SLIM SEMIMODULAR LATTICES. I. A VISUAL APPROACH

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Abstract. A finite lattice L is called *slim* if no three join-irreducible elements of L form an antichain. Slim lattices are planar. After exploring some elementary properties of slim lattices and slim semimodular lattices, we give two visual *structure theorems* for slim semimodular lattices.

1. Introduction

By a *slim lattice* we mean a finite lattice M such that $J(M)$, the poset (partially ordered set) of its non-zero join-irreducible elements, contains no three-element antichain. By R. P. Dilworth [4], a finite lattice M is slim iff $J(M)$ is the union of two chains. By Lemma 6 of [3], slim lattices are *planar*. So they are relatively easy objects to understand.

A lattice L is called (upper) *semimodular*, if $b \vee c$ covers or equals $a \vee c$ for all $a, b, c \in L$ with $a \prec b$. Because of their links to combinatorics and geometry, the study of these lattices is an important branch of Lattice Theory; see M. Stern [11] for a survey. See also [1] and [2] for recent developments.

Semimodular lattices have recently proved to be useful in strengthening a classical group theoretical result, namely, the Jordan-Hölder theorem. G. Grätzer and J. B. Nation [9] proved that given two composition series of a group, there is a matching between their factors such that the corresponding factors are isomorphic for a very specific reason: they are related by the composite of a down-perspectivity with an up-perspectivity. In [3], this matching is shown to be unique. The main role in [3] is played by *slim* semimodular lattices (introduced in G. Grätzer and E. Knapp [6]), due to the fact that any two finite maximal chains of a semimodular lattice generate a join-subsemilattice that is a slim semimodular lattice.

As it has been pointed out by G. Grätzer and E. Knapp [6] (see Proposition 9 below), planar semimodular lattices can easily be obtained from slim ones. This way slim semimodular lattices play an important role in a series of papers by G. Grätzer and E. Knapp [6]–[8] on the Congruence Lattice Representation problem.

These developments motivate a deeper study of slim semimodular lattices. Our *main results*, the twin Theorems 11 and 12, are constructive visual structure theorems of these lattices. While it seems to be difficult to provide many examples of small (and, preferably, planar) semimodular lattices when one is getting acquainted with Lattice Theory, this should not be a problem using Theorems 11 and 12. Some easy results on slim lattices and planar lattices are also surveyed or proved.

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All lattices in this paper are assumed to be *finite*. We will rely, sometimes only implicitly, on the rigorous study of *planar lattices* by D. Kelly and I. Rival [10].

2. Definitions and elementary facts

A finite lattice L is *planar* if it has a planar diagram, that is a diagram in which the edges intersect can only have their endpoints in common. A planar lattice is finite by definition. The edges of a planar diagram divide the plane into *regions*. The minimal regions are called *cells*, exemplified by the five-element non-distributive lattices: N_5 has only one cell while M_3 has two. Note that a planar lattice has no cell iff it is a chain.

L is a *4-cell lattice* (see G. Grätzer and E. Knapp [6]) if it is planar and each cell is formed by exactly four edges. Then for each cell there are elements $a, b \in L$, called the *left corner* and the *right corner* of the cell, such that the cell is surrounded by its *lower edges* $a \wedge b \prec a$ and $a \wedge b \prec b$ and its *upper edges* $a \prec a \vee b$ and $b \prec a \vee b$, and a is to the left of b. The elements $a \wedge b$ and $a \vee b$ are called the *bottom* and the *top* of the cell, respectively. If $x_1 \prec y_1$ and $x_2 \prec y_2$ are edges of a 4-cell such that $\{x_1, y_1\} \cap \{x_2, y_2\} = \emptyset$, then these two edges are called *opposite edges* of the 4-cell. By a *covering square* we mean a subset $\{a \wedge b, a, b, a \vee b\}$ such that $a \wedge b \prec a$, $a \wedge b \prec b$, $a \prec a \vee b$, and $b \prec a \vee b$. Note that 4-cells are covering squares but, as it is exemplified by M_3, not conversely. For $a \in L$, the principal ideal $[0, a] = \{x \in L : x \leq a\}$ and the principal filter $[a, 1]$ will be denoted by $\downarrow a$ and $\uparrow a$, respectively.

The left boundary and the right boundary of L are denoted by $\mathcal{B}_{\text{left}}(L)$ and $\mathcal{B}_{\text{right}}(L)$, respectively. Their meaning should be clear, or see D. Kelly and I. Rival [10] for a rigorous technical definition. Note that $\mathcal{B}_{\text{left}}(L)$ and $\mathcal{B}_{\text{right}}(L)$ are maximal chains in L. The common name for $\mathcal{B}_{\text{left}}(L)$ and $\mathcal{B}_{\text{right}}(L)$ is *boundary chain*. The union $\mathcal{B}(L) := \mathcal{B}_{\text{left}}(L) \cup \mathcal{B}_{\text{right}}(L)$ of the boundary chains is the *boundary* of L.

Proposition 1 ([3] and, mainly, G. Grätzer and E. Knapp [6])**.** *For every finite lattice L, the following five conditions are equivalent:*

(i) *L is a slim semimodular lattice;*
(ii) *L is a slim semimodular 4-cell lattice;*
(iii) *L is a planar semimodular lattice without cover-preserving M_3-sublattices;*
(iv) *L is a planar semimodular lattice in which 4-cells and covering squares are the same.*
(v) *L is a 4-cell lattice in which no two distinct 4-cells have the same bottom.*

Proof. The equivalence of the first four conditions is stated in Lemma 7 of [3], whose proof heavily relies on G. Grätzer and E. Knapp [6]. Note that third condition is clearly equivalent with the definition of a slim semimodular lattice given in [6].

The first four conditions imply the fifth one by Lemma 7 of [6].

Assume the fifth condition. Then L is semimodular by Lemma 5 of [6]. If L had a cover-preserving M_3, then it would clearly have two distinct 4-cells with the same bottom. Hence the third condition follows. □

Semimodularity is not assumed in the next seven statements.

Lemma 2. *Each element of a slim lattice L has at most two covers.*

For a slim *semimodular* lattice, this is Lemma 6 of G. Grätzer and E. Knapp [8].

Proof of Lemma 2. Assume that $u \in L$ is covered by three distinct elements, v_1, v_2, and v_3. Then we can choose an element $p_i \in \big(J(L) \cap \downarrow v_i\big) \setminus \downarrow u$, for $i \in \{1, 2, 3\}$. Since $v_i = u \vee p_i$, we conclude that $\{p_1, p_2, p_3\}$ is a three-element antichain in $J(L)$, a contradiction. □

Let us recall the following lemma.

Lemma 3 (Lemma 1.2 of D. Kelly and I. Rival [10]). *Let $x \le y$ in a planar lattice L. If x and y are on different sides of a maximal chain C in L, then there is a $z \in C$ such that $x \le z \le y$.*

We will also need the following lemma.

