b'\leqslant x\wedge u_2$. This implies that $a\leqslant\beta_{u_1,v_1}(x\wedge u_1)=m_1(x)$ and $b\leqslant\beta_{u_2,v_2}(x\wedge u_2)=m_2(x)$, therefore

$$t(x)=f(m_1(x),m_2(x))=a=m_1(x)\wedge a.$$

Thus $t(x)=m_1(x)\wedge a$ for all $x\in L$, showing that $t=\varphi_a\circ m_1\in M$. This proves that the binary operation f is in the stabilizer of M. Hence, M is not collapsing.

Now suppose that the monoid M is not collapsing. We will show that there is an element of $\mathcal{J}\setminus\mathcal{A}$ that dwarfs a nonzero element of L.

If the lattice $\mathbf{L}$ has only one atom then let the element b be the unique atom of $\mathbf{L}$ and let a be an upper cover of b. Then a is not an atom, but join-irreducible and $b \ll a$ holds.

From now on, we will suppose that the lattice $\mathbf{L}$ has at least two (distinct) atoms. For an arbitrary element u of L define a set F_u as follows:

$$F_u = \bigl\{ f \in \mathrm{Sta}(M)^{(2)} \bigm| f|_{\mathcal{A}_0} \text{ does not depend on its second variable, and} \\ \text{there are elements } y_1, y_2 \in L \text{ such that } f(u, y_1) \neq f(u, y_2) \bigr\}.$$

Furthermore, let W be the set $\{u \in L \mid F_u \neq \emptyset\}$. By the result of Grabowski [2], the stabilizer of M contains an essentially binary operation, which ensures the set W to be non-empty. By Lemma 2.2 (c) $F_0 = \emptyset$, therefore $0 \notin W$. Notice that every operation $f \in F_u$ ($u \in W$) is essentially binary. Indeed, f depends on its second variable because there are elements $y_1, y_2 \in L$ such that $f(u, y_1) \neq f(u, y_2)$. In view of Lemma 2.2 (a) and (c) we have $f(0, y_1) = f(0, y_2) = 0$. Since $f(u, y_1) \neq f(u, y_2)$, at least one of the sets $\{f(0, y_1), f(u, y_1)\}$ and $\{f(0, y_2), f(u, y_2)\}$ has more than one element, which proves that f depends on its first variable.

Choose a minimal element a from W, and let p be a minimal element of the set $\{h(a, 0) \mid h \in F_a\}$, which is not empty, since $F_a \neq \emptyset$. Hence, $\{g \in F_a \mid g(a, 0) = p\}$ is a non-empty finite set, so let f be an element of this set that is minimal with respect to the pointwise order of operations in F_a. Finally, let b be a minimal element of the set $\{d \in L \mid f(a, 0) \neq f(a, d)\}$. The elements a, b and the operation f selected this way will be fixed for the rest of the proof. Some of their basic properties are summarized in the next claim.

Claim 3.3. *We have*

(a) $0 < a$ *and* $0 < b$*;*
(b) $f(x, y) = f(x, 0)$ *whenever* $x, y \in L$ *and* $x < a$*;*
(c) $h(a, 0) < f(a, 0)$ *for no* $h \in F_a$*;*
(d) *if* $g \in F_a$ *is such that* $g(a, 0) = f(a, 0)$ *and* $g(x, y) \leqslant f(x, y)$ *for all* $x, y \in L$*, then* $g = f$*;*
(e) *if* $c < b$ *then* $f(a, c) = f(a, 0)$*.*

In (a) $0 < b$ follows from the choice of b, and $0 < a$ from the fact that $0 \notin W$. The minimality of a implies that $F_x = \emptyset$ for every elements $x < a$ ($x \in L$). In particular, for each such x we have $f \notin F_x$ although $f \in \mathrm{Sta}(M)^{(2)}$ and $f|_{\mathcal{A}_0}$ does not depend on its second variable. Thus, for such an x, $f(x, y)$ cannot depend on its variable y. This proves (b). Properties (c), (d) and (e) are immediate consequences of the minimality of $f(a, 0) = p$, the minimality of f, and the minimality of b, respectively.

Claim 3.4. *If c is an element of L such that $c \nleqslant f(a, 0)$ and $c \leqslant f(a, b)$ then the operation $\varphi_c \circ f$ is in F_a.*

Let $\bar{f}$ be the binary operation $\varphi_c \circ f \in \mathrm{Sta}(M)$. Then $\bar{f}(x, y) = f(x, y) \wedge c$ for all $x, y \in L$. Therefore $\bar{f}|_{\mathcal{A}_0}$ does not depend on its second variable, because $f|_{\mathcal{A}_0}$ has this property. Furthermore,

$$\bar{f}(a, 0) = f(a, 0) \wedge c \leqslant f(a, 0) \wedge f(a, b) < c = \bar{f}(a, b).$$

Thus $\bar{f} \in F_a$, completing the proof of Claim 3.4.

Claim 3.5. *The elements $f(a,0)$ and $f(a,b)$ are comparable.*

Suppose that $f(a,0) \parallel f(a,b)$, and let $\bar{f}$ be the binary operation $\varphi_{f(a,b)} \circ f \in \mathrm{Sta}(M)$. By Claim 3.4, $\bar{f} \in F_a$. However, $\bar{f}(a,0) = f(a,0) \wedge f(a,b) < f(a,0)$, which contradicts Claim 3.3 (c). This completes the proof of Claim 3.5.

Let m and n be the unary operations $f(\varphi_a, \varphi_0)$ and $f(\varphi_a, \varphi_b)$, respectively; that is, $m(x) = f(a \wedge x, 0)$ and $n(x) = f(a \wedge x, b \wedge x)$ for all $x \in L$. For the operation n we get that

$$f(a, a \wedge b) = n(a) \leqslant n(1) = f(a,b). \tag{3}$$

Claim 3.6. *For the operation $g(x,y) = f(\varphi_a(x), \varphi_b(y))$ we have that*

$$g(x,y) = f(x \wedge a, y \wedge b) = \begin{cases} f(x \wedge a, 0) \leqslant f(a,0) & \text{if } a \nleqslant x, \\ f(a,0) & \text{if } a \leqslant x,\ b \nleqslant y, \\ f(a,b) & \text{if } a \leqslant x,\ b \leqslant y. \end{cases}$$

Indeed, if $a \nleqslant x$ then $x \wedge a < a$. Hence by Claim 3.3 (b), $f(x \wedge a, y \wedge b) = f(x \wedge a, 0) = m(x) \leqslant m(1) = f(a,0)$. If $a \leqslant x$, $b \nleqslant y$ then $x \wedge a = a$, $y \wedge b < b$. Then by Claim 3.3 (e), $f(x \wedge a, y \wedge b) = f(a, y \wedge b) = f(a,0)$. Finally, if $a \leqslant x$, $b \leqslant y$ then $f(x \wedge a, y \wedge b) = f(a,b)$. This proves Claim 3.6.

