Intervals of Collapsing Monoids

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June 14, 2001

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Abstract [1]

We present some new families of collapsing monoids. These monoids form large intervals in the submonoid lattices of the full transformation semigroups. Some of these intervals have cardinalities $\geq 2^{2^{cn}}$ where n is the size of the base set.

1 Introduction

A set of operations on a set A is said to be a *clone* if it contains the projections and is closed under superposition. It is easy to see that the unary operations in a clone form a transformation monoid. This monoid will be called the unary part of the clone.

Throughout this paper we will assume that the base set A is finite. It is well known that, in this case, for an arbitrary transformation monoid M on A the clones whose unary part is M form an interval in the lattice of clones. This interval will be denoted by $I(M)$. Note that the clone of essentially unary operations generated by M is always a member of $I(M)$, so $I(M)$ is not empty. As there are only finitely many transformation monoids on A, the intervals $I(M)$ partition the lattice of clones on A into finitely many blocks. Since the lattice of clones on A has cardinality $2^{\aleph_0}$ if $|A| \geq 3$, one expects that "in most cases" $I(M)$ contains uncountably many clones. However, it turns out that for many interesting transformation monoids M the interval $I(M)$ is finite. So, studying these intervals may lead to a better understanding of some parts of the lattice of clones.

[1] 2000 *Mathematics Subject Classification*: 08A40

The problem of classifying those transformation monoids M for which $I(M)$ is finite (or uncountable) was posed in [16]. The one-element transformation monoid is an example for an M such that $I(M)$ has cardinality $2^{\aleph_0}$ (cf. [12]). The first explicit construction of a transformation monoid M with $|I(M)| = \aleph_0$ is due to Krokhin [7]. The full transformation semigroup on A is an example of a monoid M such that $I(M)$ is finite; in fact, in this case $I(M)$ is an $(|A|+1)$-element chain (Burle [1]). A large family of monoids M with $I(M)$ finite is provided by Pálfy's theorem ([13]): if M consists of all constants and some permutations, then $|I(M)| \leq 2$; moreover, $|I(M)| = 1$ unless M coincides with the monoid of all unary polynomial operations of a finite vector space over a finite field.

If the interval $I(M)$ has only one element, then the transformation monoid M is called *collapsing*. In this case the only element of $I(M)$ is the clone of essentially unary operations generated by M. The aim of this paper is to construct large intervals of transformation monoids such that all members of these intervals are collapsing. The construction is presented in Section 2. In Section 3 we give a complete list of collapsing transformation monoids on a three-element set. A computer study of these monoids led to the construction discussed in Section 2.

2 Intervals of collapsing monoids

Let M be a transformation monoid on a finite set A. A k-ary operation $f \in \mathcal{O}_A$ can be a member of a clone with unary part M only if $f(m_1(x), \ldots, m_k(x))$ belongs to M for all $m_1, \ldots, m_k \in M$. It is not hard to see that the operations f satisfying this condition form a clone with unary part M. Hence this is the largest clone with unary part M, and is called the *stabilizer of* M. We say that a binary operation f is *essentially binary* if f depends on both of its variables. By a result of Grabowski [5], a transformation monoid M is collapsing if and only if its stabilizer contains no essentially binary operations.

Let A be a finite set with at least four elements. Let P, Q, and R be pairwise disjoint nonempty subsets of A such that $|R| \geq 2$. Let $T(P,Q,R)$ be the set of all transformations $t \in T(A)$ such that for all $p \in P, q \in Q$ and $r, r' \in R$ with $r \neq r'$, if $t(r) = t(r')$ then $t(p) \in \{t(q), t(r)\}$. Let M be an arbitrary transformation monoid on A. The monoid M is said to be *rich* with respect to P, Q, R if for some $s \in A$, and for all $a, b \in A$ such that $a \neq b$ and $s \in \{a, b\}$, M contains transformations t_1 and t_2 such that $t_1(P) = t_1(Q) = \{a\}$, $t_1(R) = \{b\}$ and $t_2(P) = t_2(R) = \{a\}$, $t_2(Q) = \{b\}$. If P, Q, R are clear from the context, then we will simply say that M is rich.