Lemma 4. *If L is a planar lattice, a and b belong to the same boundary chain of L and $a \prec b$, then either a is meet-irreducible or b is join-irreducible.*

Proof. Suppose the contrary, and let $a, b \in \mathcal{B}_{\text{left}}(L)$ with $a \prec b$. Then there are elements a' and b' in L such that $a \prec b' \parallel b$ and $b \succ a' \parallel a$. Let $A = \downarrow a \cap \mathcal{B}_{\text{left}}(L)$ and $B = \uparrow a \cap \mathcal{B}_{\text{left}}(L)$; they are chains. Extend $\{a, b'\}$ to a maximal chain C of $\uparrow a$. The maximal chains B and C of $\uparrow a$ surround a region R of L. By Lemma 1.3 of D. Kelly and I. Rival [10], a is the least element of R. Hence $a' \notin R$, whence b and a' are on different sides of the maximal chain $A \cup C$. Lemma 3 yields an element $x \in A \cup C$ such that $x \in [a', b] = \{a, b\}$. This is a contradiction, because $a' \notin A \cup C$ and $b \notin A \cup C$. □

Proposition 5 (Lemmas 5 and 6 in [3]).

(i) *Slim lattices are planar.*
(ii) *Let $E = \{0 = e_0 \prec e_1 \prec \cdots \prec e_n\}$ and $F = \{0 = f_0 \prec f_1 \prec \cdots \prec f_m\}$ be non-empty chains of a finite lattice L such that $J(L) \subseteq E \cup F$. Then L has a planar diagram such that $\mathcal{B}_{\text{left}}(L) = E \cup \uparrow e_n$ and $\mathcal{B}_{\text{right}}(L) = F \cup \uparrow f_m$.*
(iii) *If e is a maximal element of $J(L)$, then $\uparrow e$ is a chain and $\uparrow e \subset \mathcal{B}(L)$.*

Let us call a finite lattice L *linearly indecomposable* if for each $x \in L \setminus \{0, 1\}$ there is a $y \in L$ such that x and y are incomparable. It follows easily from Lemma 1.3 of D. Kelly and I. Rival [10] that an arbitrary planar lattice L is linearly indecomposable iff $\mathcal{B}_{\text{left}}(L) \cap \mathcal{B}_{\text{right}}(L) = \{0, 1\}$.

Lemma 6. *If L is a slim lattice, then, for every planar diagram, $J(L) \subseteq \mathcal{B}(L)$.*

Proof. By way of contradiction, we assume that $p \in J(L) \setminus \mathcal{B}(L)$. Let q stand for the unique lower cover of p. Let u be the greatest element of $\downarrow p \cap \mathcal{B}_{\text{left}}(L)$, and let u^+ stand for its upper cover in $\mathcal{B}_{\text{left}}(L)$. Similarly, v denotes the greatest element of $\downarrow p \cap \mathcal{B}_{\text{right}}(L)$, and let $v^+ \in \mathcal{B}_{\text{right}}(L)$ such that $v \prec v^+$. Since $u^+ \not\le p$ and $p \ne u \in \mathcal{B}_{\text{left}}(L)$, the equation $u = u^+ \wedge p$ shows that u is meet-reducible. Hence Lemma 4 yields that $u^+ \in J(L)$, and $v^+ \in J(L)$ follows similarly.

If we had $p < u^+$, then $u \le q < p < u^+$ would contradict $u \prec u^+$. Hence $p \parallel u^+$, and the same reasoning yields that $p \parallel v^+$. Since L is slim, $\{p, u^+, v^+\}$ is not a three-element antichain. So we conclude that either $u^+ = v^+$ or, say, $u^+ < v^+$. If $u^+ = v^+$, then L is linearly decomposable at u^+, and $u^+ \not\parallel p$ is a contradiction. If $u^+ < v^+$, then the join-irreducibility of v^+ gives that $u^+ \le v < p$, a contradiction again. □

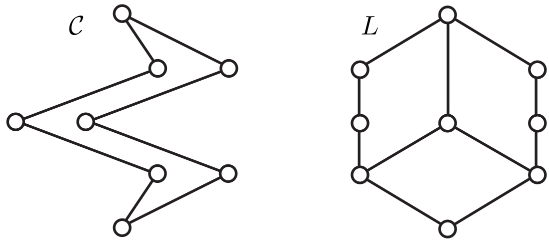

FIGURE 1. A contour $\mathcal{C}$ and a slim lattice L

Lemma 7. *Let L be a slim lattice. Then $\mathcal{B}(L)$ is uniquely determined. If, in addition, L is linearly indecomposable, then the boundary chains of L are also uniquely determined.*

Proof. If $c \in L$ is comparable with any other element of L, then c belongs to all boundary chains, since they are maximal chains. Thus, we can assume that L is linearly indecomposable and $|L| \geq 3$.

Since L is linearly indecomposable, it has exactly two atoms by Lemma 2. Let a_1 and b_1 be these atoms. They must belong to different boundary chains. Now we have a choice: let, say, a_1 belong to $\mathcal{B}_{\text{left}}(L)$. We intend to show that no more choice remains and

$$\mathcal{B}_{\text{left}}(L) = \{0 \prec a_1 \prec a_2 \prec \cdots \prec 1\},$$

$$\mathcal{B}_{\text{right}}(L) = \{0 \prec b_1 \prec b_2 \prec \cdots \prec 1\}$$

are uniquely defined. (Note that these boundary chains may be of different length.)

We prove by induction on k that, say, a_k is uniquely determined. Assume that $k > 1$ and a_{k-1} is uniquely determined. If a_{k-1} is meet-irreducible, then it has a unique cover y. Since $\mathcal{B}_{\text{left}}(L)$ is a maximal chain, $a_k = y$.

Next, assume that a_{k-1} is meet-reducible. Then it has exactly two covers, x and y by Lemma 2. We know from Proposition 1 that L is planar, so there is a left boundary chain and it contains x or y. Invoking Lemma 4 we infer that x or y is join-irreducible. If both x and y are join-irreducible, then they are on the boundary by Lemma 6, but they belong to different boundary chains, because $x \parallel y$. Their unique lower cover, the common a_{k-1}, belongs to both boundary chains. Hence L is linearly decomposable at a_{k-1}, a contradiction. Consequently, exactly one of the elements x and y is join-irreducible. This element is a_k by Lemma 6. ☐

The boundary $\mathcal{B}(L)$ of a planar lattice L is a poset. Note that $\mathcal{B}(L)$ is a (planar) lattice, but not a sublattice of L, in general. By a *contour* we mean a fixed planar diagram of a planar lattice M such that $M = \mathcal{B}(M)$. For a planar lattice L, we say that the *contour of L is arbitrary*, if L has the following property:

> for each contour $\mathcal{C}$ that is order-isomorphic to the boundary of L in some planar diagram, L has a planar diagram in which $\mathcal{B}(L)$ is congruent to $\mathcal{C}$ in the Euclidean metric.

Let us say that L satisfies the *Jordan-Hölder chain condition* if all of its maximal chains have the same length. It is well-known that finite semimodular lattices satisfy this condition. This allows us to define the *height* $h(x)$ of an element in a finite semimodular lattice: it is the length of any maximal chain of $[0, x]$.