From now on the argument splits according to whether $<$ or $=$ holds in (3).

Case 1: $n(a) < n(1)$.

In this case, we have $f(a, a \wedge b) < f(a,b)$ by (3), which implies that $a \wedge b < b$, in particular, $a \neq b$. It follows from Claim 3.3 (e) that $f(a,0) = f(a, a \wedge b)$, and therefore $f(a,0) < f(a,b)$.

Claim 3.7. $0 < f(a,0)$.

Assume that $f(a,0) = 0$. This assumption implies that $m(1) = f(a,0) = 0$, and by Proposition 2.1 (d), $m = \varphi_0$. Since $n(x) = g(x,x)$ and $f(x \wedge a, 0) = m(x) = 0$ for all $x \in L$, we get from Claim 3.6 that

$$n(x) = \begin{cases} f(a,b) & \text{if } a \leqslant x,\ b \leqslant x, \\ 0 & \text{otherwise.} \end{cases}$$

By the definition of M, the range $\{0, f(a,b)\}$ of n is the ideal $(f(a,b)]$ and by Proposition 2.1 (b), there is an isomorphism $\beta_{a \vee b, f(a,b)} \colon (a \vee b] \to (f(a,b)]$ such that $n = \beta_{a \vee b, f(a,b)} \circ \varphi_{a \vee b}$. The equality $(f(a,b)] = \{0, f(a,b)\}$ implies that $f(a,b)$ is an atom, and so $a \vee b$ is an atom, as well. Therefore by Claim 3.3 (a), the elements a and b are atoms, furthermore $a \neq b$. Hence $a \vee b$ cannot be an atom. This contradiction proves Claim 3.7.

Claim 3.8. *$f(a,0) \prec f(a,b)$ and $f(a,b)$ is join-irreducible.*

Let $c \in L$ be an element such that $f(a,0) \prec c \leqslant f(a,b)$, and let $\bar{f}$ be the operation $\varphi_c \circ f \in \mathrm{Sta}(M)$. By Claim 3.4, the operation $\bar{f}$ is in F_a. Since $\bar{f}(a,0) = f(a,0)$, and $\bar{f}(x,y) = f(x,y) \wedge c \leqslant f(x,y)$ for all $x, y \in L$, we get from Claim 3.3 (d) that $\bar{f} = f$. Hence, $f(a,0) \prec c = \bar{f}(a,b) = f(a,b)$.

The element $f(a,b)$ is the join of all the join-irreducible elements u for which $u \leqslant f(a,b)$. Since $f(a,0) < f(a,b)$, there is an element $u_0 \in \mathcal{J} \cap (f(a,b)]$ such that $u_0 \nleqslant f(a,0)$, that is $f(a,0) < u_0$ or $u_0 \parallel f(a,0)$. In the latter case, by Claim 3.4, the operation $\tilde{f} = \varphi_{u_0} \circ f$ is in F_a. Moreover, $\tilde{f}(a,0) < f(a,0)$ which contradicts Claim 3.3 (c). Thus we must have $f(a,0) < u_0 \leqslant f(a,b)$ whence $f(a,b) = u_0 \in \mathcal{J}$, since $u_0 \leqslant f(a,b)$ and $f(a,0) \prec f(a,b)$. This completes the proof of Claim 3.8.

Claim 3.9. *The element b is similar to $f(a,b)$, hence it is join-irreducible, and $a < b$.*

By Claim 3.6, for the unary operation n we have that

$$n(x) = g(x,x) = \begin{cases} f(a \wedge x, 0) & \text{if } a \nleqslant x, \\ f(a,0) & \text{if } a \leqslant x,\ b \nleqslant x, \\ f(a,b) & \text{if } a \leqslant x,\ b \leqslant x. \end{cases}$$

Since $f(a,0) < f(a,b) = n(1)$, we get from Proposition 2.1 (b) that $n = \beta_{a\vee b, f(a,b)} \circ \varphi_{a\vee b}$ and $a \vee b \sim f(a,b) \in \mathcal{J}$. Thus $a \vee b \in \mathcal{J}$, therefore $\{a,b\} \cap \mathcal{J} \neq \emptyset$ and a, b are comparable. Since $a \wedge b < b$, we get that $a < b \in \mathcal{J}$, completing the proof of Claim 3.9.

Thus, $n = \beta_{b,f(a,b)} \circ \varphi_b$. For arbitrary element c of the interval $[a,b)$ we get from Claim 3.3 (e) that $f(a,c) = f(a,0) = f(a,a)$ since $a, c < b$. Therefore

$$n(c) = f(c \wedge a, c \wedge b) = f(a,c) = f(a,a) = n(a).$$

Thus $c = a$ since $a \leqslant c < b$ and $\beta_{b,f(a,b)}$ is an isomorphism. This proves that a is the unique lower cover of b. Hence $\beta_{b,f(a,b)}$ maps b to $f(a,b)$ and a to $f(a,0)$.

Now, we will prove that the element a is isolated, that is, $a' \sim a$ implies $a' = a$. Suppose that the element $a' \in L$ is similar to a, and set $s = g(\beta_{a',a} \circ \varphi_{a'}, \varphi_b)$ where g is the operation defined in Claim 3.6. Then $s \in M$ and $s(1) = g(a,b) = f(a,b)$. Hence $s = \beta_{b',f(a,b)} \circ \varphi_{b'}$ for some element $b' \in L$, which is similar to $f(a,b)$. Then

$$f(a,b) = s(b') = g(\beta_{a',a}(b' \wedge a'), b' \wedge b).$$

Hence Claim 3.6 implies that $a \leqslant \beta_{a',a}(b' \wedge a')$ and $b \leqslant b' \wedge b$. Since $\beta_{a',a}(b' \wedge a') \leqslant a$ and $b' \wedge b \leqslant b$, we have that $\beta_{a',a}(b' \wedge a') = a$ and $b' \wedge b = b$. The second equality shows that $b \leqslant b'$, but since $b' \sim f(a,b) \sim b$ we get that $b = b'$. The first equality implies that $b' \wedge a' = a'$. Thus $a' \leqslant b'$, and equality cannot hold because that would imply that $a \sim a' = b' = b$, which is impossible. Hence, $a' < b' = b$. Since a is the unique lower cover of b and $a \sim a'$, we get that $a' = a$. This proves that the element a is isolated. Since $a \sim f(a,0)$, as witnessed by $\beta_{b,f(a,b)}$, we conclude that $f(a,0) = a$.

Since a is isolated, the join of elements similar to a is a. Since $0 < a$, the element b cannot be an atom. Therefore, for the elements $b \in \mathcal{J} \setminus \mathcal{A}$, $a \in L \setminus \{0\}$ we have that $a \ll b$.

Case 2: $n(a) = n(1)$.

Claim 3.10. $b \leqslant a$.

