On the mathematics of simple juggling patterns*

Gábor Czédli

held in **Szeged, 19 April 2013**,
within **Budapest Semester of Mathematics**

2013. április 19.

*http://www.math.u-szeged.hu/∼czedli/

Facts to attract outsiders; Leverrier (1811-1877): Neptune

Details for outsiders ??? :

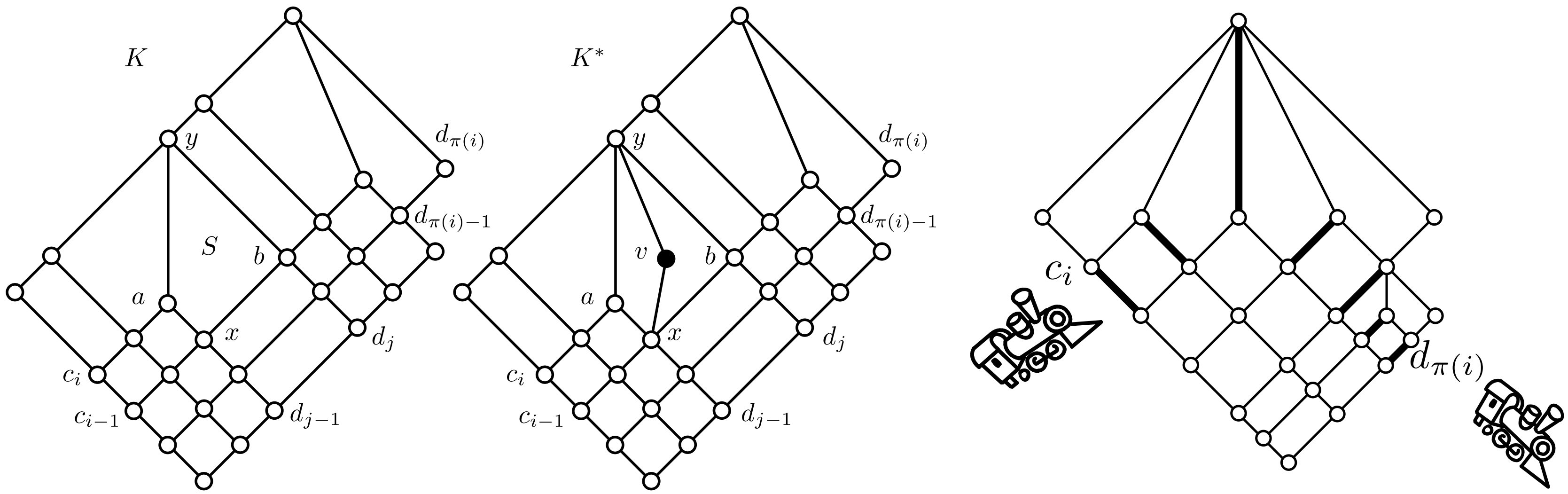
Juggling patterns invented by means of mathematics!

Burkard Polster: The mathematics of Juggling, Springer-Verlag, New York, 2003, the **Book**

Claude E. **Shannon**, 1916-2001: first math. thm. on juggling.

Ronald L. **Graham** (president of AMS in 1993, president of the International Juggler's Association, 1972; 28 joint papers with Paul Erdős.): more mathematical juggling-related theorems.

To introduce myself:



These figures, taken from my joint works with E.T. Schmidt, belong to **Lattice Theory**.

Apologies: At present, I do not know any essential connection between lattices and the mathematics of juggling. Lattice Theory is very beautiful, but I was suggested to select a less special topic, distinct from Lattice Theory.