While $\mathcal{C}$ and L in Figure 1 indicate that the contour of a planar lattice is *not* arbitrary, in general, we have the following statement.

Proposition 8. *Let L be a planar lattice satisfying the Jordan-Hölder chain condition. Then the contour of L is arbitrary.*

Proof. We prove the statement by induction on $|L|$. We can assume that $|L| \geq 4$, L is linearly indecomposable, and the statement holds for all lattices with less than $|L|$ elements. Consider a planar diagram of L, and let $\mathcal{C}$ be an arbitrary contour that is order isomorphic with $\mathcal{B}(L)$; let $\varphi : \mathcal{B}(L) \to \mathcal{C}$ be an order-isomorphism. By Theorem 2.5 of D. Kelly and I. Rival [10], we can choose a doubly irreducible element $b \in L$ such that $b \in \mathcal{B}_{\text{left}}(L)$. Since $\mathcal{B}_{\text{left}}(L)$ is a maximal chain, the unique lower cover a and the unique upper cover c of b belong to $\mathcal{B}_{\text{left}}(L)$. By the chain condition and the assumption on linear indecomposability, the cell containing a, b, c is a 4-cell with left corner b. Let d denote the right corner of this cell. Removing b from the diagram, we get a planar diagram of the sublattice $L' = L \setminus \{b\}$ such that $a, d, c \in \mathcal{B}_{\text{left}}(L')$. If $d \notin \mathcal{B}_{\text{right}}(L)$, then we obtain a new contour $\mathcal{C}'$ from $\mathcal{C}$ by moving $\varphi(b)$ slightly, horizontally towards the interior of the polygon $\mathcal{C}$ and keeping other vertices unchanged. If $d \in \mathcal{B}_{\text{right}}(L)$, then, to obtain $\mathcal{C}'$, we move $\varphi(b)$ to $\varphi(d)$. By the induction hypothesis, L' has a diagram whose boundary is congruent with $\mathcal{C}'$. Clearly, if we put $\varphi(b)$ back to $\mathcal{C}'$, we get a planar diagram of L whose boundary is congruent with $\mathcal{C}$. □

Let L be a planar semimodular lattice, and let $C_4(L)$ be the collection of all 4-cells of L (with respect to a fixed planar diagram). For each 4-cell S, we insert $n_S \geq 0$ new elements $c_{S,1}, \ldots, c_{S,n_S}$, called "eyes", into the interior of S such that $0_S \prec c_{S,i} \prec 1_S$ for $i = 1, \ldots, n_S$. This way we obtain a new lattice, which is called an *anti-slimming* of L. If $n = \sum_{S \in C_4(L)} n_S$, then we speak of an n-step anti-slimming. This terminology is motivated by G. Grätzer and E. Knapp [6]. For example, M_3 is a 1-step anti-slimming of the four-element Boolean lattice—which is slim.

Proposition 9 (G. Grätzer and E. Knapp [6])**.** *Every anti-slimming of a planar semimodular lattice is a planar semimodular lattice. Conversely, every planar semimodular lattice is an anti-slimming of a slim semimodular lattice.*

This statement shows that, in a sense, the description of planar semimodular lattices reduces to that of slim semimodular lattices. The rest of the paper deals only with *slim* semimodular lattices.

3. Forks, corners, and visual constructions

Let d be a doubly irreducible element of a slim semimodular lattice L. Then d is on a boundary chain of L by Lemma 6. Clearly, the unique lower cover d^- and the unique upper cover d^+ of d belong to the same boundary chain. If d^- is meet-reducible and d^+ is join-reducible, then the doubly irreducible element d is called a *weak corner* of L. It is clear by Lemma 2 that d^- has exactly two upper covers, provided that d is a weak corner. This motivates the following definition: by a *corner* of L we mean a weak corner d such that d^+ has exactly two lower covers. For example, the grey-filled element d of L_2 is a weak corner of L_2 in Figure 2, while the black-filled element is a corner of L_3. (It will be evident by Proposition 10 or

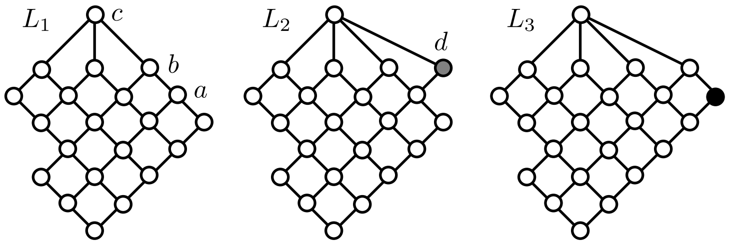

FIGURE 2. Weak corner and corner

Theorem 11 that the lattices in Figures 2, 3, 5, 6 and 7 are semimodular, but we do not need this fact now.)

A corner or a weak corner can be *removed* and a sublattice remains. The reverse procedure will be called *adding a weak corner* and *adding a corner*, respectively. More exactly, if L is a slim semimodular lattice, $a \prec b \prec c$ are elements of one of its boundary chains and a is meet-irreducible, then we can add a new element d to L such that $a \prec d \prec c$; we say that the lattice $L \cup \{d\}$ is obtained from L by adding a weak corner. If, in addition, $c \in J(L)$, then we say that $L \cup \{d\}$ is obtained from L by adding a corner. For example, L_2 in Figure 2 is obtained from L_1 by adding a weak corner, while L_3 is is obtained from L_2 by adding a corner.

Proposition 10.

(i) *If we add a weak corner (or, in particular, a corner) to a slim semimodular lattice, then we obtain a slim semimodular lattice.*
(ii) *If we remove a weak corner (or, in particular, a corner) from a slim semimodular lattice, then we obtain a slim semimodular lattice.*
(iii) *Each slim semimodular lattice can be obtained from a chain by adding weak corners, one by one, in a finite number of steps.*

Proof. Clearly, if L' is obtained from a 4-cell lattice L by adding a weak corner, then L' is again a 4-cell lattice. If L has no two distinct 4-cells with a common bottom, then neither has L'. Hence the first part of the statement follows from Proposition 1.

The second part follows analogously.