In this case, (3) shows that $f(a, a \wedge b) = f(a, b)$, so by Claim 3.3 (e), we have $a \wedge b = b$, i.e., $b \leqslant a$.

Since $m, n \in M$, there are elements $u, v, u', v' \in L$ and isomorphisms $\beta_{u,v} \colon (u] \to (v]$, $\beta_{u',v'} \colon (u'] \to (v']$ such that

$$m = f(\varphi_a, \varphi_0) = \beta_{u,v} \circ \varphi_u, \qquad \text{and} \qquad n = f(\varphi_a, \varphi_b) = \beta_{u',v'} \circ \varphi_{u'}.$$

We will denote the isomorphisms $\beta_{u,v}$ and $\beta_{u',v'}$ by β and β', respectively. Thus

$$f(x \wedge a, 0) = m(x) = \beta(x \wedge u),$$

and

$$f(x \wedge a, x \wedge b) = n(x) = \beta'(x \wedge u')$$

for all $x \in L$.

Claim 3.11. *If $x \in L$ is such that $a \nleqslant x$ then $m(x) = n(x)$.*

If $a \nleqslant x$ then $x \wedge a < a$ and by Claim 3.3 (b),

$$n(x) = f(x \wedge a, x \wedge b) = f(x \wedge a, 0) = m(x)$$

holds. This completes the proof of Claim 3.11.

Since $m(a) = f(a, 0) = m(1)$ and $n(a) = f(a, a \wedge b) = f(a, b) = n(1)$ we have that $u \leqslant a$ and $u' \leqslant a$, by Proposition 2.1 (b). Now we distinguish cases according to whether $=$ or $<$ holds. The followings are easy consequences of the definition of m and n:

$$\begin{aligned} f(a, 0) = m(1) = v = m(u) = \beta(u) \sim u, \\ f(a, b) = n(1) = v' = n(u') = \beta'(u') \sim u'. \end{aligned} \tag{4}$$

Case 2.1: $u, u' < a$.
Then $a \nleqslant u, u'$, and by Claim 3.11 and (4),

$$\begin{aligned} f(a, 0) = m(u) = n(u) = \beta'(u \wedge u') \leqslant v' = f(a, b), \\ f(a, b) = n(u') = m(u') = \beta(u' \wedge u) \leqslant v = f(a, 0). \end{aligned}$$

Hence $f(a, 0) = f(a, b)$, which contradicts the choice of the elements a and b.

Case 2.2: $u = a$ and $u' < a$.
Then $a \nleqslant u'$, and by Claim 3.11 and (4),

$$f(a, b) = n(u') = m(u') = \beta(u' \wedge u) = \beta(u') < \beta(u) = v = f(a, 0).$$

Next we want to show that $a \in \mathcal{J}$. Suppose that there are distinct elements $r_1, r_2 \in L$ such that $r_1, r_2 \prec a = u$. Then $m(r_1) = \beta(r_1) \neq \beta(r_2) = m(r_2)$ since β is an isomorphism, and $n(r_j) \leqslant n(1) = f(a, b)$ $(j = 1, 2)$ by the monotonicity of n

and by (4). Furthermore, $m(r_j) = n(r_j)$ $(j = 1, 2)$ by Claim 3.11. Thus, combining these with (4) and the fact that β is an isomorphism we get that

$$\begin{aligned} f(a, 0) &= \beta(u) = \beta(r_1 \vee r_2) = \beta(r_1) \vee \beta(r_2) \\ &= m(r_1) \vee m(r_2) = n(r_1) \vee n(r_2) \leqslant f(a, b). \end{aligned}$$

Since $f(a, b) < f(a, 0)$, this is impossible. Hence $a \in \mathcal{J}$, and for the the unique lower cover $\bar{a}$ of a we have $u' \leqslant \bar{a} \prec a = u$, therefore by (4) and Claim 3.11

$$f(a, b) = \beta'(u') = \beta'(\bar{a} \wedge u') = n(\bar{a}) = m(\bar{a}) = \beta(\bar{a}) \prec \beta(a) = f(a, 0),$$

where $\beta(a) \sim a$. Hence, $f(a, b) \prec f(a, 0) \sim a \in \mathcal{J}$.

From Claim 3.10 we know that $b \leqslant a$. First we will argue the case $a = b$. Suppose $a = b$. Then $f(a, a) \prec f(a, 0) \sim a \in \mathcal{J}$. We want to show that the element a is isolated. Let z be an arbitrary element that is similar to a, and set $t = f(\varphi_a, \beta_{z,a} \circ \varphi_z) \in M$. Then $t(1) = f(a, a)$ and by Claim 3.3 (e), $t(a) = f(a, \beta_{z,a}(a \wedge z)) \in \{f(a, 0), f(a, a)\}$. The monotonicity of t implies that $t(a)$ must be equal to $f(a, a)$, and so $\beta_{z,a}(a \wedge z) = a$. Since $\beta_{z,a}$ is an isomorphism, we get that $a \wedge z = z$, i.e., $z \leqslant a$. As $z \sim a$, we have that $z = a$. Therefore, a is isolated, and it cannot be an atom, because our assumption that $\mathbf{L}$ has at least two atoms ensures that atoms are not isolated in $\mathbf{L}$. Hence $b = a \in \mathcal{J} \setminus \mathcal{A}$, and $b \ll a$.

Now we suppose that $b < a$. Let b' be an arbitrary elements of L such that $b' \sim b$, and let t be the unary operation $f(\varphi_a, \beta_{b',b} \circ \varphi_{b'}) \in M$. Suppose that $b' \nleqslant a$. Then $a \wedge b' < b'$ implies that $\beta_{b',b}(a \wedge b') < b$ and so $t(a) = f(a, \beta_{b',b}(a \wedge b')) = f(a, 0)$ by Claim 3.3 (e). Furthermore, $t(1) = f(a, b)$. The monotonicity of t implies that $f(a, 0) = t(a) \leqslant t(1) = f(a, b)$, however, this is impossible. Hence, $b' \leqslant a$, proving that $\bigvee\{b' \in L \mid b' \sim b\} \leqslant a$. Since $0 < b < a \in \mathcal{J}$, this proves that a is not an atom and $b \ll a$.

Case 2.3: $u < a$ and $u' = a$.