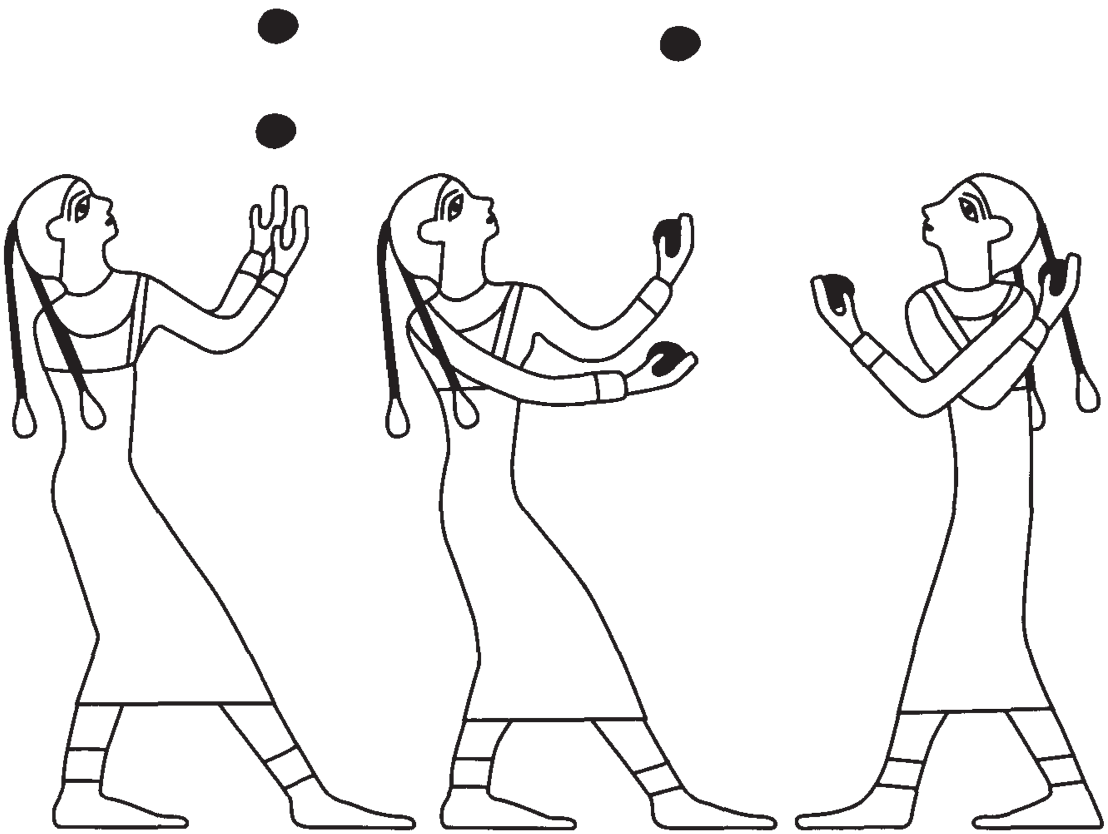


FIGURE 1.1. The earliest known record of juggling is about 4000 years old.

Wall painting, Beni Hassan, Egypt, Middle Kingdom

For stories and history, see the Book.

To set up a model, we simplify. Juggling (in narrow sense): alternately throwing and catching objects. Only **balls**. Juggler's performance: a sequence of 'elementary' parts: **juggling patterns**. We study these patterns, with a lot of simplifications. There are **simple juggling patterns** and others (like multiplex patterns); we deal only with the simple ones, whose axioms are:

(J1) **Constant beat**: throws occur at discrete, equally spaced moments in time, also called **seconds** (although the time unit is not necessarily a real second);

(J2) **Periodic**; $(-\infty, \infty)$ (in principle);

(J3) **Single hot potato axiom**: on every beat, at most one ball is caught, and the same ball is thrown with no delay.

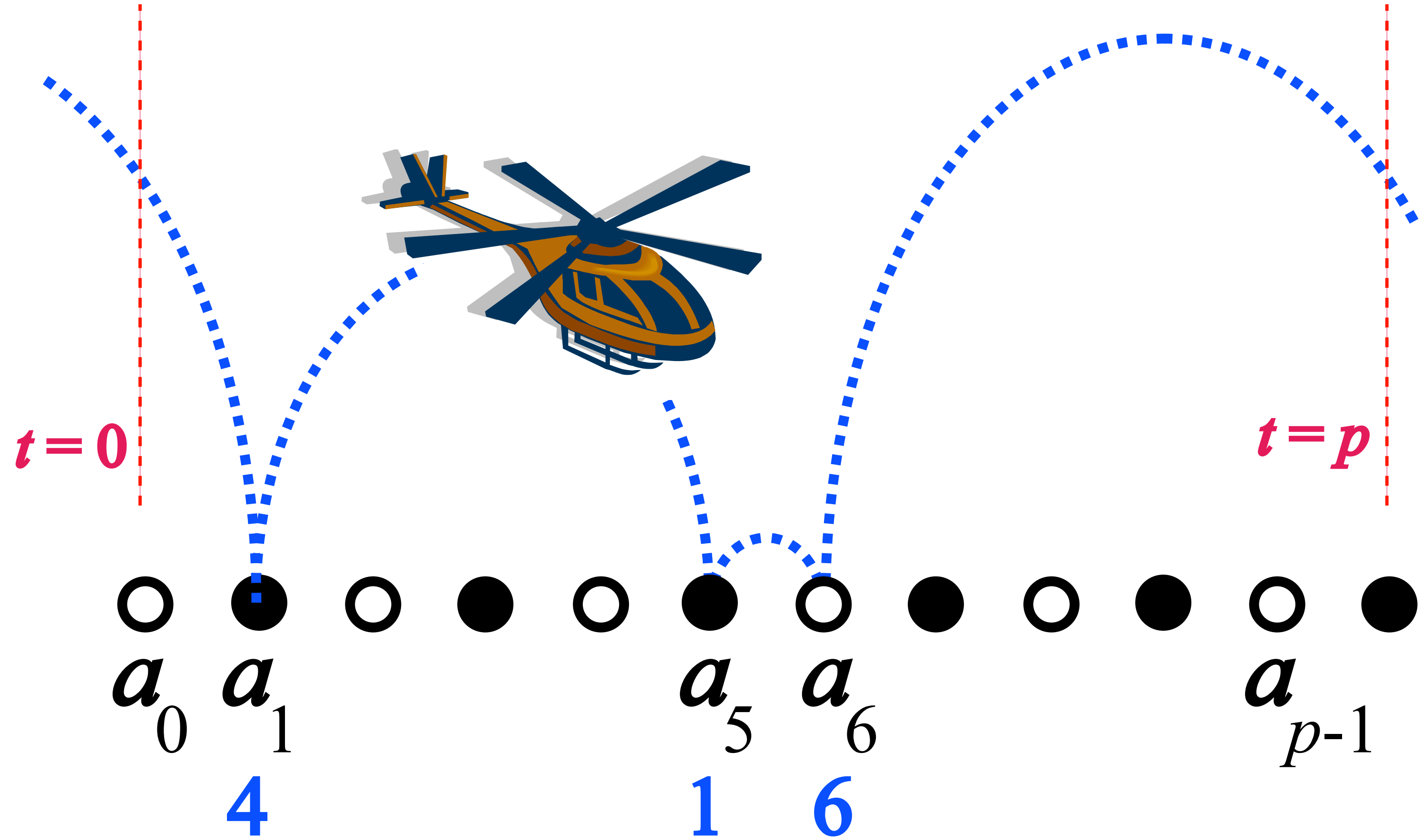
Non-simple: **exorcist, yoyo, multiplex, box, robot, columns**

