ON THE GEOMETRIC CONSTRUCTIBILITY OF CYCLIC POLYGONS WITH EVEN NUMBER OF VERTICES

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Abstract. We deal with convex cyclic polygons with even order, that is, with inscribed n-gons where n is even. We prove that these polygons are in general not constructible with compass and ruler, provided n is at least six and even. We conjecture that the statement also holds for odd orders. Some related questions are also discussed.

1. Introduction and the main results

A (*convex*) *cyclic polygon* is an n-gon inscribed in a circle. Here n denotes the *order*, that is the number of vertices, of the polygon. *Constructibility* is always understood as the classical geometric constructibility with compass and ruler. Our main goal is to prove the following theorem.

Theorem 1.1. *If $6 \leq n$ and n is even, then the cyclic n-gon is in general not constructible from its sides with compass and ruler. For $n = 4$, it is constructible.*

Also, we formulate the following conjecture.

Conjecture 1.2. Let $n \geq 3$ be a natural number. The cyclic n-gon is in general constructible from its side lengths if and only if $n \in \{3, 4\}$.

Besides Theorem 1.1, there are some other results that support this conjecture. For $n = 5$, the cyclic pentagon is in general not constructible by Schreiber [5, Theorem 2]. Also, there is a more involved approach for $n = 5$ in Varfolomeev [6]. However, none of these two approaches for $n = 5$ seems to carry over for larger odd numbers. In particular, according to the overview given by Pak [4], the polynomials used by Varfolomeev [6] would be rather complicated for this purpose. The case $n = 3$ is evident. The conjecture is evidently true for all those n for which the regular n-gon is not constructible; these n are well-known from the Gauss-Wantzel theorem [7]. Actually, Conjecture 1.2 has been verified for all $n \leq 770$, see Section 3 for details.

Note that Schreiber [5, Theorem 3] formulated Conjecture 1.2 as a theorem. However, his proof is wrong; Section 4 will explain this in three different ways. One of our arguments in Section 4 relies on cases $n \in \{3, 4\}$ of the following statement,

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which we recall from Czédli and Szendrei [2, IX.1.26–27 and 2.13]. An illustration with $n = 4$ is given in Figure 1. For a stronger statement, see Proposition 3.2 later.

Proposition 1.3 ([2]). *Assume that we want to construct a cyclic n-gon P_n from the distances $d_1, \dots, d_n$ of its sides from the center of its circumscribed circle. For $n \in \{3, 4, \dots, 100\}$, P_n is in general constructible from $\langle d_1, \dots, d_n \rangle$ if and only if $n = 4$. In particular, P_4 is constructible but P_3 is not.*

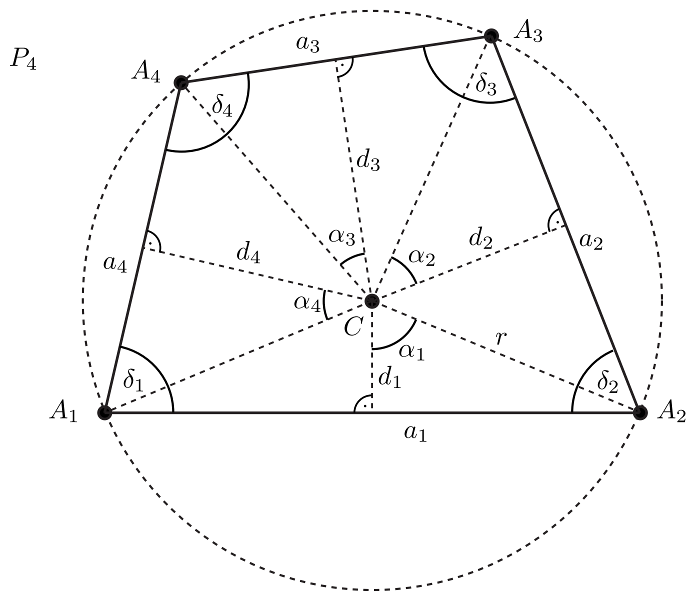

FIGURE 1. A cyclic n-gon for $n = 4$

The case $n = 3$ is somewhat surprising. For $n \in \{5, \dots, 100\} \setminus \{15, 17, 51, 85\}$, Proposition 1.3 follows from the Gauss-Wantzel theorem and Proposition 1.4 below. For $n \in \{15, 17, 51, 85\}$, Proposition 1.3 was proved with computer force in [2]. Since [2] is only available in Hungarian, we will recall the proofs from it for $n \in \{3, 4, 5\}$.