Then $a \nleqslant u$, and by Claim 3.11 and (4),

$$f(a, 0) = m(u) = n(u) = \beta'(u \wedge u') = \beta'(u) \leqslant \beta'(u') = f(a, b).$$

We want to show that $a \in \mathcal{J}$. Suppose that there are distinct elements $r_1, r_2 \in L$ such that $r_1, r_2 \prec a = u'$. Then $n(r_1) = \beta'(r_1) \neq \beta'(r_2) = n(r_2)$ since β' is an isomorphism, and $m(r_j) \leqslant m(1) = f(a, 0)$ $(j = 1, 2)$ by the monotonicity of m and by (4). Moreover, $m(r_j) = n(r_j)$ $(j = 1, 2)$ by Claim 3.11. Thus, in the same way as in Case 2.2, we get that

$$\begin{aligned} f(a, b) &= \beta'(u') = \beta'(r_1 \vee r_2) = \beta'(r_1) \vee \beta'(r_2) \\ &= n(r_1) \vee n(r_2) = m(r_1) \vee m(r_2) \leqslant f(a, 0). \end{aligned}$$

Since $f(a, 0) < f(a, b)$, this is impossible. Hence $a \in \mathcal{J}$, and for the unique lower cover $\bar{a}$ of a we have $u \leqslant \bar{a} \prec a = u'$, therefore by (4) and Claim 3.11

$$f(a, 0) = \beta(u) = \beta(\bar{a} \wedge u) = m(\bar{a}) = n(\bar{a}) = \beta'(\bar{a}) \prec \beta'(a) = f(a, b),$$

where $\beta'(a) \sim a$. Hence, $f(a, 0) \prec f(a, b) \sim a \in \mathcal{J}$. Since $b \leqslant a$ by Claim 3.10, we get that either $a = b$ or $b < a$.

If $a = b$ then $f(a, 0) \prec f(a, a) \sim a \in \mathcal{J}$. Let z be an arbitrary element that is similar to a, and set $t = f(\varphi_a, \beta_{z,a} \circ \varphi_z) \in M$. Let x be an arbitrary element of

L. If $a \nleqslant x$ then $x \wedge a < a$ and by Claim 3.3 (b), $t(x) = f(x \wedge a, \beta z, a(x \wedge z)) = f(x \wedge a, 0) = m(x) \leqslant m(1) = f(a, 0)$. If $a \leqslant x$ and $z \nleqslant x$ then $x \wedge a = a$ and $x \wedge z < z$. The latter implies that $\beta_{z,a}(x \wedge z) < a$ since $\beta_{z,a}$ is an isomorphism. Hence by Claim 3.3 (e), $t(x) = f(x \wedge a, \beta_{z,a}(x \wedge z)) = f(a, \beta_{z,a}(x \wedge z)) = f(a, 0)$. Finally, if $a, z \leqslant x$ then $t(x) = f(a, a)$. Since $t(1) = f(a, a)$, we have by Proposition 2.1 (b) that $t = \beta_{a \vee z, f(a,a)} \circ \varphi_{a \vee z}$ with $a \vee z \sim f(a, a)$. Thus $a \vee z \sim f(a, a) \sim a$. Since a is join-irreducible and $a \sim z$, the element z must be equal to a. Hence, a is a join-irreducible isolated element of $\mathbf{L}$, and $b \ll a$. The element a is not an atom, because our assumption that $\mathbf{L}$ has at least two atoms ensures that atoms are not isolated in $\mathbf{L}$.

Now assume that $b < a$. Let b' be arbitrary element of L such that $b' \sim b$, and set $t = f(\varphi_a, \beta_{b',b} \circ \varphi_{b'})$. Then $t(1) = f(a, b)$, and by Proposition 2.1 (b), there is an element $a' \sim t(1)$ such that $t(a') = t(1)$. Thus $t(a') = f(a, b)$. Hence, by the definition of t we have that

$$f(a, b) = t(a') = f(a' \wedge a, \beta_{b',b}(a' \wedge b')). \tag{5}$$

Since $\sim$ is an equivalence relation and $a' \sim t(1) = f(a, b) \sim a$, we have that $a' \sim a$. Therefore either $a' = a$ or $a' \parallel a$. Assume that $a' \parallel a$. Then $a' \wedge a < a$. Applying first (5), then Claim 3.3 (b), and finally the definition and the monotonicity of m, we get that

$$f(a, b) = f(a' \wedge a, \beta_{b',b}(a' \wedge b')) = f(a' \wedge a), 0) = m(a' \wedge a) \leqslant m(a) = f(a, 0).$$

This is a contradiction since $f(a, 0) < f(a, b)$. Therefore, $a' = a$, and so

$$f(a, b) = t(a') = t(a) = f(a, \beta_{b',b}(a \wedge b')).$$

Since $\beta_{b',b}(a \wedge b') \leqslant b$, we get from Claim 3.3 (e) that $\beta_{b',b}(a \wedge b')$ must equal b', which implies that $b' \leqslant a \wedge b'$, i.e., $b' \leqslant a$. Hence, $b \ll a$. Here a is not an atom, because $0 < b < a$.

Case 2.4: $u = u' = a$.

Then by (4), $f(a, 0) \sim u \sim u' \sim f(a, b)$. Since $f(a, 0) \neq f(a, b)$, this implies that $f(a, 0) \parallel f(a, b)$, which contradicts Claim 3.5.

This completes the proof of Theorem 3.1. □

As a corollary of Theorem 3.1, we show that for all atomistic lattices $\mathbf{L}$ with at least three elements the inverse monoids $\mathrm{IS}(\mathbf{L})$ are collapsing. Furthermore, we describe all lattices $\mathbf{L}$ with at most 6 elements for which the inverse monoids $\mathrm{IS}(\mathbf{L})$ are collapsing.

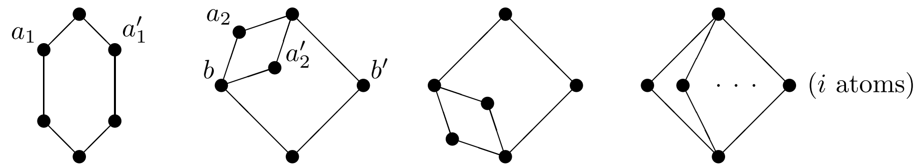
From now on, we will assume $\mathbf{L}$ to be a finite lattice with at least 3 elements. The lattice $\mathbf{L}$ will be called **atomistic** if every element of $L \setminus \{0\}$ is a join of atoms.

Corollary 3.12. *If* $\mathbf{L}$ *is an atomistic lattice then* $\mathrm{IS}(\mathbf{L})$ *is collapsing.*

Proof. If $\mathbf{L}$ is an atomistic lattice then the set of join-irreducible elements coincides with the set of atoms. Therefore by Theorem 3.1, the monoid $\mathrm{IS}(\mathbf{L})$ is collapsing. □

As another application of Theorem 3.1 we determine all lattices $\mathbf{L}$ with at most six elements for which the inverse monoids $\mathrm{IS}(\mathbf{L})$ are collapsing.