To prove the third part by induction on the size, let L be a slim semimodular lattice. We know that L is planar, and we can assume that it is not a chain. By Theorem 2.5 of D. Kelly and I. Rival [10], L has a doubly irreducible element $d \in \mathcal{B}_{\text{right}}(L) \setminus \{0, 1\}$. We can assume that $d \notin \mathcal{B}_{\text{left}}(L)$, because otherwise L would be linearly decomposable at d and the induction hypothesis would apply to $\downarrow d$ and $\uparrow d$. Clearly, d belongs to a unique 4-cell, which is a covering square $S = \{a = b \wedge d, b, d, c = b \vee d\}$. Removing d from L means that S is removed from the set of 4-cells. Hence

$$K = L \setminus \{d\} \text{ is a slim semimodular lattice} \tag{1}$$

by Proposition 1. By the induction hypothesis, K can be obtained from a chain by adding weak corners finitely many times. One of the upper covers of a, namely d, is removed, whence d is a meet-irreducible element in K by Lemma 2. So, L is obtained from K by adding a weak corner. □

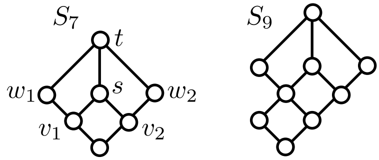

FIGURE 3. The lattice S_7 (on the left) and S_9 (on the right)

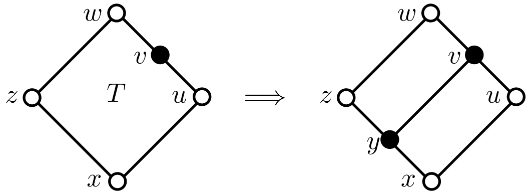

FIGURE 4. The downward-going procedure

Usually, adding a corner results in a more aesthetic diagram than adding a weak corner, see Figure 2. Unfortunately, we cannot drop "weak" from Proposition 10. Indeed, the lattice S_7 depicted in Figure 3, which has a crucial role in this paper, cannot be obtained from a chain by adding corners. Figure 3 shows the notation for its elements. The only meet-irreducible but join-reducible element of S_7 will be called the *middle element* of S_7, denoted by s. The lower covers of s are denoted by v_1 and v_2. The upper cover of s is the *top* of this S_7, it is denoted by t. The double irreducible cover of v_i is denoted by w_i.

We are now in the position of giving one of the crucial definitions. Let S be a 4-cell of a slim semimodular lattice L. Then S is a covering square $\{a = b_1 \wedge b_2, b_1, b_2, c = b_1 \vee b_2\}$. We change L to a new lattice L' as follows.

Firstly, we replace S by a copy of S_7. This way we get three new 4-cells instead of S.

Secondly, as long as there is a chain $u \prec v \prec w$ such that v is a new element and $T = \{x = u \wedge z, z, u, w = z \vee u\}$ is a 4-cell in the original lattice L but $x \prec z$ at the present stage, see Figure 4, we insert a new element y such that $x \prec y \prec z$ and $y \prec v$. (This way we get two 4-cells to replace the cell T.) When this "downward-going" procedure terminates, we obtain L'. The collection of all new elements, which is a poset, will be called a *fork*. We say that L' is obtained from L by *adding a fork to* L (*at the* 4-*cell* S), see Figure 5 for an illustration. Adding forks to L means to add several forks to L one by one.

Theorem 11. *Each slim semimodular lattice can be obtained from a chain by using the following two operations*

(i) *adding a fork*
(ii) *adding a corner*

finitely many times. Moreover, the class of slim semimodular lattices is closed with respect to these operations.

Notice that neither of the two operations can be omitted from Theorem 11. For example, S_9 in Figure 3 cannot be obtained from a distributive lattice by

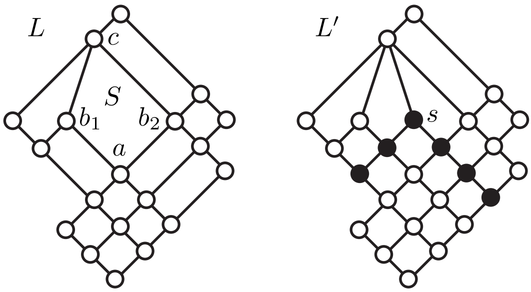

FIGURE 5. Adding a fork to L

adding fork(s). Similarly, S_7 cannot be obtained from a distributive lattice by adding corner(s).

A slim lattice L is called a *rectangular lattice*, if $J(L)$ is the union of two disjoint chains C and D such that every element of C is incomparable with all elements of D. Note that rectangular lattices are at least four-element. Although the definition of rectangular lattices given by G. Grätzer and E. Knapp [7] is different from ours, for slim lattices the two definitions are equivalent. The advantage of starting from a rectangular slim lattice is that rectangular lattices can be depicted in a very aesthetic "rectangular" way; see several figures in [7] or see the lattice on the right-hand side of Figure 7. A chain with more than one element is called a *nontrivial chain.*

Theorem 12. *Let L be a slim semimodular lattice consisting of at least three elements. Then L can be obtained from the direct product of two nontrivial finite chains such that*

(i) *first we add finitely many forks one by one,*
(ii) *and then we remove corners, one by one, finitely many times.*

4. PROOFS AND FURTHER LEMMAS

The proofs of Theorems 11 and 12 require some lemmas. Two lower covers of an element are called *neighboring* if one of them is immediately to the right of the other one in a fixed planar diagram.

Lemma 13. *Let x and y be two neighboring lower covers of z in a 4-cell lattice. Then $\{x \wedge y, x, y, z\}$ is a 4-cell.*

Although this lemma looks quite evident visually, we give a formal rigorous proof in the style of D. Kelly and I. Rival [10].

Proof. Let $b := x \wedge y$, and assume that y is on the right of x. Let D be the rightmost chain between b and x. That is, we consider the interval $[b, x]$, which is a region by Lemma 1.5 of D. Kelly and I. Rival [10], and D is the right boundary of this interval. Similarly, let E be the leftmost chain between b and y. Choose maximal chains D' and E' such that $D' \supseteq D \cup \{z\}$ and $E' \supseteq E \cup \{z\}$. By the definition of a meet, $D \cap E = \{b\} = E \cap \downarrow x$. This together with Lemma 3 easily implies that every element of $D \setminus \{b\}$ is on the left-hand side of E'. Similarly, every element of $E \setminus \{b\}$ is on the right-hand side of D'. Hence $D \cup \{z\}$ and $E \cup \{z\}$ are the left

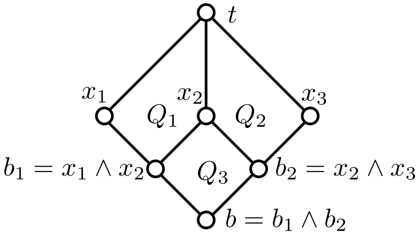

FIGURE 6. Constructing an S_7 in the proof of Lemma 15

and right boundary chains of a region T, respectively, and the intersection of these boundary chains of T is $\{b, z\}$.

We now suppose, by way of contradiction, that there is an element u in the interior of T. Since b and z are the least and the greatest elements of T by Lemma 1.3 of D. Kelly and I. Rival [10], we know that $b < u < z$. Observe that $u \not< x$, because otherwise taking a maximal chain from b to u inside T and continuing it from u to x inside T we would get a new maximal chain from b to x on the right of D, a contradiction. Similarly, $u \not< y$. Therefore, if we take a maximal chain from u to z inside T, then the last but one element of this chain is a lower cover of z strictly on the right of x and strictly on the left of y. This contradicts the assumption that y is an immediate right neighbor of x. Therefore, T is a cell. Hence it is a 4-cell, because L is a 4-cell lattice. ☐

Lemma 14. *Let L be a slim semimodular lattice. Let t be an element of L such that t has at least three lower covers, and suppose that t is minimal with respect to this property. Then t is the top of a cover-preserving S_7 sublattice.*

Proof. Since L is planar by Proposition 1, we fix a planar diagram of L. Let x_1, x_2, x_3 be three *neighboring* lower covers of t such that x_{i+1} is immediately to the right of x_i, for $i = 1, 2$. Lemma 13 gives us two 4-cells, $Q_1 = \{b_1, x_1, x_2, t\}$ and $Q_2 = \{b_2, x_2, x_3, t\}$, see Figure 6. The Jordan-Hölder condition gives $h(t) - 1 = h(x_1) = h(x_2) = h(x_3) = h(b_1) + 1 = h(b_2) + 1$. So, if we had $b_1 \leq x_3$, then x_1, x_2 and x_3 would be three distinct covers of b_1, which would contradict Lemma 2. Hence $b_1 \not\leq x_3$ and $b_2 \not\leq x_1$. In particular, $b_1 \neq b_2$. Since t was minimal with more than two lower covers, b_1 and b_2 are the only lower covers of x_2. Let $b = b_1 \wedge b_2$. Lemma 13 yields that $Q_3 := \{b, b_1, b_2, x_2\}$ is a covering square.