Next, in connection with Theorem 1.1 and Proposition 1.3, we formulate our second result.

Proposition 1.4. *With the notation of Proposition 1.3, if $n \geq 6$ and n is even, then P_n is in general not constructible from $\langle d_1, \dots, d_n \rangle$.*

The following statement, which we recall from Czédli and Szendrei [2, IX.2.14], extends the scope of Proposition 1.4 to circumscribed polygons.

Remark 1.5 ([2]). Let $n \in \{3, 4, 5, \dots\}$. With the notation of Proposition 1.3, a circumscribed n-gon T_n is in general constructible from the distances of its vertices from the center of the inscribed circle if and only if the inscribed polygon P_n is constructible from $\langle d_1, \dots, d_n \rangle$ in general.

2. PROOFS

Our approach is based on the following well-known statement from classical algebra. Its Part (C) is the Eisenstein-Schönemann criterion, see Cox [1] for our terminology. The degree of a polynomial $f(x)$ in the variable x is denoted by $\deg_x(f)$. Usually, we assume that we are given some complex numbers, and we want to construct an additional complex number depending on the given ones.

Proposition 2.1.

(A) *Let $u \in \mathbb{C}$ be the number that we want to construct in general from $v_1, \dots, v_s \in \mathbb{C}$. Let $f(x; y_1, \dots, y_s)$ in $\mathbb{Q}[x; y_1, \dots, y_s]$ be an irreducible polynomial such that $\deg_x(f)$ is not a power of 2. If $f(u, v_1, \dots, v_s) = 0$, then u is not constructible.*

(B) *If $f(x; y_1, \dots, y_s) \in \mathbb{Z}[x; y_1, \dots, y_s]$ and there exist $c_1, \dots, c_s \in \mathbb{Z}$ such that $g(x) = f(x; c_1, \dots, c_s)$ is irreducible in $\mathbb{Q}[x]$and $\deg_x(g) = \deg_x(f)$, then $f(x; y_1, \dots, y_s)$ is irreducible in $\mathbb{Q}[x; y_1, \dots, y_s]$.*

(C) *If $f(x) = \sum_{j=0}^{k} a_j x^j \in \mathbb{Z}[x]$ and p is a prime number such that $p \nmid a_k$, $p^2 \nmid a_0$, and $p \mid a_j$ for $j \in \{0, \dots, k-1\}$, then $f(x)$ is irreducible in $\mathbb{Q}[x]$.*

For $k \in \{0, 1, 2, \dots\} = \{0\} \cup \mathbb{N}$, we need the following two known formulas, which are easily derived from de Moivre's formula and the binomial theorem. For brevity, the conjunction of "$2 \mid j$" and "j runs from 0" is denoted by $2 \mid j = 0$, while $2 \nmid j = 1$ is understood analogously.

$$\sin(k\gamma) = \sum_{2 \nmid j=1}^{k} (-1)^{(j-1)/2} \binom{k}{j} (\cos\gamma)^{k-j} \cdot (\sin\gamma)^j \tag{2.1}$$

$$\cos(k\gamma) = \sum_{2 \mid j=0}^{k} (-1)^{j/2} \binom{k}{j} (\cos\gamma)^{k-j} \cdot (\sin\gamma)^j. \tag{2.2}$$

A prime p is a *Fermat prime*, if $p-1$ is a power of 2. A Fermat prime is necessarily of the form $p_k = 2^{2^k} + 1$. We know that $p_0 = 3$, $p_1 = 5$, $p_2 = 17$, $p_3 = 257$, and $p_4 = 65\,537$ are primes, but it is an open problem if there exists any other Fermat prime.

Lemma 2.2. *If $n = 5$ or $8 \le n \in \mathbb{N}$, then there exists a prime p such that $n/2 < p < n$ and p is not a Fermat prime.*

Proof. We know from Nagura [3] that, for each $25 \le x \in \mathbb{R}$, there exists a prime in the open interval $(x, 6x/5)$. Applying this result twice, we obtain two distinct primes in $(x, 36x/25)$. Hence, for $25 \le n \in \mathbb{N}$, there are at least two primes in the interval $(n, 2n)$. Since the ratio of two consecutive Fermat primes above 25 is more than 2, this gives the lemma for $50 \le n$. For $n \le 50$, appropriate primes are given in the following table.

n	5	8–13	14–25	26–45	46–85
p	3	7	13	23	43

□

Proof of Theorem 1.1. It suffices to find an appropriate $p \in \{1, 2, \dots, n-1\}$ and $a, b \in \mathbb{N}$ such that P_n is not constructible even if p of the given n side lengths are equal to a and the rest $n - p$ side lengths are equal to b. Let r and C be the radius and the center of the circumscribed circle, respectively.