Corollary 3.13. *For a lattice* $\mathbf{L}$ *such that* $3 \leqslant |L| \leqslant 6$, $\mathrm{IS}(\mathbf{L})$ *is collapsing if and only if* $\mathbf{L}$ *is isomorphic to one of the lattices in Figure 1.*



$\mathbf{L}_1$ $\mathbf{L}_2$ $\mathbf{L}_3$ $\mathbf{M}_i$ $(i = 2, 3, 4)$

Figure 1

Proof. Since the lattices $\mathbf{L}_3$, $\mathbf{M}_2$, $\mathbf{M}_3$, and $\mathbf{M}_4$, are atomistic, we get from Corollary 3.12, that the inverse monoids $\mathrm{IS}(\mathbf{L}_3)$, $\mathrm{IS}(\mathbf{M}_2)$, $\mathrm{IS}(\mathbf{M}_3)$, and $\mathrm{IS}(\mathbf{M}_4)$ are collapsing.

In the lattice $\mathbf{L}_1$ there are exactly two join-irreducible elements that are not atoms: a_1 and a_1'. Furthermore, these elements are similar, and $a_1 \wedge a_1' = 0_{\mathbf{L}_1}$. Hence there is no element other then 0 which is strongly smaller than a. In the lattice $\mathbf{L}_2$ the join-irreducible elements that are not atoms are a_2 and a_2'. These elements are similar, and $a_2 \wedge a_2' = b \succ 0$. Since $b \sim b'$ and $b \vee b' = 1_{\mathbf{L}_2}$ we get that there is no element other than 0 which is strongly smaller than a. Thus by Theorem 3.1, the inverse monoids $\mathrm{IS}(\mathbf{L}_1)$ and $\mathrm{IS}(\mathbf{L}_2)$ are collapsing.

It is straightforward to check that every lattice L $(3 \leqslant |L| \leqslant 6)$ that is not isomorphic to either of the lattices in Figure 1 is isomorphic to one of the four lattices at the top of Figure 2 or has exactly one atom, or has exactly one coatom, that is, it has the form shown at the bottom of Figure 2.

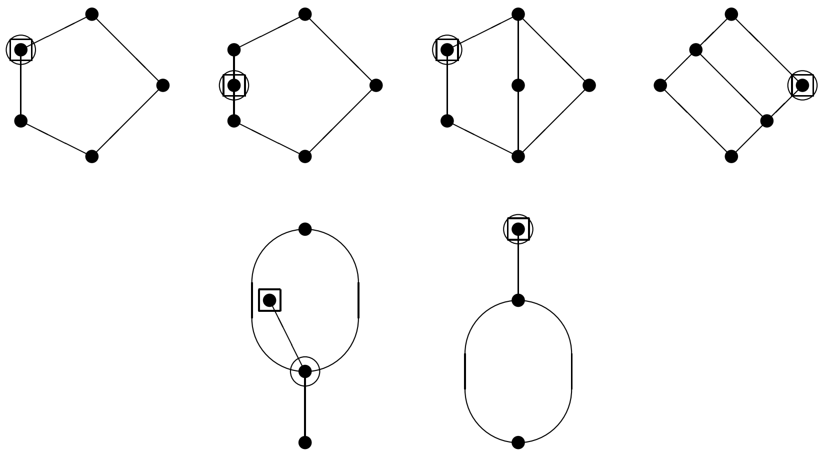


Figure 2

To prove that for these lattices $\mathbf{L}$ the inverse monoids $\mathrm{IS}(\mathbf{L})$ is not collapsing, we provide elements $a \in \mathcal{J}(\mathbf{L}) \setminus \mathcal{A}(\mathbf{L})$ and $b \in L \setminus \{0\}$ such that $b \ll a$. In Figure 2 the boxed element is a, and the encircled element is b. This completes the proof of Corollary 3.13. $\square$

4. Examples when $\mathrm{Int}(\mathrm{IS}(\mathbf{L}))$ is large

In this section we present some examples of lattices $\mathbf{L}$ for which the interval $\mathrm{Int}(\mathrm{IS}(\mathbf{L}))$ are infinite.

On the 2-element set $A = \{0, 1\}$, there is only one lattice, up to isomorphism, namely the 2-element chain $\mathbf{C}_2$ with the partial order $0 < 1$. Then $\mathrm{IS}(\mathbf{C}_2)$ consists of the unary operations $\varphi_1 = \mathrm{id}_A$ and φ_0. Using Post's results (cf. Post [9]), we get the following.

Theorem 4.1. *The monoidal interval corresponding to* $\mathrm{IS}(\mathbf{C}_2)$ *contains countably infinite clones.*

From now on, we will assume $\mathbf{L}$ to be a finite lattice with at least 3 elements.

Theorem 4.2. *For a 3-element chain* $\mathbf{L}$ *we have* $|\mathrm{Int}(\mathrm{IS}(\mathbf{L}))| = 2^{\aleph_0}$.

Proof. Let $\mathbf{L}$ be the chain $0 < 1 < 2$ on $\{0, 1, 2\}$. Then $M = \mathrm{IS}(\mathbf{L}) = \{\varphi_0, \varphi_1, \varphi_2\}$. Now we describe the operations in $\mathrm{Sta}(M)$.

Claim 4.3. *An n-ary operation $f \in \mathcal{O}_L$ belongs to the stabilizer of M if and only if $f(0, \dots, 0) = 0$ and $f(1 \wedge s_1, \dots, 1 \wedge s_n) = 1 \wedge f(s_1, \dots, s_n)$ holds for all elements $s_1, \dots, s_n \in L$.*

Let $f \in \mathcal{O}_L$ be an n-ary operation satisfying the requirements of the claim, and let $s_1, \dots, s_n \in L$ be arbitrary elements of L. Set $t = f(\varphi_{s_1}, \dots, \varphi_{s_n})$. Then $t(0) = f(0, \dots, 0) = 0 = 0 \wedge f(s_1, \dots, s_n)$, and $t(2) = f(2 \wedge s_1, \dots, 2 \wedge s_n) = f(s_1, \dots, s_n) = 2 \wedge f(s_1, \dots, s_n)$. The assumption on f and these equalities imply that $t = \varphi_{f(s_1,\dots,s_n)}$, whence $t \in M$. This proves that $f \in \mathrm{Sta}(M)$.

Conversely, if $f \in \mathrm{Sta}(M)$ is an n-ary operation then for all $s_1, \dots, s_n \in L$ we have that $t = f(\varphi_{s_1}, \dots, \varphi_{s_n}) \in M$. Since $t(2) = f(2 \wedge s_1, \dots, 2 \wedge s_n) = f(s_1, \dots, s_n)$, we get that $t = \varphi_{f(s_1,\dots,s_n)}$. Hence, $f(0, \dots, 0) = t(0) = 0$, and $f(1 \wedge s_1, \dots, 1 \wedge s_n) = t(1) = 1 \wedge f(s_1, \dots, s_n)$. This completes the proof of Claim 4.3.