Finally, knowing that Q_1, Q_2, and Q_3 are covering squares, it is routine to check that $\{b, b_1, b_2, x_1, x_2, x_3, t\}$ is a cover-preserving S_7 sublattice of L. ☐

Lemma 15. *Let L be a slim semimodular lattice. Then L is distributive if and only if S_7 is not a cover-preserving sublattice of L.*

Proof. The "only if" part trivially follows from the fact that S_7 is non-distributive.

Conversely, assume that L is a slim semimodular non-distributive lattice. We know from Lemma 3 of G. Grätzer and E. Knapp [6] that L is not modular. But L is semimodular, so Corollary IV.2.3 of G. Grätzer [5] implies that L is not dually (=lower) semimodular. There exist two distinct 4-cells with the same top, because otherwise L would be dually semimodular by the dual of Proposition 1. Consequently, there is an element $t \in L$ with at least three lower covers. Hence Lemma 14 applies. ☐

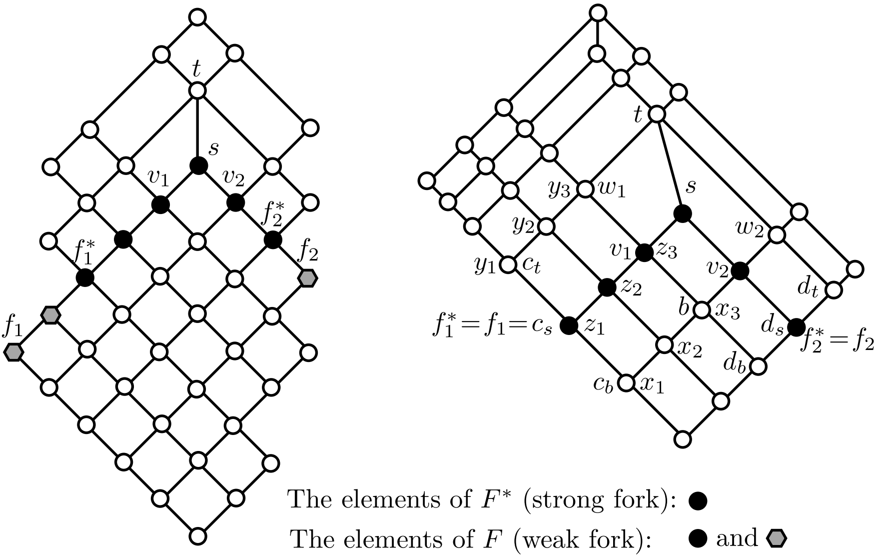

FIGURE 7. Weak and strong forks

Lemma 16. *Every slim distributive lattice is dually slim.*

Proof. Lemma 14 together with distributivity imply that no element has three or more lower covers. Hence no two distinct 4-cells have the same top, and the lattice is dually slim by the dual of Proposition 1. □

In a semimodular lattice L, let s be the middle element of a cover-preserving S_7 such that the top t of this S_7 is *minimal*. (Note that there can be several cover-preserving S_7 sublattices with minimal top, even with the same top.) As usual, see Figure 3, the left and the right lower covers of s are denoted by v_1 and v_2, respectively. Define

$$F = F(s) := \{x \in L : x \le s \text{ and the interval } [x, s] \text{ is a chain}\},$$
$$F_i = F_i(s) := F \cap \downarrow v_i \quad (i = 1, 2), \text{ and}$$
$$K = K(s) := L \setminus F.$$

F defined above is called the *weak fork* determined by the middle element s. The *strong fork* determined by s is defined as

$$F^* = F^*(s) := \{x \in F : x = s \text{ or } x \text{ is meet-reducible}\}.$$

For an illustration in a slim semimodular lattice and in a slim semimodular rectangular lattice, see Figure 7. Let us summarize the terminology: s determines a weak fork or a strong fork (always with adjective), but we add a fork (without adjective) to L. If we add finitely many forks one by one, then we speak of adding forks. For $i = 1, 2$, let

$$F_i^* := F^* \cap \downarrow v_i, \quad f_i^* := \bigwedge F_i^*, \quad f_i = \bigwedge F_i.$$

Lemma 17. *$\downarrow s$ is a slim and dually slim distributive sublattice of L.*

Proof. Since t was minimal, Lemmas 15 and 16 apply. □

The following lemma justifies the appearance of Figure 7.

Lemma 18. *For $i = 1, 2$, F_i is the chain $[f_i, v_i]$ and F_i^* is the chain $[f_i^*, v_i]$. Further, F is the disjoint union of F_1, F_2 and $\{s\}$ while F^* is the disjoint union of F_1^*, F_2^* and $\{s\}$.*

Proof. If $x \in F_1 \cap F_2$, then $v_1, v_2 \in [x, s]$ shows that $[x, s]$ is not a chain, whence $x \notin F$, a contradiction. This shows that the union $F_1 \cup F_2 \cup \{s\}$ is a disjoint union, and therefore the same holds for $F_1^* \cup F_2^* \cup \{s\}$. Note that s has only two lower covers: v_1 and v_2. So, if $x \in F \setminus \{s\}$, then $x \leq v_1$ or $x \leq v_2$ implies $x \in F_1 \cup F_2$. Hence $F \subseteq F_1 \cup F_2 \cup \{s\}$, while the converse inclusion is trivial. This also yields that $F^* = F_1^* \cup F_2^* \cup \{s\}$.

Suppose, by way of contradiction, that F_1 is not a chain. Then there are $x, y \in F_1$ such that $x \parallel y$. Let $z = x \vee y$, and consider an arbitrary $w \in [x, z]$. Since $w \leq v_1$ and $[w, s] \subseteq [x, s]$, we obtain that $[w, s]$ is a chain and $w \in F_1$. In particular, $z \in F_1$ and there is an x' with $x \leq x' \prec z$ such that $x' \in F_1$. Similarly, there is an y' with $y \leq y' \prec z$ such that $y' \in F_1$. Clearly, $z = x' \vee y'$. We know that neither of x', y', z is in $\downarrow v_2$, because otherwise v_1 and v_2 would be two incomparable elements in the chain, say, $[x', s]$. Hence the distributivity of $\downarrow s$, see Lemma 17, yields that $z \wedge v_2 \prec z \wedge s = z$, and z has three distinct lower covers: x', y' and $z \wedge v_2$. Hence Lemma 14 yields a cover-preserving S_7 in $\downarrow s$, which contradicts the minimality of t (or Lemmas 15 and 17). Thus, F_1 is a chain. So is F_2, and so are their subsets F_1^* and F_2^*.