Simple ones: **cascade (3), fountain (4)**

Height of a throw: how many seconds (=beats) the ball will fly. 0: no ball is thrown. **Juggling sequence** of a pattern := the sequence of its heights. Usually, only one period is written (and it can be repeated to obtain longer sequences.) In some sense, the sequence determines the pattern. (But **3-ball cascade, tennis, Mill’s mess**.) The best example, because it is easy for me: **4413**. Some others (match them to the sequences):

3, 51, 4, 555000, 53. (Soon we intend a new one.) (Transition, flash start.)

51 describes the same as 5151 or 515151; usually we only take **minimal** sequences. **Juggling sequence** = a sequence associated with a simple juggling pattern.

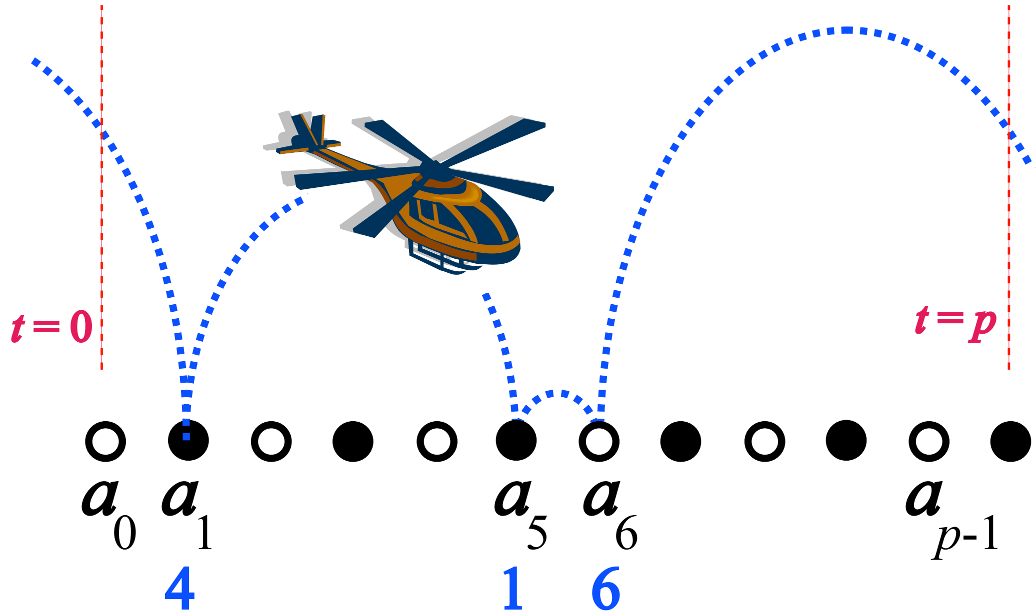


How to prove the Average Theorem?

The Average Theorem. The average $(a_0 + \ldots + a_{p-1})/p$ of a juggling sequence $a_0a_1 \ldots a_{p-1}$, that is, $(a_0, a_1, \ldots, a_{p-1})$, is an integer; namely, it is the number of balls.

Do not practice a pattern whose sequence is, say, 434 !