Let $U_k, V_k, W_k \subseteq L^k$ ($k \in \mathbb{N},\ k \geqslant 3$) denote the following sets

$$U_k = \{(0, 2, 2, \dots, 2), (2, 0, 2, \dots, 2), \dots, (2, 2, 2, \dots, 2, 0)\},$$
$$V_k = \{(0, 1, 1, \dots, 1), (1, 0, 1, \dots, 1), \dots, (1, 1, 1, \dots, 1, 0)\},$$
$$W_k = \{1, 2\}^k \setminus \{(2, 2, 2, \dots, 2)\}.$$

Define an n-ary operation f_n and an m-ary relation ρ_m $(m, n \in \mathbb{N},\ m, n \geqslant 3)$ on L as follows:

$$f_n(x_1, \dots, x_n) = \begin{cases} 0 & \text{if } x_1 = x_2 = x_3 = \cdots = x_n = 0, \\ 2 & \text{if } (x_1, x_2, x_3, \dots, x_n) \in U_n, \\ 1 & \text{otherwise,} \end{cases}$$

and

$$\rho_m = \big\{(0, 0, 0, \dots, 0)\big\} \cup U_m \cup V_m \cup W_m \subseteq L^m.$$

Next, we summarize some easy observations on the relation ρ_m for later reference. All these facts are simple consequences of the definition of ρ_m.

Claim 4.4. *Let $(a_1, a_2, a_3, \dots, a_m)$ be an arbitrary element of ρ_m. Then*

(a) *if $0 \in \{a_1, a_2, a_3, \dots, a_m\}$ then $(a_1, \dots, a_m) \in \big\{(0, \dots, 0)\big\} \cup U_m \cup V_m$;*
(b) *if there exist indices $1 \leqslant i < i' \leqslant m$ such that $a_i = a_{i'} = 0$ then $a_j = 0$ for all j $(1 \leqslant j \leqslant m)$.*

From the definition of f_n we get that

$$1 \wedge f(s_1, \dots, s_n) = f(1 \wedge s_1, \dots, 1 \wedge s_n) = \begin{cases} 0 & \text{if } s_1 = \cdots = s_n = 0, \\ 1 & \text{otherwise,} \end{cases}$$

for all elements $s_1, \dots, s_n \in L$. Hence by Claim 4.3, the operation f_n belongs to $\mathrm{Sta}(M)$ $(n \in \mathbb{N},\ n \geqslant 3)$.

Claim 4.5. $M \subseteq \mathrm{Pol}(\rho_m)$ $(m \in \mathbb{N},\ m \geqslant 3)$.

Let $(a_1, \dots, a_m) \in L^m$ be an arbitrary element of ρ_m, and let $t = \varphi_s$ be an arbitrary element of M. If $a_1 = \cdots = a_m = 0$ or $s = 0$ then $(t(a_1), \dots, t(a_m)) = (0, \dots, 0) \in \rho_m$. If $t = \varphi_2$ then $(t(a_1), \dots, t(a_m)) = (2 \wedge a_1, \dots, 2 \wedge a_m) = (a_1, \dots, a_m) \in \rho_m$. Finally, if $t = \varphi_1$ and $(a_1, \dots, a_m) \neq (0, \dots, 0)$ then $(a_1, \dots, a_n) \in U_m \cup V_m \cup W_m$. If $(a_1, \dots, a_n) \in U_m \cup V_m$ then $(t(a_1), \dots, t(a_m)) \in V_m \subseteq \rho_m$, while if $(a_1, \dots, a_m) \in W_m$ then $(t(a_1), \dots, t(a_m)) = (1, \dots, 1) \in \rho_m$. This proves Claim 4.5.

Claim 4.6. *For $m, n \geqslant 3$, the operation f_n preserves the relation ρ_m if and only if $m \neq n$.*

Clearly, the operation f_n preserves the relation ρ_m if and only if for every $n \times m$ matrix whose rows belong to ρ_m, the m-tuple of column values of f_n belongs to ρ_m, as well.

The rows of the $m \times m$ matrix

$$\begin{pmatrix} 0 & 2 & 2 & \cdots & 2 \\ 2 & 0 & 2 & \cdots & 2 \\ 2 & 2 & 0 & \cdots & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2 & 2 & 2 & \cdots & 0 \end{pmatrix}$$

belong to ρ_m, however, the m-tuple of the column values of f_m is $(2, 2, 2, \dots, 2)$ which is not in ρ_m. Therefore the operation f_m does not preserve the relation ρ_m.

From now on, we will suppose that $m \neq n$. Let $H = (h_{i\,j})_{n\times m}$ be an arbitrary $n \times m$ matrix whose rows belong to ρ_m. The rows and the transposed columns of H will be denoted by r_i and c_j $(1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant m)$, and we set $h = (f_n(c_1), \dots, f_n(c_m))$. Our aim is to prove that $h \in \rho_m$.

If there exist indices $1 \leqslant j < j' \leqslant m$ such that $c_j = c_{j'} = (0, 0, 0, \dots, 0)$ then by Claim 4.4 (b), $r_i = (0, 0, 0, \dots, 0)$ for all i $(1 \leqslant i \leqslant n)$. Then $h = (0, 0, 0, \dots, 0) \in \rho_m$.

If there is exactly one $j_0 \in \{1, \dots, m\}$ for which $c_{j_0} = (0, 0, 0, \dots, 0)$ then by Claim 4.4 (a), the rows of H belong to $\big\{(0, \dots, 0)\big\} \cup U_m \cup V_m$. If every row of H belongs to $U_m \cup V_m$ then $c_j \in \{1, 2\}^n$ for all j $(1 \leqslant j \leqslant m,\ j \neq j_0)$. Therefore $f_n(c_j) = 1$ for all j $(1 \leqslant j \leqslant m,\ j \neq j_0)$. Since $f_n(c_{j_0}) = 0$ we get that $h \in V_m \subseteq \rho_m$. If there exist indices $1 \leqslant i < i' \leqslant n$ such that $r_i = r_{i'} = (0, 0, 0, \dots, 0)$ then for all j $(1 \leqslant j \leqslant m,\ j \neq j_0)$ we get that $f_n(c_j) = 1$. Hence, $h \in V_m \subseteq \rho_m$ since $f_n(c_{j_0}) = 0$. If there is exactly one $i_0 \in \{1, \dots, n\}$ such that $r_{i_0} = (0, 0, 0, \dots, 0)$ then either there is an $i_1 \in \{1, \dots, n\}$, $i_1 \neq i_0$, such that $r_{i_1} \in V_m$ or $r_1 = \cdots = r_{i_0-1} = r_{i_0+1} = \cdots = r_n \in U_m$. In the first case H is of the form

$$\begin{array}{c c} & \begin{array}{cccccc} & & & j_0 & & \ \end{array} \\ \begin{array}{c} \\ i_0 \\ \\ i_1 \\ \\ \end{array} & \begin{pmatrix} \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 1 & \cdots & 1 & 0 & 1 & \cdots & 1 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \end{pmatrix} \end{array},$$

and so, $h = (1, \dots, 1, 0, 1, \dots, 1) \in V_m \subseteq \rho_m$. In the second case we get that H is of the form

$$\begin{array}{c c} & \begin{array}{cccccc} & & & j_0 & & \ \end{array} \\ \begin{array}{c} \\ \\ \\ i_0 \\ \\ \\ \\ \end{array} & \begin{pmatrix} 2 & \cdots & 2 & 0 & 2 & \cdots & 2 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 2 & \cdots & 2 & 0 & 2 & \cdots & 2 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 2 & \cdots & 2 & 0 & 2 & \cdots & 2 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 2 & \cdots & 2 & 0 & 2 & \cdots & 2 \end{pmatrix} \end{array},$$

and so, $h = (2, \dots, 2, 0, 2, \dots, 2) \in U_m \subseteq \rho_m$.