Since F_i is a chain, its smallest element is f_i. Hence $F_i \subseteq [f_i, v_i]$. Conversely, if $z \in [f_i, v_i]$, then $[z, s] \subseteq [f_i, s]$ yields that $[z, s]$ is a chain, whence $z \in F_i$. This shows that $F_i = [f_i, v_i]$.

Finally, it suffices to prove that

$$F_i^* \text{ is a filter of } F_i. \tag{2}$$

Clearly, v_i, the greatest element of F_i, belongs to F_i^*. Suppose that $x \in F_i^* \setminus \{v_i\}$, $y \in F_i$ and $x \prec y$; we have to show that $y \in F_i^*$, that is, y is meet-reducible or $y = s$. We can assume that $y \neq s$. Since x is meet-reducible and $[x, s]$ is a chain, there is an $a \in L \setminus [x, s]$ such that $x \prec a$. Let $b = a \vee y$; it covers y by semimodularity. Notice that $b \not\leq s$, because otherwise $a \leq s$, which is not the case. Hence $y = b \wedge s$ shows that y is meet-reducible. Thus, $y \in F_i^*$. □

We continue to use the notation introduced right before Lemma 17.

Lemma 19. $f_1, f_2 \in J(L)$.

Proof. Assume, by way of contradiction, that, say, f_1 is join-reducible. Since $\downarrow s$ is distributive by Lemma 17, $f_1 \wedge v_2 \prec f_1$. Hence f_1 has a lower cover $a \prec f_1$ such that $a \neq f_1 \wedge v_2$. We know that $[a, s]$ is not a chain, because $a \notin F$. Hence there are $u_1, u_2 \in [a, s]$ such that $u_1 \parallel u_2$. If u_1 is comparable with all elements of $F_1 \cup \{a, s\}$, which is a maximal chain in $[a, s]$, then, by the maximality of this chain, $u_1 \in F_1 \cup \{a, s\}$. Therefore either u_1 or u_2 is incomparable with some element of $F_1 \cup \{a, s\}$.

Consequently, we can choose a maximal element $y \in F_1 \cup \{a, s\}$ such that y is incomparable with some element of $[a, s]$. Clearly, $y \in F_1$. Let y^+ denote the unique upper cover of y in $F_1 \cup \{s\}$. Choose a maximal element $x \in [a, s]$ such that $x \parallel y$. The maximality of y yields that $x < y^+$, and then the maximality of x gives that $x \prec y^+$.

If $x < z \le s$, then the maximality of y implies that z is comparable with all elements of the chain $[y^+, s] \cup \{x\}$, which is a maximal chain in $[x, s]$, so $z \in [y^+, s] \cup \{x\}$. This shows that $[x, s]$ is a chain, whence $x \in F$. Since F_1 is a chain by Lemma 18, $y \in F_1$ and $x \parallel y$, we obtain that $x \notin F_1$. Clearly, $x \neq s$. Consequently, $x \in F_2 = [f_2, v_2]$. This yields that $a \le x \le v_2$. Therefore, $a \le f_1 \wedge v_2$. This together with $a \prec f_1$ and $f_1 \wedge v_2 \prec f_1$ imply $a = f_1 \wedge v_2$, a contradiction. □

Lemma 20. *$K = L \setminus F$ is sublattice of L, and it is a slim semimodular lattice. Moreover, L can be obtained from K by adding a fork and then adding $|F \setminus F^*|$ corners.*

Proof. Suppose $a_1, a_2 \in K$ but $a_1 \vee a_2 \notin K$. Then $a_1 \vee a_2 \in F$, so a_1 and a_2 belong to $\downarrow s$, which is a distributive lattice by Lemma 17. Since $F = F_1 \cup F_2 \cup \{s\}$, there is an $i \in \{1, 2\}$ such that $f_i \le a_1 \vee a_2 \le s$. Since f_i is join-irreducible by Lemma 19, there is a $j \in \{1, 2\}$ such that $f_i \le a_j \le s$. Then $a_j \in F_i \cup \{s\} \subseteq F$ contradicts $a_j \in K$. This shows that K is closed with respect to joins.

Suppose, seeking a contradiction, that K is not closed with respect to meets. Then we can choose a maximal element z such that $z \in F$ and z is the meet of some $a, b \in K$. Since s is meet-irreducible, we can assume that $z \in F_1$. Since $v_1 = x \wedge y$ clearly implies $s \in \{x, y\}$ and $s \notin K$, we can also assume that $z < v_1$. We know that z is meet-reducible, so it has exactly two covers by Lemma 2. One of its covers, denoted by z^+, is in the chain F_1. The other cover c of z is not in F_1, because $z \in F_1$ and F_1 is a chain. Let, say $a \ge c$. Then $b \ge z^+$, because the other possibility would lead to $z = a \wedge b \ge c \wedge c = c$.

Let $d := c \vee z^+$. Then $z^+ \prec d$ by semimodularity. We have $d \in K$, because otherwise $z, d \in F$ and $z \le c \le d$ would imply $c \in F$. We also have $d \not\le b$, because otherwise $z = a \wedge b \ge c$. Using the covering $z^+ \prec d$ and the relation $z^+ \le b$, we obtain $z^+ = d \wedge b$. Since $d, b \in K$, this contradicts the maximality of z. Thus, K is a sublattice of L.

The next plan is to omit the minimal element(s) of $F \setminus F^*$ one by one, and to show that this procedure preserve semimodularity and slimness. So assume that $F^* \subset F$, and, say, $f_1 < f_1^*$. Then, by definition and Lemma 19, f_1 is a doubly irreducible element. Let f_1^- and f_1^+ be its unique lower cover and upper cover, respectively. If f_1^- was meet-irreducible, then $[f_1^-, s] = \{f_1^-\} \cup [f_1, s]$ would be a chain and f_1^- would belong to $F_1 = [f_1, v_1]$, a contradiction. Hence f_1^- is meet-reducible. If f_1^+ was join-irreducible, then the distributivity of $\downarrow s$ (by Lemma 17) would imply $f_1^+ \wedge v_2 \prec f_1^+$, whence $f_1 = f_1^+ \wedge v_2 \le v_2 \le s$, so $v_1, v_2 \in [f_1, s]$ would contradict the fact that $[f_1, s]$ is a chain. Hence f_1^+ is join-reducible, and f_1 is a weak corner. In fact, for $i = 1, 2$,

$$f_i \text{ is a corner of } L, \text{ provided } f_i < f_i^*, \tag{3}$$

by Lemma 17 and the dual of Lemma 2.