The half of the central angle for a and b are denoted by α and β, respectively; see the α_i in Figure 1 for the meaning of half central angles. Clearly, P_n is constructible iff so is $u = 1/(2r)$. Since we will choose a and b nearly equal, C is in the interior of P_n, and we have

$$p\alpha + (n-p)\beta = \pi. \tag{2.3}$$

It follows from (2.3) that $\sin(p\alpha) - \sin((n-p)\beta) = 0$. Therefore, using (2.1),

$$\sin\alpha = au,\ \sin\beta = bu,\ \cos\alpha = \sqrt{1-a^2u^2},\ \text{and}\ \cos\beta = \sqrt{1-b^2u^2}, \tag{2.4}$$

we obtain that u is a root of the following function:

$$\begin{aligned} f_p^{(1)}(x) = & \sum_{2\nmid j=1}^{p} (-1)^{(j-1)/2}\binom{p}{j}(1-a^2x^2)^{(p-j)/2}\cdot(ax)^j \\ & - \sum_{2\nmid j=1}^{n-p} (-1)^{(j-1)/2}\binom{n-p}{j}(1-b^2x^2)^{(n-p-j)/2}\cdot(bx)^j \\ & = \Sigma_1^f - \Sigma_2^f. \end{aligned} \tag{2.5}$$

Observe that $f_p^{(1)}(x)$ is a polynomial since $p-j$ and $n-p-j$ are even for j odd. In fact, $f_p^{(1)}(x) \in \mathbb{Z}[x]$ for all $a, b \in \mathbb{N}$. Besides $f_p^{(1)}(x) = \Sigma_1^f - \Sigma_2^f$, we also consider the polynomial $f_p^{(2)}(x) = \Sigma_1^f + \Sigma_2^f$.

From now on, we assume that $8 \le n$ is even and p is chosen according to Lemma 2.2. We know from Schreiber [5] that P_n exists if and only if each of the given side lengths is smaller than the sum of the rest. Hence, obviously, we can choose a and b such that

$$a \equiv 1 \pmod{p^2}, \quad b \equiv 0 \pmod{p^2}, \tag{2.6}$$

and a/b is so close to 1 that P_n exists and C is in the interior of P_n. The inner position of C is convenient but not essential, because we can allow a central angle larger than π; then (2.4) still holds and the sum of half central angles is still π.

Let $v \in \{1, 2\}$. The assumption $n/2 < p < n$ gives $\deg_x(f_p^{(v)}) = p$. Hence, we can write

$$f_p^{(v)}(x) = \sum_{s=0}^{p} c_s^{(v)} x^s, \quad \text{where } c_0^{(v)}, \dots, c_p^{(v)} \in \mathbb{Z}.$$

We have $c_0^{(v)} = 0$ since $j > 0$ in (2.5). Our plan is to apply Proposition 2.1(C) to the polynomial $f_p^{(v)}(x)/x$. Hence, we are only interested in the coefficients $c_s^{(v)}$ modulo p^2. Note that this congruence extends to the polynomial ring $\mathbb{Z}[x]$ in the usual way. The presence of $(bx)^j$ in Σ_2^f yields that all coefficients in Σ_2^f are congruent to 0 modulo p^2. Therefore, $f_p^{(v)}(x) \equiv \Sigma_1^f \pmod{p^2}$, and we can assume that the all the $c_s^{(v)}$ come from Σ_1^f. Each summand of Σ_1^f is of degree p. Therefore, computing modulo p^2, the leading coefficient $c_p^{(v)}$ satisfies the following:

$$\begin{aligned} c_p^{(v)} \equiv & \sum_{2\nmid j=1}^{p} (-1)^{(j-1)/2}\binom{p}{j}(-1)^{(p-j)/2}\,(a^2)^{(p-j)/2}\,a^j \\ & = (-1)^{(p-1)/2}\sum_{2\nmid j=1}^{p}\binom{p}{j}a^p \equiv (-1)^{(p-1)/2}\sum_{2\nmid j=1}^{p}\binom{p}{j} \\ & = (-1)^{(p-1)/2}\,2^{p-1} = (-1)^{(p-1)/2} + pt_p \pmod{p^2} \quad \text{for some } t_p \in \mathbb{Z}; \end{aligned} \tag{2.7}$$