If $c_j \neq (0, 0, 0, \dots, 0)$ for all j $(1 \leqslant j \leqslant m)$ then $f_n(c_j) \in \{1, 2\}$ for all j $(1 \leqslant j \leqslant m)$, and so $h \in \{1, 2\}^m$. Our aim is to show that $h \neq (2, 2, 2, \dots, 2)$. By the definition of f_n, the equality $h = (2, 2, 2, \dots, 2)$ holds if and only if $c_1, \dots, c_m \in U_n$. Since $U_n \subseteq \{0, 2\}^n$ we get that if H has a 1 entry then $h \neq (2, 2, 2, \dots, 2)$. Further on we may suppose that all entries of H are 0 or 2. If there is an $i_0 \in \{1, \dots, n\}$ such that $r_{i_0} = (0, 0, 0, \dots, 0)$ then there must be a column c_{j_0} of H such that $c_{j_0} \notin U_n$.

Then $f_n(c_{j_0}) = 1$ implies that $h \in W_m \subseteq \rho_m$. Otherwise, $r_i \neq (0,0,0,\dots,0)$ for all i $(1 \leqslant i \leqslant n)$. Therefore, every row of H contains at most one 0 entry. Since $m \neq n$, there is a column c_{j_0} of H which contains either no 0 entries or more than two 0 entries. Then $f_n(c_{j_0}) = 1$, and so $h \in W_m \subseteq \rho_m$. This completes the proof of Claim 4.6.

Let I be an arbitrary subset of $\{k \in \mathbb{N} \mid k \geqslant 3\}$, and set $F_I = \langle \{f_i \mid i \in I\} \cup M \rangle$. Then $F_I \subseteq \mathrm{Sta}(M)$. If $I_1, I_2 \subseteq \{k \in \mathbb{N} \mid k \geqslant 3\}$, $I_1 \neq I_2$, then we may suppose, without restricting generality, that there is an element $i \in \mathbb{N}$ such that $i \in I_1$ and $i \notin I_2$. Therefore $F_{I_2} \neq F_{I_1}$, since by Claims 4.5 and 4.6, $F_{I_2} \subseteq \mathrm{Pol}(\rho_i)$ but $F_{I_1} \nsubseteq \mathrm{Pol}(\rho_i)$. Hence,

$$2^{\aleph_0} = \left|\left\{F_I \,\middle|\, I \subseteq \{k \in \mathbb{N} \mid k \geqslant 3\}\right\}\right| \leqslant |\mathrm{Int}(\mathrm{IS}(\mathbf{L}))| \leqslant 2^{\aleph_0},$$

which proves that $|\mathrm{Int}(\mathrm{IS}(\mathbf{L}))| = 2^{\aleph_0}$. □

We conclude the paper with a discussion of lattices $\mathbf{L}$ for which the monoidal interval $\mathrm{Int}(\mathrm{IS}(\mathbf{L}))$ has cardinality $2^{\aleph_0}$. For elements $u \leqslant v$ of $\mathbf{L}$, we will use the notation $[u,v]$ for the interval $\{x \in L \mid u \leqslant x \leqslant v\}$. We will call a lattice **L pinched** if L contains an element $b \in L \setminus \{0,1\}$ such that $L = [0,b] \cup [b,1]$.

Theorem 4.7. *Let* $\mathbf{L}$ *be a pinched lattice, and let* $b \in L \setminus \{0,1\}$ *be an element such that* $L = [0,b] \cup [b,1]$. *Then* $|\mathrm{Int}(\mathrm{IS}([0,b]))| \leqslant |\mathrm{Int}(\mathrm{IS}(\mathbf{L}))|$.

Proof. It is easy to see that we have $c \leqslant b$ or $b \leqslant c$ for every element $c \in L$. Hence, the element b is isolated. Let M and $\widetilde{M}$ be the inverse monoids $\mathrm{IS}(\mathbf{L})$ and $\mathrm{IS}([0,b])$, respectively.

Claim 4.8. $[0,b]$ *is closed under each operation* $f \in \mathrm{Sta}(M)$.

Let $c_1,\dots,c_n$ be arbitrary elements of $[0,b]$, and set $t = f(\varphi_{c_1},\dots,\varphi_{c_n})$. Then by Proposition 2.1 (b), there are similar elements $u,v \in L$ and an isomorphism $\beta\colon (u] \to (v]$ such that $t = \beta_{u,v} \circ \varphi_u$. Since $c_1,\dots,c_n \leqslant b$, we get that

$$t(1) = f(c_1,\dots,c_n) = f(b \wedge c_1,\dots,b \wedge c_n) = t(b).$$

Then by Proposition 2.1 (b), $f(c_1,\dots,c_n) = v \leqslant b$. Hence, $f(c_1,\dots,c_n) \in [0,b]$. This proves Claim 4.8.

Claim 4.9. *For all transformations* $m \in M$ *and for all elements* $c \in [b,1]$ *we have that* $m(c) \wedge b = m(b) \wedge b$.

By Proposition 2.1 (b), there are similar elements $u,v \in L$ and isomorphism $\beta_{u,v}\colon (u] \to (v]$ such that $m = \beta_{u,v} \circ \varphi_u$. If $u \leqslant b$ then again by Proposition 2.1 (b), we get that $m(c) = m(b)$ for all $c \in [b,1]$. Hence, $m(c) \wedge b = m(b) \wedge b$ for all $c \in [b,1]$. On the other hand, if $u > b$ then by Proposition 2.1 (a) and by the fact that b is isolated, we get that $b = m(b) < m(u) = v$. Hence, $b = m(b) \wedge b \leqslant m(c) \wedge b \leqslant b$ implies that $m(c) \wedge b = m(b) \wedge b$ for all $c \in [b,1]$. This proves Claim 4.9.

Let u be an arbitrary element of the interval $[0,b]$. By Claim 4.8, the unary operation φ_u can be restricted to $[0,b]$, the restriction $\varphi_u|_{[0,b]}$ will be denoted by $\widetilde{\varphi}_u$.

Claim 4.10. $M|_{[0,b]} = \widetilde{M}$.