Let c denote the upper cover of f_1^- distinct from f_1; note that $c \prec f_1^+$. Since the distributivity of $\downarrow s$ gives $f_1^+ \wedge v_2 \prec f_1^+$, $f_1 \not\le v_2$ and f_1^+ has only two lower covers, we conclude that $c = f_1^+ \wedge v_2 \le v_2$. Let $L' := L \setminus \{f_1\}$; it is a slim semimodular lattice by Proposition 10. Since c, the only lower cover of f_1^+ in L', is below v_2, the weak fork determined by s in L' is $F \setminus \{f_1\}$ but the strong fork determined by s remains the same. Repeating this procedure in $|F \setminus F^*|$ steps we arrive at a slim semimodular sublattice of L in which the weak fork and the strong fork determined by s are the same.

Therefore, by changing the notation if necessary, we can assume that

$$F = F^*.$$

We claim that $K = L \setminus F = L \setminus F^*$ is a slim semimodular lattice and, in addition, L can be obtained from K by adding a fork.

We start from $F_1 = F_1^* = \{f_1^* = z_1 \prec \dots \prec z_n = v_1\}$, where $n \in \mathbb{N}$; see Figure 7 with $n = 3$. Define $x_i = z_i \wedge v_2$. Since $v_2 \prec s$, the distributivity of $\downarrow s$ yields that $x_i \prec z_i$ and $x_i \prec x_{i+1}$, that is, $T_i := \{x_i, z_i, x_{i+1}, z_{i+1}\}$ is a covering square for $1 \le i < n$. By Lemma 2, $f_1^* = z_1$ has a unique upper cover y_1 outside F_1^*. Define $y_i = z_i \vee y_1$ for $1 < i \le n$. Although the y_i are not in $\downarrow s$, the semimodularity of L yields that $P_i := \{z_i, y_i, z_{i+1}, y_{i+1}\}$ is a covering square for $1 \le i < n$. Covering squares of L are 4-cells.

Clearly, when we delete the elements of F_1^*, then, for each $i \in \{1, \dots, n\}$, two 4-cells, T_i and P_i, are replaced by a single 4-cell, $\{x_i, y_i, x_{i+1}, y_{i+1}\}$. The same happens when we delete the elements of F_2^*. Finally, when we delete the middle element s, then we get a single 4-cell instead of three old ones. This shows that $L \setminus F^*$ remains a 4-cell lattice. The bottom of each new 4-cell is the bottom of some old 4-cell. Thus, no two distinct 4-cells of K have the same bottom, and Proposition 1 implies that K is a slim semimodular lattice. Finally, the consideration above shows that L can be obtained from K by adding back the (strong) fork we have just deleted. □

Proof of Theorem 11. By Proposition 10, the class $\mathfrak{S}_{\mathrm{sm}}$ of all slim semimodular lattices is closed with respect to adding a corner. When we add a fork, then all the new cells are 4-cells, no two new cells have the same bottom, and if a new cell has the same bottom as an old one, then the old cell is deleted. Hence Proposition 1 implies that $\mathfrak{S}_{\mathrm{sm}}$ is closed with respect to adding a fork.

We have to prove that each $L \in \mathfrak{S}_{\mathrm{sm}}$ can be obtained from a chain by the two permitted operations. We prove this by induction on $|L|$. We can assume that $|L| \ge 3$ and the statement holds for every slim semimodular lattice with size smaller than $|L|$.

If L happens to be distributive, then Theorem 2.5 of D. Kelly and I. Rival [10] allows us to choose a doubly irreducible element $d \in \mathcal{B}_{\mathrm{right}}(L) \setminus \{0, 1\}$. Lemma 2 together with its dual and Lemma 16 yield that d is a corner of L. Consider the sublattice $K = L \setminus \{d\}$. It is a slim semimodular (in fact, distributive) lattice by (1). So, the induction hypothesis yields that K can be obtained by the two permitted operations. The same holds for L, because L is obtained from K by adding a corner.

Thus, we can assume that L is not distributive. By Lemma 15, we can choose a cover-preserving S_7 sublattice with minimal top. This determines a weak fork F, see right before Lemma 17. Then $K = L \setminus F$ is a slim semimodular lattice by Proposition 20. So, the induction hypothesis implies that K can be obtained from a chain by the two permitted operations. The same holds for L by Proposition 20. □

The proof of Theorem 12 is divided into the following two lemmas, both also of separate interest.

Lemma 21. *Let L be a slim semimodular lattice consisting of at least three elements. Then L can be obtained from a rectangular slim semimodular lattice by removing a corner finitely many times.*

Proof. Let L_0 be a slim semimodular lattice of length $n \geq 2$, that is, of size at least 3. If we add corners to L_0, each after each, then we obtain a slim semimodular lattice L of the same length by Theorem 11. However, Lemma 6 yields that $|L| \leq 2^{2n}$. Hence the procedure of adding new and new corners terminates in a finite number of steps. So we can assume that L is a slim semimodular lattice such that no corner can be added to L; we have to show that L is rectangular.

Let c_1 and d_1 be the largest element of $\mathcal{B}_{\text{left}}(L) \cap J(L)$ and $\mathcal{B}_{\text{right}}(L) \cap J(L)$, respectively. Define $C = \mathcal{B}_{\text{left}}(L) \cap \downarrow c_1$ and $D = \mathcal{B}_{\text{right}}(L) \cap \downarrow d_1$.

We claim that $J(L) = (C \cup D) \setminus \{0\}$. Lemma 6 implies that $J(L) \subseteq (C \cup D) \setminus \{0\}$. Assume, by way of contradiction, that the converse inclusion fails. Then some element of, say, $C \setminus \{0\}$ is join-reducible; let x be the largest such element. Let $x^- \in \mathcal{B}_{\text{left}}(L)$ and $x^+ \in \mathcal{B}_{\text{left}}(L)$ be the lower cover and the upper cover of x on the left boundary, respectively. Then $x^+ \in J(L)$ by the maximality of x, and Lemma 4 yields that x^- is meet-irreducible. Hence we can add a corner d to L such that $x^- \prec d \prec x^+$, a contradiction. This shows that $J(L) = (C \cup D) \setminus \{0\}$.

Clearly, L is not a chain, because otherwise a corner could be added to it. Therefore, $C \neq D$.

Assume, seeking a contradiction, that $C \cap D \neq \{0\}$. If $x \prec y \in C \cap D$ and $x \in C$, then $x \in C \cap D$, because otherwise y would not be join-irreducible. Therefore, there is an atom $a \in C \cap D$. Since a belongs to both boundary chains, a is the only atom in L. Hence 0 is meet-irreducible. Let a^+ be the unique cover of a in C. It is join-irreducible, because a is the only atom. Hence we can add a corner d to L such that $0 \prec d \prec a^+$, a contradiction. This shows that $C \cap D = \{0\}$.