Proof. *Helicopters*:= balls; b: their number. *Gasoline stations* := hands. A helicopter lands with empty tank, takes only the minimum amount of gasoline (in 0 time) to land with empty tank next time, and launches immediately. It consumes 1 gallon of gasoline per second. Total consumption of all helicopters in p seconds?

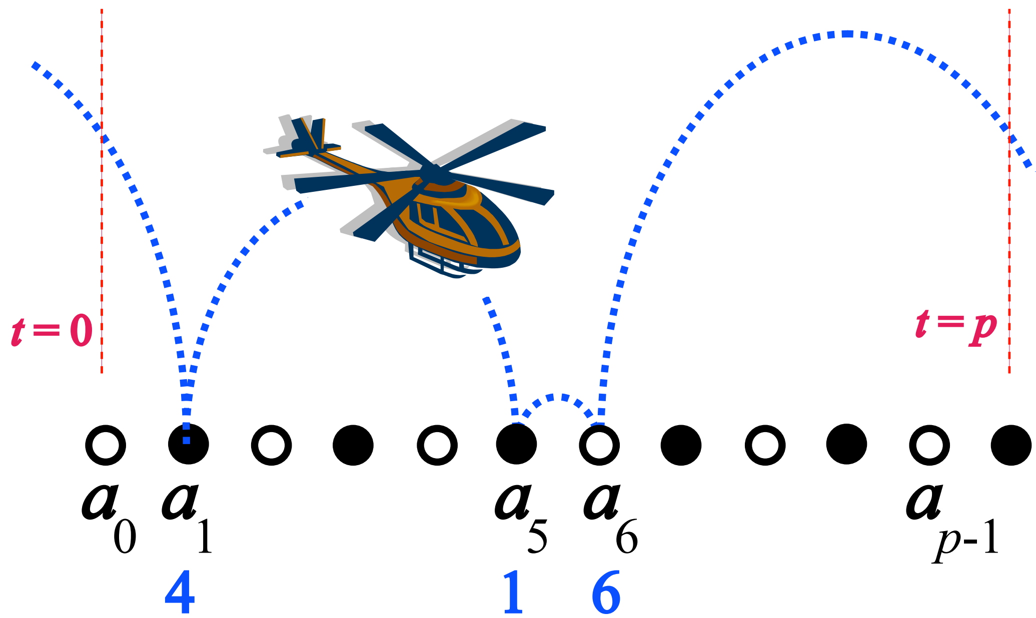


How to prove the Average Theorem?

Each pilot says: 'I flew in p seconds (with 0-time stops only), so I used p gallons.' There are b helicopters, so the total consumption is bp gallons.

The staff of the gasoline stations say: in the i-th second, a helicopter launched for an a_i-second-long trip, so we gave it a_i gallons. Hence, the total consumption is $a_0 + a_1 + \ldots + a_{p-1}$ gallons.

Thus, $a_0 + a_1 + \ldots + a_{p-1} = bp$ implies the theorem.



How to prove the Average Theorem?

Well, some helicopters were in the air at the beginning or remained in the air at the end, but no problem: the gasoline left at the end = the initial gasoline needed at the beginning.

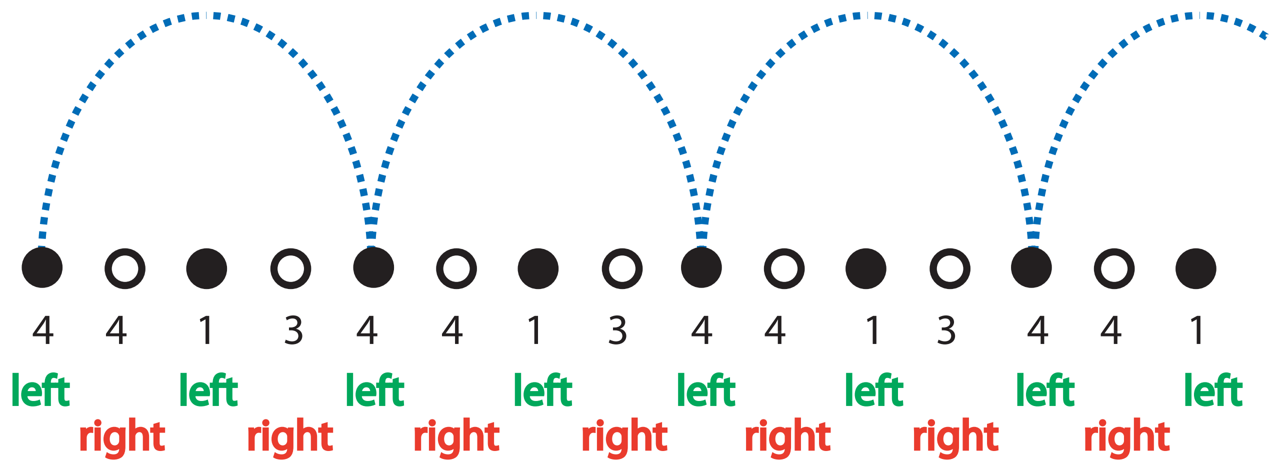
This proof was clear for many outsiders. (My experience.) Hopefully, they got closer to mathematics.