Example 2.1 Let B be the set $\{0, 1, 2, 3\}$, and M be the monoid of all transformations $t \in T(B)$ such that $t(2) = t(3)$ and $t(0) \in \{t(1), t(2)\}$, or t is the identity operation. If we choose the sets $P = \{0\}$, $Q = \{1\}$ and $R = \{2, 3\}$, then it is obvious that the monoid M is rich, and it is contained in $T(P, Q, R)$.

The main result of this paper is the following.

Theorem 2.2 *Let A be a finite set with at least four elements, and let P, Q, R be disjoint nonempty subsets of A such that $|R| \geq 2$. Then every rich monoid $M \subseteq T(P, Q, R)$ is collapsing.*

The proof of Theorem 2.2 is based on the lemma below, which states that in the operation table of an essentially binary operation a particular configuration always occurs.

Lemma 2.3 *Let $f \in \mathcal{O}_A$ be an essentially binary operation on a finite set A with at least two elements. Then, for every element $s \in A$ there exist $a, b, c, d \in A$ such that $s \in \{a, b\} \cap \{c, d\}$ and $f(a, d) \neq f(b, d) \neq f(b, c)$.*

Proof. Let s be a fixed element of A. Suppose first that the unary operation $f(s, x)$ is constant. Since f depends on its second variable, there are elements $u, v \in A$ such that $f(u, s) \neq f(u, v)$. Thus, if $f(s, v) = f(u, v)$, then $f(s, s) = f(s, v) = f(u, v) \neq f(u, s)$, so the elements $a = s, b = u, c = v, d = s$ satisfy the requirements. Otherwise we have $f(s, v) \neq f(u, v) \neq f(u, s)$, hence the choice $a = s, b = u, c = s, d = v$ is appropriate. The same argument yields suitable elements if the unary operation $f(x, s)$ is constant. Finally, if none of the unary operations $f(s, x)$ and $f(x, s)$ are constant, then there are elements $a, c \in A$ such that $f(a, s) \neq f(s, s) \neq f(s, c)$, hence the elements $a, b = s, c$, and $d = s$ satisfy the requirements of the lemma. ■

Proof of Theorem 2.2 Let M be a monoid which is rich with respect to P, Q, R, and let $s \in A$ be an element witnessing the richness of M.

Choose elements $p, q, r, r' \in A$ such that $p \in P,\ q \in Q$, and $r, r' \in R$ with $r \neq r'$. We will show that the stabilizer of M contains no essentially binary operation. By Grabowski's result in [5] this will imply that M is collapsing. Let $f \in \mathcal{O}_A$ be an essentially binary operation. By Lemma 2.3, there are elements a, b, c, d such that $s \in \{a, b\} \cap \{c, d\}$ and $f(a, d) \neq f(b, d) \neq f(b, c)$. Since M is a rich monoid, there are transformations $t', t'' \in M$ such that

$$t'(p) = t'(q) = b,\ t'(r) = t'(r') = a$$

and

$$t''(p) = t''(r) = t''(r') = d,\ t''(q) = c.$$

Let $t(x) = f(t'(x), t''(x))$ $(x \in A)$. Then

$$t(p) = f(b, d),\ t(q) = f(b, c) \quad \text{and} \quad t(r) = t(r') = f(a, d).$$

If $f(a, d) \neq f(b, c)$ then $t(p), t(q)$ and $t(r)$ are pairwise distinct, while, if $f(a, d) = f(b, c)$ then $t(p) \neq t(q) = t(r)$. Hence, in both cases, the transformation t is not in $T(P, Q, R)$, thus, t is not in M. This proves that the operation f doesn't belong to the stabilizer of M. ■

Now we show that for a finite set A with $|A| \geq 6$ in the submonoid lattice of $T(A)$ there are large intervals which contain only collapsing monoids. In the sequel, the monoid generated by a set H of transformations will be denoted by $\langle H \rangle$. If $H = \{t_1, \ldots, t_r\}$ then we will write $\langle t_1, \ldots, t_r \rangle$ instead of $\langle \{t_1, \ldots, t_r\} \rangle$.