Next, by way of contradiction, we suppose that L is not rectangular. Then, up to C-D symmetry, there is a minimal $y \in D$ such that $(C \setminus \{0\}) \cap \downarrow y \neq \emptyset$. Let $x \in (C \setminus \{0\}) \cap \downarrow y$. Since y is not an atom, it has a unique lower cover $y^- \in D$. Since $x \not\leq y^-$, we have $y = x \vee y^-$, which contradicts $y \in D \subseteq J(L)$. Consequently, L is rectangular. $\square$

Lemma 22. *Each rectangular slim semimodular lattice L can be obtained from the direct product of two nontrivial chains by adding forks finitely many times.*

Proof. We prove the lemma by induction on L. If there is no cover-preserving S_7 sublattice in L, then L is distributive by Lemma 15. Moreover, since $J(L)$ determines L in this case, L is the direct product of two chains and there is nothing to do.

Next, we assume that L contains a cover-preserving S_7 sublattice. Choose one with minimal top t, see Figure 7. Besides the notation of Figure 3, the bottom element of this S_7 is denoted by b. Let $C = \mathcal{B}_{\text{left}}(L) \cap J(L)$ and $D = \mathcal{B}_{\text{right}}(L) \cap J(L)$. Observe that

$$C \cup \{0\} \text{ and } D \cup \{0\} \text{ are ideals in } L. \tag{4}$$

Indeed, if $c \in C$, $x \leq c$, and $x \notin C \cup \{0\}$, then $d \leq x$ for some $d \in D$ and $d \leq c$ would contradict the rectangularity of L. Hence $C \cup \{0\}$ is an ideal, and so is $D \cup \{0\}$.

For $x \in L$, let c_x and d_x denote the largest element of $(C \cup \{0\}) \cap \downarrow x$ and $(D \cup \{0\}) \cap \downarrow x$, respectively. Note that the mappings $\varphi_C \colon L \to C \cup \{0\}$, $x \mapsto c_x$ and $\varphi_D \colon L \to D \cup \{0\}$, $x \mapsto d_x$ are order-preserving. Further, $x = c_x \vee d_x$.

Let q and r be distinct upper covers of an arbitrary element $a \in L$, and let $b = q \vee r$. Then $\{a, q, r, b\}$ is a covering square, and we assert that

$$c_a < c_b \text{ and } d_a < d_b. \tag{5}$$

Indeed, let c_a^+ and d_a^+ be the (unique) covers of c_a and d_a in C and D, respectively. They exist, because otherwise a could not have two distinct covers. We infer from semimodularity that $c_a^+ \vee d_a$ and $c_a \vee d_a^+$ are covers of $a = c_a \vee d_a$, and clearly they are the only covers of a. Hence, up to q-r symmetry, $q = c_a^+ \vee d_a$ and $r = c_a \vee d_a^+$. This gives $b = q \vee r = c_a^+ \vee d_a^+$, implying (5).

Let X be a maximal chain that includes $\{b, v_1, s\}$. Then c_s, like any element of $\mathcal{B}_{\text{left}}(L)$, is on the left of X and v_2 is on the right of X. If we had $c_s \leq v_2$, then Lemma 3 and $v_1 \parallel v_2$ would imply $c_s \leq v_1 \wedge v_2 = b$, whence $c_s \leq c_b$, although (5) applied to $\{b, v_1, v_2, s\}$ gives $c_b < c_s$. Therefore, $c_s \not\leq v_2$. However, $c_s < s$ and s has only two lower covers, v_1 and v_2, whence $c_s \leq v_1$. This implies that $c_s \leq c_{v_1}$. The reverse inequality also holds, because φ_C is order-preserving. Hence $c_s = c_{v_1}$. So, applying (5) to the covering squares $\{b, v_1, v_2, s\}$ and $\{v_1, w_1, s, t\}$, see Figure 7, and using C-D symmetry, we conclude that

$$c_b < c_s = c_{v_1} < c_t \text{ and } d_b < d_s = d_{v_2} < d_t. \tag{6}$$

The minimality of t together with Lemma 15 yield that $\downarrow s$ is a distributive lattice. Since $b \wedge c_s \in C$ by (4), we get that $b \wedge c_s \leq c_b$. The reverse inequality is evident, so we get that $b \wedge c_s = c_b$. On the other hand, $c_s = c_{v_1} \leq v_1$, $b \prec v_1$ and $c_s \not\leq b$ by (6). Therefore, $b \vee c_s = v_1$. So, the distributivity of $\downarrow s$ yields that $c_b \prec c_s$. By (6), there are a unique $\tilde{c} \in C$ and a unique $\tilde{d} \in D$ such that $c_s \prec \tilde{c} \leq c_t$ and $d_s \prec \tilde{d} \leq d_t$. Taking the C-D symmetry into account, (6) strengthens to

$$c_b \prec c_{v_1} = c_s \prec \tilde{c} \leq c_t \text{ and } d_b \prec d_{v_2} = d_s \prec \tilde{d} \leq d_t. \tag{7}$$

Since L is rectangular, we know that $c \parallel d$ for all $c \in C$ and $d \in D$. Hence we easily obtain that $[c_s, s]$ and $[c_d, s]$ are chains. This means that c_s and d_s belong to the weak fork $F = F(s)$. Since $c_s \in J(L)$, its only lower cover is c_b. From $v_1, v_2 \in [c_b, s]$ we infer that $c_b \notin F$. Hence $c_s = f_1$, the least element of F_1. Since $s \wedge \tilde{c} = c_s$ indicates that c_s is meet-reducible, $c_s = f_1 = f_1^*$ by (2). Similarly, $d_s = f_2 = f_2^*$. Therefore, F coincides with the strong fork F^*. Thus, by Lemma 20, L can be obtained from the slim semimodular lattice $K = L \setminus F^*$ by adding a fork.

Finally, we claim that

$$J(K) = J(L) \setminus \{c_s, d_s\}. \tag{8}$$

This will clearly imply that K is rectangular, whence the induction hypothesis applies to it. To prove (8), it suffices to show that, for all $x \in K$, $c_x \neq c_s$ and $d_x \neq d_s$. Suppose the contrary. Then, say, $c_x = c_s$ for some $x \in K$. Let $y := x \wedge s$ and $z := y \vee \tilde{c}$. Observe that $y \neq s$, because otherwise $s < x$ would lead to $t \leq x$, yielding $c_t \leq c_x = c_s$, contradicting (7). Hence $y \in [c_s, v_1] = [f_1, v_1] = F_1$, and y has a unique cover y^+ in the chain $F_1 \cup \{s\} = [c_s, s]$. On the other hand, $\tilde{c} \not\leq y$, because otherwise $\tilde{c} \leq c_y \leq c_x = c_s$ would contradict (7) again. Hence $y \prec z$ by semimodularity. Note that $\tilde{c} \not\leq s$ implies that $z \not\leq s$. Hence z and y^+ are distinct, so they are the only covers of y by Lemma 2. Clearly, $y < x$ follows from $y \leq x$, $x \in K$, and $y \in F$. Consequently, one of the two covers of y is less than or equal to x. However, $y^+ \leq x$ would lead to $y^+ \leq x \wedge s = y < y^+$, a contradiction. The other possibility, $z \leq x$, would lead to $\tilde{c} \leq c_z \leq c_x = c_s$, contradicting (7). Thus, we have shown that $c_x \neq c_s$, while $d_x \neq d_s$ follows by symmetry. □

Proof of Theorem 12. By Lemmas 21 and 22. □

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