Before inventing a new juggling pattern, we need:

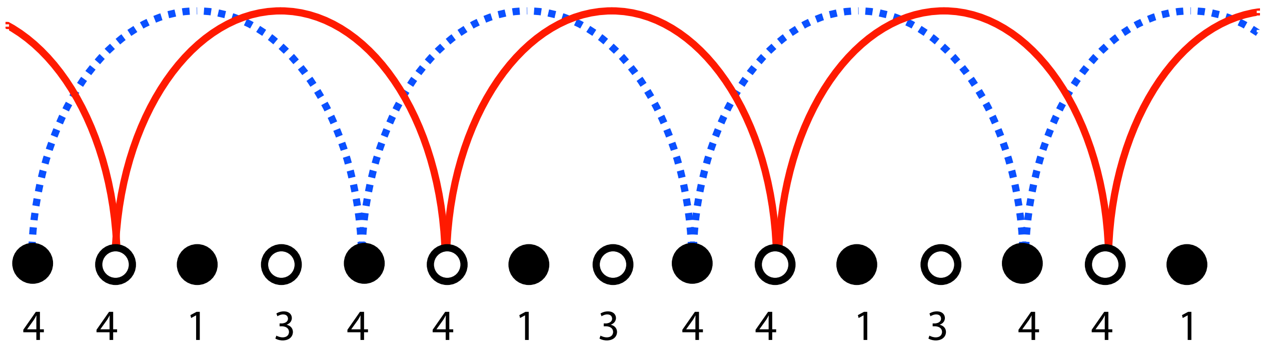
Permutation Test Theorem A sequence $(a_0, a_1, \dots, a_{p-1})$ of nonnegative integers is a juggling sequence if and only if the integers $a_0 + 0$, $a_1 + 1$, $a_2 + 2$, $\dots$, $a_{p-1} + p - 1$ are pairwise incongruent modulo p. It is a **minimal** juggling sequence iff, in addition, its period length is p.

Equivalently (and this is where the name comes from): the sequence above is a juggling sequence iff $(a_0 \bmod p, (a_1 + 1) \bmod p, \dots, (a_{p-1} + p - 1) \bmod p)$ is a permutation of $(0, 1, \dots, p-1)$ where the binary operation mod gives the remainder when we divide by its second argument.

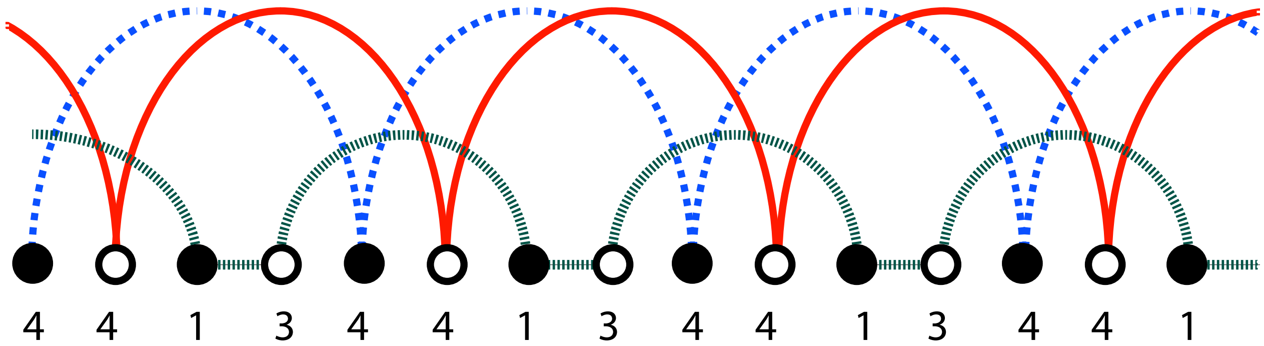
Proof. We need a graphical tool; we illustrate it for 4413 :