Let A be a finite set with $|A| \geq 6$. Let the elements $p, q, r, r' \in A$ be pairwise distinct, and let $P = \{p\},\ Q = \{q\},\ R = \{r, r'\}, A' = A \backslash (P \cup Q \cup R)$. We define the monoid N on A to be the monoid of all transformations $t \in T(P, Q, R)$ for which $t(r) = t(r')$ and the restriction of t onto A' is the identity operation on A', or t is the identity operation. For an arbitrary monoid $K \in T(A')$ we will denote by $\hat{K}$ the monoid which consists of all transformations from $T(A)$ whose restriction onto A' is a member of K, and whose restriction onto the set $P \cup Q \cup R$ is the identity transformation. Since $t \in \langle N \cup \hat{K} \rangle$ implies that $t|_{A'} \in K$, we get that if K_1, K_2 are submonoids of $T(A')$ and $K_1 \neq K_2$ then $\langle N \cup \hat{K}_1 \rangle \neq \langle N \cup \hat{K}_2 \rangle$. Furthermore, $\langle N \cup \widehat{T(A')} \rangle \subseteq T(P, Q, R)$, and any element of A witnesses the richness of N.

Proposition 2.4 *Let A be a finite set with $|A| = n \geq 6$. Then all members of the interval $[N, \langle N \cup \widehat{T(A')} \rangle]$ are collapsing, and this interval has cardinality greater than $2^{2^{cn}}$ for some positive constant c.*

To prove Proposition 2.4, we need a simple lower bound of the cardinality of the subsemigroup lattice of $T(A)$.

Lemma 2.5 *Let A be a finite set with $|A| = n \geq 2$. Then the full transformation semigroup $T(A)$ has at least $2^{2^{Cn}}$ subsemigroups for some positive constant C.*

Proof. If $|A| = 2$ then $|T(A)| = 9 > 2^{2^{\frac{3}{4} \cdot 2}}$. For $|A| = n \geq 3$ let $P'(A)$ be the set of all subsets of A whose cardinality is $[n/2]$. Let U be an arbitrary element of $P'(A)$, and let M_U be the semigroup of all transformations whose ranges are contained in U. It is easy to see that if H is a subset of $P'(A)$ then $T_H = \bigcup_{U \in H} M_U$ is a subsemigroup of $T(A)$. Furthermore, if $H_1, H_2 \subseteq P'(A)$ and $H_1 \neq H_2$ then $T_{H_1} \neq T_{H_2}$. Thus, we have that the subsemigroups T_H $(H \subseteq P'(A))$ are pairwise distinct, and

$$|\{T_H \mid H \subseteq P'(A)\}| = 2^{|P'(A)|} = 2^{\binom{n}{[n/2]}} \geq 2^{4^{[n/2]}/(2[n/2])} \geq 2^{2^{c_1 n}}$$

for some positive constant c_1. If we choose $C = \min\{\frac{3}{4}, c_1\}$ then C will satisfy the requirement of the lemma. ■

Proof of Proposition 2.4. Since the monoid N is rich and $\langle N \cup \widehat{T(A')} \rangle \subseteq T(P, Q, R)$, we see from Theorem 2.2 that every monoid in the interval $[N, \langle N \cup \widehat{T(A')} \rangle]$ is collapsing. Furthermore, this interval has cardinality greater than the number of subsemigroups of $T(A')$, hence by Lemma 2.5 we have $|[N, \langle N \cup \widehat{T(A')} \rangle]| \geq 2^{2^{C(n-4)}} \geq 2^{2^{cn}}$ for some positive constant c. ■

3 Collapsing monoids on a three element set

In this section we will describe all collapsing monoids on a 3-element set.

Let A be a 3-element set. We will define two sets of transformations on A. Let $p, s \in A$ be arbitrary elements of A. Let T_p denote the set of all transformations $t \in T(A)$ such that either t is a permutation fixing p or t is not a permutation, and $t(p) \in \{t(q), t(r)\}$ for $\{p, q, r\} = A$. Furthermore, let $M_{p,s}$ be the set of all transformations $t \in T_p$ such that $t(A) \subseteq \{s, a\}$ for some $a \in A \setminus \{s\}$ or t is the identity transformation. It is easy to see that both T_p and $M_{p,s}$ are transformation monoids on A. For these monoids we get a result analogous to Theorem 2.2.