the last but one equality is well-known while the last one follows from Fermat's little theorem. Since Σ_1^f gives a linear summand only for $j = 1$, we have

$$c_1^{(v)} \equiv \binom{p}{1} \cdot a = pa \equiv p \pmod{p^2}. \tag{2.8}$$

Next, let $1 \le s < p$. For $j = p$, the j-th summand of Σ_1^f is $\pm(ax)^p$, which cannot influence $c_s^{(v)}$. Hence, modulo p^2, $c_s^{(v)}$ comes from the $\sum_{2\not| j=1}^{p-2}$ part of Σ_1^f. However, for $j \in \{1, \dots, p-2\}$, the binomial coefficient $\binom{p}{j}$ is divisible by p. Hence, we conclude that there exist integers $t_1, \dots, t_{p-1}$ such that

$$c_s^{(v)} \equiv pt_s \pmod{p^2} \quad \text{for } s \in \{1, \dots, p-1\}. \tag{2.9}$$

Now, (2.7), (2.8), (2.9), $c_0^{(v)} = 0$, and Proposition 2.1(C) imply that

$$\text{for } v = 1, 2, \quad f_p^{(v)}(x)/x \text{ is irreducible.} \tag{2.10}$$

By the choice of p, $\deg_x(f_p^{(v)}(x)/x) = p - 1$ is not a power of 2. Since $a, b \in \mathbb{Z}$, we can apply Proposition 2.1(A) (with $s = 0$ since no parameter is given) to $f_p^{(1)}(x)/x$ to conclude that P_n is not constructible. Alternatively, we can apply Proposition 2.1(A) and (B).

Next, assume $n = 4$. With the notation of Figure 1 and using the fact that $\cos\delta_3 = \cos(\pi - \delta_1) = -\cos\delta_1$, the law of cosines gives

$$a_1^2 + a_3^2 - 2a_1a_3\cos\delta_1 = \overline{A_2A_4}^2 = a_2^2 + a_4^2 + 2a_2a_4\cos\delta_1,$$

which yields an easy expression for $\cos\delta_1$. This implies that $\cos\delta_1$ is constructible, and so is the cyclic quadrangle P_4. This settles the case $n = 4$.

Finally, the case $n = 6$ needs a bit more work, which we quote from Czédli and Szendrei [2, IX.2.7]. Using the cosine angle addition identity, it is easy to conclude that, for all $\kappa_1, \kappa_2, \kappa_3 \in \mathbb{R}$ such that $\kappa_1 + \kappa_2 + \kappa_3 = \pi$,

$$(\cos\kappa_1)^2 + (\cos\kappa_2)^2 + (\cos\kappa_3)^2 + 2\cos\kappa_1 \cdot \cos\kappa_2 \cdot \cos\kappa_3 - 1 = 0\,. \tag{2.11}$$

Assume that the side lengths are given as follows: $a_1 = a_2 = \sqrt{2}$, $a_3 = a_4 = \sqrt{3}$, and $a_5 = a_6 = \sqrt{5}$. It follows from Schreiber [5, Theorem 1], or from an easy reasoning based on continuity, that these data determine a cyclic polygon P_6. Note that $\sqrt{2}$, $\sqrt{3}$, and $\sqrt{5}$ are constructible from 0 and 1, so we will apply Proposition 2.1 with $s = 0$ (no data is given). Let $\alpha_1, \dots, \alpha_6$ be the corresponding central half angles. Define $\kappa_1/2 = \alpha_1 = \alpha_2$, $\kappa_2/2 = \alpha_3 = \alpha_4$, $\kappa_3/2 = \alpha_5 = \alpha_6$, and $u = (1/2r)^2$, where r is the radius of the circumscribed circle. We have $\cos\kappa_1 = \cos(2\alpha_1) = 1 - 2\cdot(\sin\alpha_1)^2 = 1 - 2(a_1/2r)^2 = 1 - 2a_1^2u = 1 - 4u$. We obtain $\cos\kappa_2 = 1 - 6u$ and $\cos\kappa_3 = 1 - 10u$ similarly. Since $\kappa_1 + \kappa_2 + \kappa_3 = \pi$, we can substitute these equalities into (2.11). Hence, we obtain that u is a root of the cubic polynomial $h_1(x) = 120x^3 - 100x^2 + 20x - 1$. Since the Schönemann-Eisenstein Theorem with the prime 5 implies the irreducibility of the "mirror polynomial" $h_2(y) = y^3h_1(1/y) = -y^3 + 20y^2 - 100y + 120$, h_1 is irreducible. Therefore, P_6 is not constructible. □