We start the figure of 4413: the trajectory of the 1st ball



The trajectories of the 1st and 2nd balls



The figure of 4413 with trajectories, completed

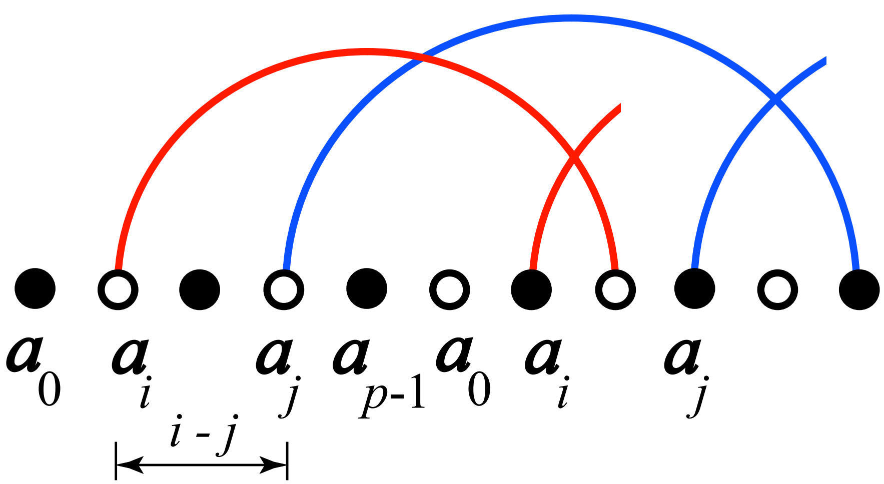
If $\vec{a} := (a_0, a_1, \ldots, a_{p-1})$ is a juggling sequence, then we can make an analogous figure such that *the trajectories never collide*. That is, at each beat, at most one trajectory lands and launches. Moreover, $(a_0, a_1, \ldots, a_{p-1})$ is a juggling sequence if there exists a corresponding figure with non-colliding trajectories! Here, of course, we continue $(a_0, a_1, \ldots, a_{p-1})$ to an infinite sequence $(a_0, a_1, \ldots, a_{p-1}, a_p = a_0, a_{p+1} = a_1, \ldots, a_n = a_{n \bmod p}, \ldots)$.

The ball thrown at time i lands at time $a_i + i$. Hence,
$\vec{a}$ is **not** a juggling sequence iff
there exist two colliding trajectories iff
there exist i, j such that $a_i + i = a_j + j$ and $a_i \neq a_j$ iff
$\exists\ i_0, i_1, j_0, j_1$ such that $a_{pi_1+i_0} + pi_1 + i_0 = a_{pj_1+j_0} + pj_1 + j_0$ and $i_0, j_0 \in \{0, \dots, p-1\}$ and $a_{pi_1+i_0} \neq a_{pj_1+j_0}$ iff
$\exists\ i_0, i_1, j_0, j_1$ such that $a_{i_0} + i_0 = a_{j_0} + j_0 + p(j_1 - i_1)$ and $i_0, j_0 \in \{0, \dots, p-1\}$ and $a_{i_0} \neq a_{j_0}$ iff
$\exists\ i_0 \neq j_0 \in \{0, \dots, p-1\}$ such that $a_{i_0} + i_0 \equiv a_{j_0} + j_0 \pmod{p}$ iff
the condition of the theorem **fails**. Q.e.d.

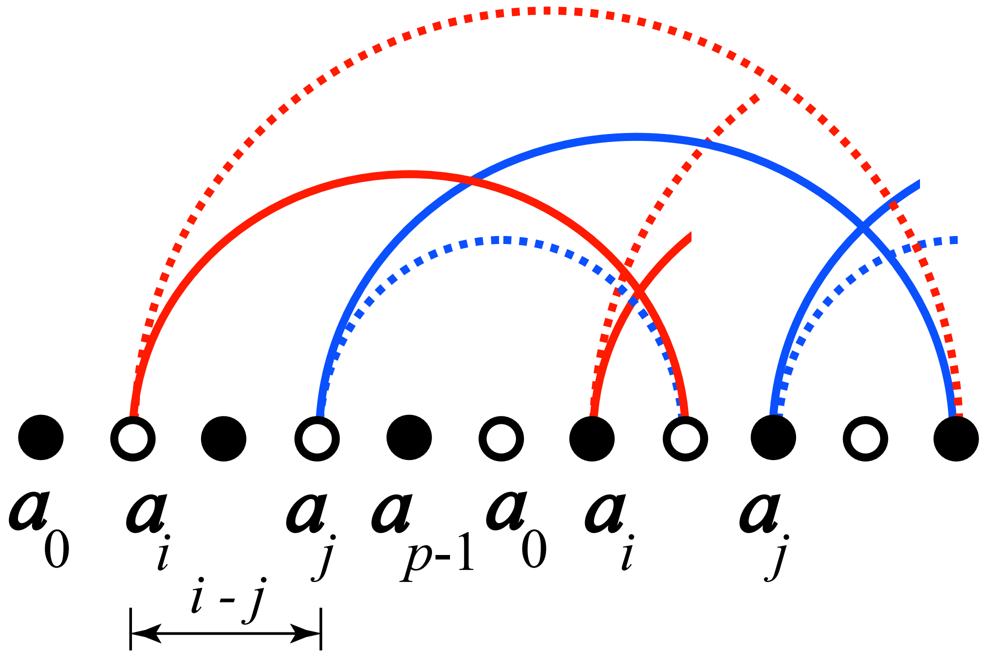
Corollary If $\vec{s} = (a_0, a_1, \dots, a_{p-1})$ is a juggling sequence, $k_0, \dots, k_{p-1}$ are integers, and
$\vec{s}' := (a_0 + k_0 p, a_1 + k_1 p, \dots, a_{p-1} + k_{p-1} p)$ consists of non-negative integers, then $\vec{s}'$ is also a juggling sequence.