Theorem 3.1 *Let A be a 3-element set. Then each monoid $M \subseteq T(A)$ for which there are elements $p, s \in A$ such that $M_{p,s} \subseteq M \subseteq T_p$ is collapsing.*

Proof. Let M be a monoid on A such that $M_{p,s} \subseteq M \subseteq T_p$ for some $p, s \in A$. As in Theorem 2.2, it suffices to show that the stabilizer of M contains no essentially binary operation. Let $f \in \mathcal{O}_A$ be an essentially binary operation, and suppose that f is in the stabilizer of M. By Lemma 2.3, there are elements $a, b, c, d \in A$ such that $s \in \{a, b\} \cap \{c, d\}$ and $f(a, d) \neq f(b, d) \neq f(b, c)$. Since $M_{p,s} \subseteq M$, there are transformations $t_1, t_2 \in M$ such that

$$t_1(p) = t_1(q) = b,\ t_1(r) = a \quad \text{and} \quad t_2(p) = t_2(r) = d,\ t_2(q) = c.$$

Let $t(x) = f(t_1(x), t_2(x))$ $(x \in A)$. Then $t \in M$, and

$$t(p) = f(b, d),\ t(q) = f(b, c) \quad \text{and} \quad t(r) = f(a, d).$$

Thus $t(p) \notin \{t(q), t(r)\}$. By the assumptions on M, we have that $t \in T_p$, therefore we get that t is a permutation which fixes the element p. Hence $p = f(b, d)$. We will show that this leads to a contradiction. Since the elements $f(b, d), f(b, c)$, and $f(a, d)$ are pairwise distinct, we have that $f(a, c) \in \{f(b, d), f(b, c), f(a, d)\}$.

Case 1: $f(a, c) = f(a, d)$ or $f(a, c) = f(b, c)$. Assume the first equality holds, and let $t_3, t_4 \in M_{p,s}$ be transformations such that

$$t_3(p) = t_3(r) = b,\ t_3(q) = a \quad \text{and} \quad t_4(p) = t_2(q) = c,\ t_4(r) = d.$$

Let $t'(x) = f(t_3(x), t_4(x))$ $(x \in A)$. Then t' is a permutation and $t'(p) = f(b, c) \neq p$, which is a contradiction. Mutatis mutandis for the case $f(a, c) = f(b, c)$.

Case 2: $f(a,c) = f(b,d)$. Let $t_5, t_6 \in M_{p,s}$ be transformations such that

$$t_5(p) = t_5(q) = b,\ t_5(r) = a \quad \text{and} \quad t_6(p) = t_6(r) = c,\ t_6(q) = d.$$

Let $t''(x) = f(t_5(x), t_6(x))$ $(x \in A)$. Then $t'' \in M$, and $t''(r) = t''(q) \neq t''(p)$, which is again a contradiction. ■

Now we are in a position to give a complete list of collapsing monoids on a 3-element set.

Let A be a 3-element set. Without loss of generality, we may assume that $A = \{0,1,2\}$. Let M_1 and M_2 be submonoids of $T(A)$. We say that M_1 is *equivalent* to M_2, and we write $M_1 \bowtie M_2$, iff there is a permutation α on A such that $M_2 = \{\alpha^{-1} m \alpha \mid m \in M_1\}$, that is M_2 is the conjugate of M_1 by α. It is straightforward to check that $\bowtie$ is an equivalence relation on the set of submonoids of $T(A)$. Furthermore, if $M_1 \bowtie M_2$ then $I(M_1) \cong I(M_2)$. On a 3-element set the subsemigroups of $T(A)$ were described by Lau [10], [11]. Using this description, we get 699 submonoids on A in 160 $\bowtie$-classes. We will use several earlier results to obtain the complete list of collapsing monoids on A. These results are due to Demetrovics–Hannák [2], [3], Fearnley–Rosenberg [4], Krokhin [8], [9], Pálfy [13], Pálfy–Szendrei [14]. We will use the following notation for the constants and the permutations in $T(A)$:

x	$c_0(x)$	$c_1(x)$	$c_2(x)$	$\mathrm{id}(x)$	$\tau_0(x)$	$\tau_1(x)$	$\tau_2(x)$	$\sigma(x)$	$\sigma^2(x)$
0	0	1	2	0	0	2	1	1	2
1	0	1	2	1	2	1	0	2	0
2	0	1	2	2	1	0	2	0	1