Proof of Proposition 1.4. First, we assume $n \ge 8$ since $n = 6$ will need a separate treatment. Let p be a prime according to Lemma 2.2. Choose a and b according to (2.6) such that a/b be sufficiently close to 1. Let $d_1 = \dots = d_p = a$ and $d_{p+1} = \dots d_n = b$ be the lengths of the sides of P_n from C. Hence, P_n exists and, clearly, its interior contains the center C of circumscribed circle. (Note that the inner position of C is convenient but not essential if we allow that one of the given distances can be negative.) The radius of the circumscribed circle is denoted by r, and let $u = 1/r$. Instead of (2.4), now we have

$$\cos\alpha = au,\ \cos\beta = bu,\ \sin\alpha = \sqrt{1-a^2u^2},\ \text{and}\ \sin\beta = \sqrt{1-b^2u^2}. \tag{2.12}$$

Combining (2.2), (2.3), and (2.12), and using $2 \nmid p$ and $2 \nmid n-p$, we obtain that u is a root of the following polynomial:

$$\begin{aligned} g_p(x) = &\sum_{2|j=0}^{p-1} (-1)^{j/2}\binom{p}{j}(ax)^{p-j}(1-a^2x^2)^{j/2} \\ &+ \sum_{2|j=0}^{n-p-1} (-1)^{j/2}\binom{n-p}{j}(bx)^{n-p-j}(1-b^2x^2)^{j/2} = \Sigma_1^g + \Sigma_2^g. \end{aligned} \tag{2.13}$$

Substituting s for $p-j$ in Σ_1^g above and using the rule $\binom{p}{j} = \binom{p}{p-j}$, we obtain $\Sigma_1^g = (-1)^{(p-1)/2} \cdot \Sigma_1^f$. Similarly, substituting s for $n-p-j$ in Σ_2^g, we obtain $\Sigma_2^g = (-1)^{(n-p-1)/2} \cdot \Sigma_2^f$. Hence, $\{g_p(x), -g_p(x)\} \cap \{f_p^{(1)}(x), f_p^{(2)}(x)\} \neq \varnothing$, and (2.10) yields that $g_p(x)/x$ is irreducible. Hence, Proposition 2.1 implies that P_n is not constructible. This proves the case $2 \mid n \geq 8$.

Next, we deal with $n = 6$; our approach below is simpler than the argument given in Czédli and Szendrei [2, IX.2.13]. Let

$$d_1 = d_2 = d_3 = d_4 = 1,\ d_5 = 2,\ \text{and}\ d_6 = 3. \tag{2.14}$$

The corresponding central half angles are $\alpha_1, \dots, \alpha_6$. As usual, $\cos(\alpha_5) = d_5 u = 2u$, where $u = 1/r$, and $\cos(\alpha_6) = 3u$. We obtain from (2.2) and $\cos(\alpha_1) = u$ that $\cos(\alpha_1 + \dots + \alpha_4) = \cos(4\alpha_1) = 8u^4 - 8u^2 + 1$. These equalities, together with $4\alpha_1 + \alpha_5 + \alpha_6 = \pi$ and (2.11), imply that u is a root of $x^2(64x^6 - 32x^4 - 16x^2 + 9)$. Since $u \neq 0$, it is a root of $h_1(x) = 64x^6 - 32x^4 - 16x^2 + 9$. Let $h_2(y) = h_1(\sqrt{y+2}/2) = y^3 + 4y^2 + 1$. Clearly, since $u > 0$, $(2u)^2 - 2$ is a root of $h_2(y)$. The polynomial $h_2(y)$ is irreducible in $\mathbb{Q}[y]$ since $0 \notin \{h_2(1), h_2(-1)\}$. Hence $(2u)^2 - 2$ is not constructible. This implies that neither u, nor $r = 1/u$ is constructible.