There is another approach, by site swaps. Site swaps are of practical importance: smooth transition with any two juggling patterns with the same number of balls. For example: **3, 51, 4413**.

Site swap: we change the landing beats of two distinct trajectories. Originally we have:



Before the site swap



and we turn the solid lines into the dotted ones

I.e., if $0 \leq i < j \leq p-1$ and $\vec{s} = (a_0, a_1, \ldots, a_{p-1})$, then $\vec{s}^{(i,j)} =$
$(a_0, a_1, \ldots, \overbrace{a_j + (j-i)}^{i}, \ldots, \overbrace{a_i - (j-i)}^{j}, \ldots, a_{p-1})$.

$$\vec{\mathrm{s}}^{(i,j)} = (a_0, a_1, \dots, \overbrace{a_j + (j-i)}^{i}, \dots, \overbrace{a_i - (j-i)}^{j}, \dots, a_{p-1})$$

Exercise: perform a site swap of beats (1,3) for the sequence 4413!

Solution: Remember, the sequence starts with its 0th member. We obtain: $(4, 3+2, 1, 4-2) = \underline{\underline{(4, 5, 1, 2)}}$.

If subtraction yields a negative number, then the site swap is not allowed.

Lemma: If an (i, j)-site-swap is allowed for a finite sequence $\vec{\mathrm{s}}$ of non-negative integers, then $\vec{s}$ is a juggling sequence iff so is $\vec{\mathrm{s}}^{(i,j)}$.

Proof: Obvious, because no two trajectories of $\vec{\mathrm{s}}$ collide $\iff$ the same holds for $\vec{\mathrm{s}}^{(i,j)}$. Q.e.d

Cyclic shift: $\vec{s}^k := (a_k, a_{k+1}, \ldots, a_{p-1}, a_0, \ldots, a_{k-1})$.

Lemma $\vec{s}$ is a juggling sequence iff so is $\vec{s}^k$. **Proof:** Obvious.

Flattening algorithm: Given a sequence $\vec{s}$ of non-negative integers. Repeat the following steps:

(a) If the sequence is of form $(x, x, \ldots, x)$ (constant sequence), then print "$\vec{s}$ is a juggling sequence" and stop.

(b) Apply a cyclic shift that results in $(m, x, \ldots)$, where m is (one of) the largest member of the sequence and $x < m$.

(c) If the sequence is of form $(m, m-1, \ldots)$ then print "$\vec{s}$ is not a juggling sequence" and stop.

(d) Apply a site swap at $(0, 1)$.

For example: 4413, 4134, 2334, 4233, 3333, **yes!**

Another example: 55212, 52125, 34125, 53412, 44412, 41244, 23244, 42324, 33324, 43332, **no!**

Flattening Algorithm Theorem: this works.

Proof. Trivially, the algorithm yields the right answer. Does it stops?

At each step, either it reduces the number of maximal members, or reduces the only maximal member $\Rightarrow$ it stops. Q.e.d.

New discovery: 441. The sequence came to existence before the pattern was performed!

Remark Both the Average Thm. and the Permutation Test Thm. can be derived from the Flattening Algorithm Theorem. Furthermore, the Permutation Test Thm. implies the Average Thm.

The sequence 43233 is not a juggling one, though its average, $(4+3+2+3+3)/5$, is an integer.

Converse of the Average Theorem If $(a_0, a_1, \ldots, a_{p-1}) \in \mathbb{N}_0^p$ and $(a_0, a_1, \ldots, a_{p-1})/p \in \mathbb{N}$, then there exists a permutation σ of $\{0, 1, \ldots, p-1\}$ such that $(a_{\sigma(0)}, a_{\sigma(1)}, \ldots, a_{\sigma(p-1)})$ is a juggling sequence.

The proof, which is 4 pages in Polster’s book, will be omitted.

Next task: **count** juggling sequences.

Main Theorem of this talk: The number of minimal simple juggling patterns with b balls and period p is exactly