First we prove that $M|_{[0,b]} \subseteq \widetilde{M}$. Let t be an arbitrary transformation from M. Then by Proposition 2.1 (b), there are similar elements $u, v \in L$ and an isomorphism $\beta_{u,v}\colon (u] \to (v]$ such that $t = \beta_{u,v} \circ \varphi_u$. If $u \leqslant b$ then $v \leqslant b$ also holds, since $u \sim v$. Therefore the elements u, v are similar in the interval $[0, b]$. Hence, $t|_{[0,b]} = \beta_{u,v} \circ \widetilde{\varphi}_u \in \widetilde{M}$. If $u > b$ then $v > b$, since $u \sim v$. Furthermore, $\beta_{u,v}|_{[0,b]}$ is the isomorphism $\beta_{b,b}\colon [0, b] \to [0, b]$, $c \mapsto \beta_{u,v}(c)$, since b is isolated. Hence, for all $c \in [0, b]$ we get that $t(c) = \beta_{u,v}(c \wedge u) = \beta_{u,v}(c) = \beta_{b,b}(c) = \beta_{b,b}(c \wedge b)$. Thus, $t|_{[0,b]} = \beta_{b,b} \circ \widetilde{\varphi}_b \in \widetilde{M}$.

To prove the reverse inclusion, choose an arbitrary transformation $s \in \widetilde{M}$. Define the unary transformation f_s on L as follows:

$$f_s\colon L \to L,\ f_s(x) = s(x \wedge b).$$

By Proposition 2.1 (b), there are similar elements $u, v \in [0, b]$ such that $s = \beta_{u,v} \circ \widetilde{\varphi}_u$. If $x \leqslant b$ then $x \wedge b = x$ and $f_s(x) = s(x \wedge b) = s(x) = \beta_{u,v}(x \wedge u)$. If $x \geqslant b$ then $x \wedge b \geqslant u$ and $f_s(x) = s(x \wedge b) = \beta_{u,v}((x \wedge b) \wedge u) = \beta_{u,v}(x \wedge u)$. Hence, $f_s = \beta_{u,v} \circ \varphi_u \in M$. This concludes the proof of Claim 4.10.

Let g be an arbitrary n-ary operation from $\mathrm{Sta}(\widetilde{M})$, and define the n-ary operation f_g on L as follows:

$$f_g\colon L^n \to L,\ (a_1, \dots, a_n) \mapsto g(a_1 \wedge b, \dots, a_n \wedge b).$$

Claim 4.11. *For all $g \in \mathrm{Sta}(\widetilde{M})$ we have that $f_g|_{[0,b]} = g$ and $f_g \in \mathrm{Sta}(M)$.*

It is straightforward to check that the first statement is true. To prove the second statement choose arbitrary transformations $m_1, \dots, m_n \in M$, and set $t = f_g(m_1, \dots, m_n)$. By Proposition 2.1 (b), there are similar elements $u_i, v_i \in L$ and isomorphisms $\beta_i\colon (u_i] \to (v_i]$ for every i ($1 \leqslant i \leqslant n$) such that $m_i = \beta_{u_i,v_i} \circ \varphi_{u_i}$. As $m_1|_{[0,b]}, \dots, m_n|_{[0,b]} \in \widetilde{M}$ by Claim 4.10, we get that

$$t|_{[0,b]} = f_g|_{[0,b]}(m_1|_{[0,b]}, \dots, m_n|_{[0,b]}) = g(m_1|_{[0,b]}, \dots, m_n|_{[0,b]}) \in \widetilde{M}.$$

Then by Proposition 2.1 (b), there are similar elements $u, v \in [0, b]$ and an isomorphisms $\beta\colon (u] \to (v]$ such that $t|_{[0,b]} = \beta_{u,v} \circ \widetilde{\varphi}_u$. By Claim 4.9, for all $c \geqslant b$ we get that

$$\begin{aligned} t(c) &= f_g(m_1(c), \dots, m_n(c)) \\ &= g(m_1(c) \wedge b, \dots, m_n(c) \wedge b) \\ &= g(m_1(b) \wedge b, \dots, m_n(b) \wedge b) \\ &= t(b). \end{aligned}$$

Thus, $t = \beta_{u,v} \circ \varphi_u \in M$. This proves Claim 4.11.

For an arbitrary clone $\mathcal{D} \in \mathrm{Int}(\widetilde{M})$ define the clone $\mathcal{C}_{\mathcal{D}}$ in the following way:

$$\mathcal{C}_{\mathcal{D}} = \langle \{f_g \mid g \in \mathcal{D}\} \rangle .$$

Since $M \subseteq \{f_g \mid g \in \mathcal{D}\} \subseteq \mathrm{Sta}(M)$, we get that $\mathcal{C}_{\mathcal{D}}$ is in $\mathrm{Int}(M)$. Furthermore, by Claims 4.10 and 4.11, we get that

$$\mathcal{C}_{\mathcal{D}}|_{[0,b]} = \big\langle \big\{ f_g|_{[0,b]} \mid g \in \mathcal{D} \big\} \big\rangle = \langle \{ g \mid g \in \mathcal{D} \} \rangle = \mathcal{D}.$$

Hence, the map

$$\varphi \colon \mathrm{Int}(\widetilde{M}) \to \mathrm{Int}(M), \; \mathcal{D} \mapsto \mathcal{C}_{\mathcal{D}}$$

is an injection, which proves that $|\mathrm{Int}(\mathrm{IS}([0,b]))| \leqslant |\mathrm{Int}(\mathrm{IS}(\mathbf{L}))|$. □

Corollary 4.12. *If* $\mathbf{L}$ *is a finite lattice which has a unique atom then* $\mathrm{Int}(\mathrm{IS}(\mathbf{L}))$ *is infinite.*

Proof. Let $0 \prec b$ be the unique atom of $\mathbf{L}$. Then $\mathbf{L}$ is pinched: $L = [0,b] \cup [b,1]$. Therefore by Theorem 4.7, $\aleph_0 = |\mathrm{Int}(\mathrm{IS}([0,b]))| \leqslant |\mathrm{Int}(\mathrm{IS}(\mathbf{L}))|$. □

Corollary 4.13. *If* $\mathbf{L}$ *is a finite chain with at least 3 elements then* $\mathrm{Int}(\mathrm{IS}(\mathbf{L}))$ *has cardinality* $2^{\aleph_0}$.

Proof. We may assume that $\mathbf{L}$ is the n-element chain on the set $\{0,1,2,\ldots,n-1\}$ $(n \geqslant 3)$ with the order $0 < 1 < 2 < \cdots < n-1$. Then the lattice $\mathbf{L}$ is pinched: $L = [0,2] \cup [2,n-1]$. Therefore by Theorems 4.7 and 4.2, $2^{\aleph_0} = |\mathrm{Int}(\mathrm{IS}([0,2]))| \leqslant |\mathrm{Int}(\mathrm{IS}(\mathbf{L}))|$. Hence, $|\mathrm{Int}(\mathrm{IS}(\mathbf{L}))| = 2^{\aleph_0}$. □

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