Finally, to remedy the problem that there is no cyclic hexagon satisfying (2.14), compute $h_2(y) = h_1(\sqrt{y+2}/2)$ again with the initial assumption $\cos(\alpha_5) = d_5 u$, $\cos(\alpha_6) = d_6 u$ and $\cos(\alpha_1) = u$, where d_5 and d_6 are treated as parameters. Since we still have $\deg_y(h_2) = 3$, Parts (A) and (B) of Proposition 2.1 imply that P_6 is not constructible. This completes the proof. $\square$

Parts from the proof of Proposition 1.3 (*Czédli and Szendrei* [2]). Let $n = 3$. With $d_1 = 1$, $d_2 = 2$ and $d_3 = 3$, (2.11) and the formulas analogous to (2.12) give that $12x^3 + 14x^2 - 1 = 0$. Substituting $x = y/2$, we obtain that $2u = 2/r$ is a root of $h_3(y) = 3y^3 + 7y^2 - 2$. Since none of ± 1, ± 2, $\pm 1/3$ and $\pm 2/3$ is a root of $h_3(y)$, this polynomial is irreducible. Hence, we conclude that the triangle P_3 is not constructible.

Next, following Czédli and Szendrei [2, IX.1.27], we deal with the cyclic quadrangle P_4, see Figure 1. Since $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = \pi$, we have $\cos(\alpha_1 + \alpha_2) = -\cos(\alpha_3 + \alpha_4)$. Hence, using the cosine angle addition identity and rearranging and squaring twice, we obtain

$$\begin{aligned}&\sum_{j=1}^{4}(\cos\alpha_j)^4 - 2\cdot\sum_{1\le j<s\le 4}(\cos\alpha_j)^2(\cos\alpha_s)^2\\ &\qquad + 4\cdot\cos\alpha_1\cdot\cos\alpha_2\cdot\cos\alpha_3\cdot\cos\alpha_4\cdot\Big(-2+\sum_{j=1}^{4}(\cos\alpha_j)^2\Big)\\ &\qquad + 4\cdot\sum_{1\le j<s<t\le 4}(\cos\alpha_j)^2(\cos\alpha_s)^2(\cos\alpha_t)^2 = 0.\end{aligned}\tag{2.15}$$

Clearly, if we substitute $\cos\alpha_j$ in (2.15) by $d_j u$, for $j = 1, \ldots, 4$, and divide the equality by u^4, then we obtain that $u = 1/r$ is a root of a polynomial of the form $c_2x^2 + c_0$. A straightforward calculation (preferably, by computer algebra) shows that this polynomial is not the zero polynomial since

$$c_2 = 4(d_1d_2 + d_3d_4)(d_1d_3 + d_2d_4)(d_1d_4 + d_1d_3).$$

Thus $u = 1/r$ is constructible, and so is P_4.

Next, let $n = 5$, and let $d_1 = d_2 = 1$, $d_3 = d_4 = 2$ and $d_5 = 3$. With $u = 1/r$ as before, $\cos(2\alpha_1) = 2(\cos\alpha_1) - 1 = 2u^2 - 1$, $\cos(2\alpha_3) = 2\cdot(2u)^2 - 1$, and $\cos\alpha_5 = 3u$. Applying (2.11) to $\kappa_1 = 2\alpha_1$, $\kappa_2 = 2\alpha_3$, and $\kappa_3 = \alpha_5$, we obtain that u is a root of the polynomial $96x^5 + 68x^4 - 60x^3 - 11x^2 + 6x + 1$. Using computer algebra, we obtain that this polynomial is irreducible. Hence, P_5 is not constructible. □

3. Cyclic polygons of odd order

As we have already mentioned, Conjecture 1.2 holds for every n for which the regular n-gon is not constructible. By the well-known Gauss-Wantzel theorem [7], all those $n \in \{7, 9, 11, 13, \ldots, 5\cdot 257 - 2 = 1283\}$ for which the regular n-gon is constructible are listed in the first row of Table 3.1 below; the rest of the table will be explained soon.

(3.1)

n	15	15	17	51	85	255	257	771
$n =$	$3\cdot 5$	$3\cdot 5$	17	$3\cdot 17$	$5\cdot 17$	$15\cdot 17$	257	$3\cdot 257$
p	13	11	15	49	83	253	255	
a	1	1	1	1	1	1	1	
b	2	2	2	2	2	2	2	
$\deg_x(h(x)/x)$	12	10	14	48	82	252	254	
irreducible?	yes	yes	yes	yes	yes	yes	yes	?