Theorem 3.2 *On the 3-element set $A = \{0,1,2\}$ there are 27 collapsing monoids in 10 $\bowtie$-classes. If M is a collapsing monoid on A, then M is equivalent to exactly one of the following monoids:*

(1) $\langle c_0, \tau_2 \rangle = \{id, c_0, c_1, \tau_2\}$,

(2) $\langle c_0, c_1, c_2 \rangle = \{id, c_0, c_1, c_2\}$,

(3) $\langle c_0, c_2, \tau_2 \rangle = \{id, c_0, c_1, c_2, \tau_2\}$,

(4) $\langle c_0, \sigma \rangle = \{id, c_0, c_1, c_2, \sigma, \sigma^2\}$,

(5) S_3,

(6) $M_{2,0}$,

(7) $M_{2,2}$,

(8) $\langle M_{2,2}, \tau_2\rangle = M_{2,2} \cup \{\tau_2\}$,

(9) $T_2 \setminus \{\tau_2\}$,

(10) T_2,

where T_2 is the monoid of all transformations $t \in T(A)$ such that either t is a permutation fixing 2 or $t(2) \in \{t(0), t(1)\}$, while $M_{2,s}$ ($s \in \{0, 2\}$) is the monoid of all transformations $t \in T_2$ such that $t(A) \subseteq \{s, a\}$ for some $a \in A \setminus \{s\}$ or t is the identity transformation.

Proof. From results in [4], [13] and [14], it follows that the monoids (1), (2)–(4), and (5), respectively, are collapsing, while for the monoids (6)–(10) this property is the consequence of Theorem 3.1. To check that transformation monoids that are equivalent to neither of (1)–(10) are not collapsing we used the results in [1], [2], [3], [6], [8], [9], [12], [13], [14], and a computer program (written in PASCAL) based on the result of Grabowski [5]. ■

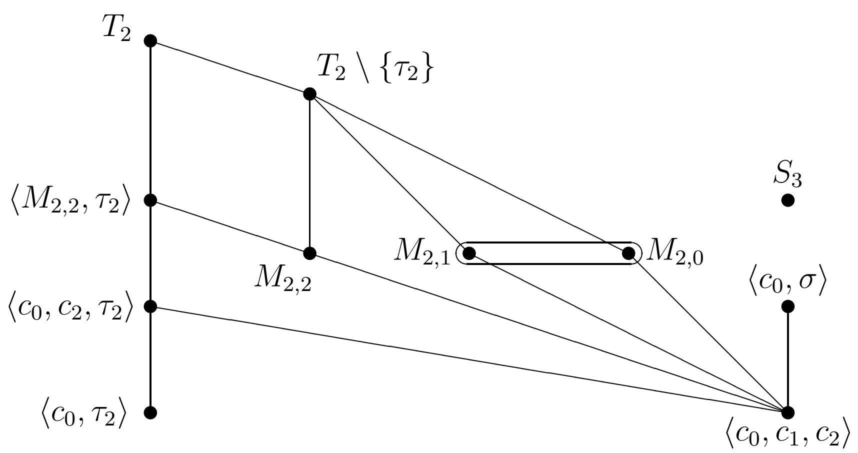

Figure 1

In Figure 1 a fragment of the poset of collapsing monoids on $\{0, 1, 2\}$ can be seen. The whole poset can be obtained by rotating this fragment about the "axis" S_3, $\langle c_0, \sigma \rangle$, $\langle c_0, c_1, c_2 \rangle$ through $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$. Rotating the whole poset through $\frac{2\pi}{3}$ corresponds to conjugating the monoids by σ, and $M_{2,1}$ is the conjugate of $M_{2,0}$ by τ_2. Hence the three monoids on the "axis" form singleton $\bowtie$-classes, the $\bowtie$-class of $M_{2,0}$ has six elements, while all the other $\bowtie$-classes contain exactly three elements.

Acknowledgements

The author wishes to express his thanks to Á. Szendrei and to the Referee for their helpful suggestions.

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