In the rest of this section, let n be odd. We do not assume that p is a prime; however, we assume that $n/2 < p < n$ and p is odd. In this case, neither $f_p^{(1)}(x)$ defined in (2.5), nor $f_p^{(2)}(x)$ is a polynomial. However, the product

$$h(x) = f_p^{(1)}(\sqrt{x})f_p^{(2)}(\sqrt{x}) = (\Sigma_1^f(\sqrt{x}))^2 - (\Sigma_2^f(\sqrt{x}))^2 \tag{3.2}$$

is a polynomial with integer coefficients. The difficulty with $h(x)$ is that even if p is a prime and a and b are chosen according to (2.6), the counterpart of (2.8) fails, because of the squares in (3.2). However, in order to make Conjecture 1.2 credible, one can use computer algebra to obtain Table 3.1 in a few seconds. Actually, we have used Maple, version V.3, and the corresponding worksheet is available from our web sites. The column for $n = 771$ is beyond the capacity of our personal computers.

The table shows that the choice $p = n - 2$, $a = 1$, and $b = 2$ of the parameters works. Generally, p can be chosen differently; this is only exemplified for $n = 15$. Observe that, at each column of the table, $h(x)/x$ is irreducible and its degree is not a power of 2. Therefore, Schreiber [5, Theorem 2] for $n = 5$, Theorem 1.1, and Table 3.1 yield the following corollary.

Corollary 3.1. *Conjecture 1.2 holds for all* $n \leq 770$ *and for all even* n.

Except that the choice $p = n - 2$ does not work in general, a straightforward modification of our approach (included in the Maple worksheet mentioned above) also yields the following statement.

Proposition 3.2. *Proposition 1.3 remains true if we replace* 100 *by* 770.

Based on Proposition 1.4 and Remark 3.2, we formulate the following counterpart of Conjecture 1.2.

Conjecture 3.3. For $2 < n \in \mathbb{N}$, the cyclic n-gon is in general constructible from the distances of its sides from the center of its circumscribed circle if an only if $n = 4$.

4. Notes on Schreiber's argument

We only deal with a small portion of Schreiber [5], which claims to prove Conjecture 1.2. With some insignificant simplifications, the argument given in [5] runs as follows.

"Suppose for contradiction that the cyclic polygon P_n is in general constructible for some $n > 5$. The radius r of its circumscribed circle is an n-ary continuous function of its side lengths $a_1, \dots, a_n$. Also, it is a "quadratic irrationality" $R = R(a_1, \dots, a_n)$ depending on $a_1, \dots, a_n$. Using the continuity of this quadratic irrationality and that of f, and letting a_n converge to 0, we conclude that the quadratic irrationality $R(a_1, \dots, a_{n-1}, 0)$ describes the construction of P_{n-1}. Thus the constructibility of P_n implies that of P_{n-1}, P_{n-2}, …, P_5, which is a contradiction since we know that P_5 is not constructible."

Although [5] does not define "quadratic irrationalities", they are expressions of their variables and the operations $+$, $-$, $\cdot$, $/$, and $\sqrt{}$. Hence, the first objection against his argument is that quadratic irrationalities are not everywhere continuous. Nothing excludes the possibility that, say, a_n is the denominator of a subterm of R above, and R is not continuous at $\langle a_1, \dots, a_{n-1}, 0\rangle$.

Second, think of the geometric construction as a precise list of elementary steps. One of these steps can be that we have to take the intersection of two lines, determined by four points constructed already. Nothing excludes the possibility that these two lines intersect for all $a_n > 0$ but they become parallel when $a_n = 0$.

Third, suppose for contradiction that the argument quoted from [5] is correct. We show that the triangle P_3 is constructible from the distances d_1, d_2, d_3 of its sides from the center of the circumscribed circle. Let $d_4^{(0)}$ denote the radius of this circle; of course, $d_4^{(0)}$ depends on d_1, d_2, d_3. Let $d_4 \in \mathbb{R}$ such that $d_4 < d_4^{(0)}$ and $d_4 \to d_4^{(0)}$. We know from Proposition 1.3 that the cyclic quadrangle P_4 determined by $\langle d_1, \dots d_4\rangle$ is constructible, and it clearly converges to P_3 if $d_4 \to d_4^{(0)}$. It follows from the argument quoted from [5] that P_3 is constructible. However, this contradicts Proposition 1.3.

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