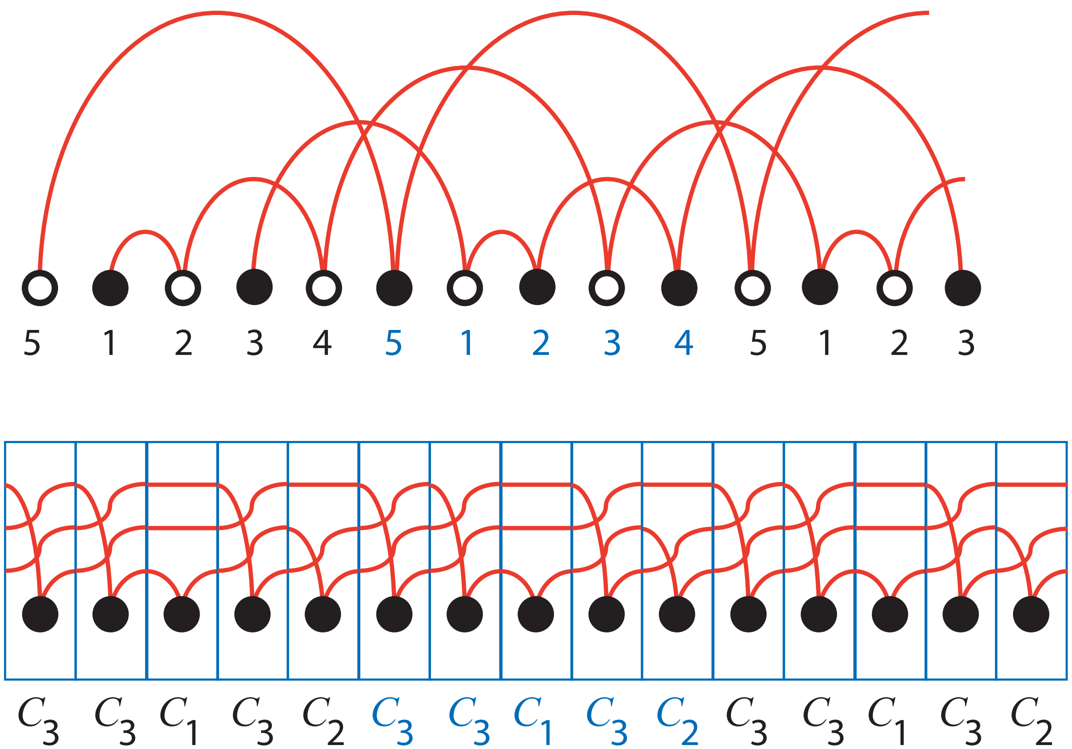
$$\frac{1}{p}\sum_{d|p}\mu(\frac{p}{d}))((b+1)^p - b^p).$$

Here μ is the Möbius function, and two patterns whose juggling sequences differ only in a cyclic shift are considered equal. The final step of the proof will use the Möbius' inversion formula.

This formula helped inventing new juggling patterns, not known by professional jugglers before.

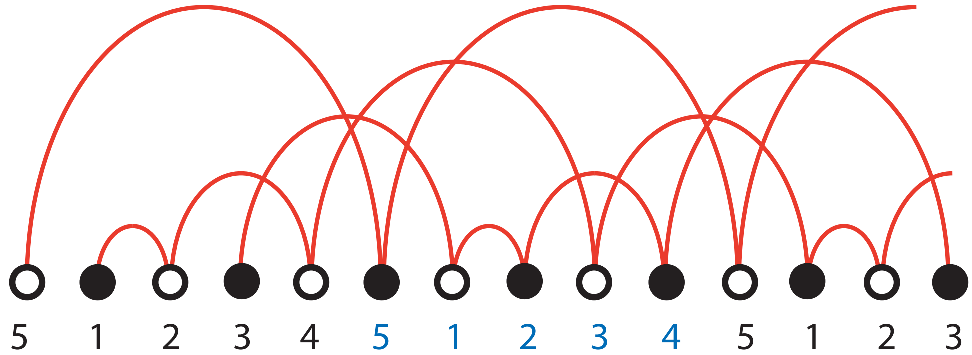
Note: **all** results included in this talk are taken from Polster's **Book**.

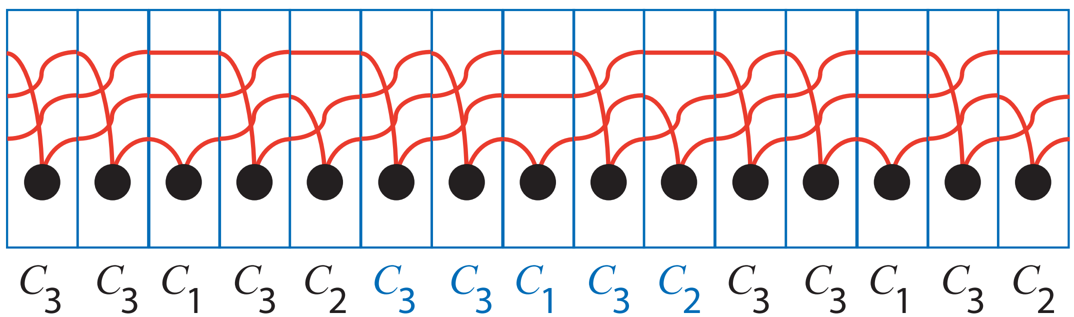
Proof First, we need the concept of **juggling cards**.



The original figure and the normalized figure of 51234

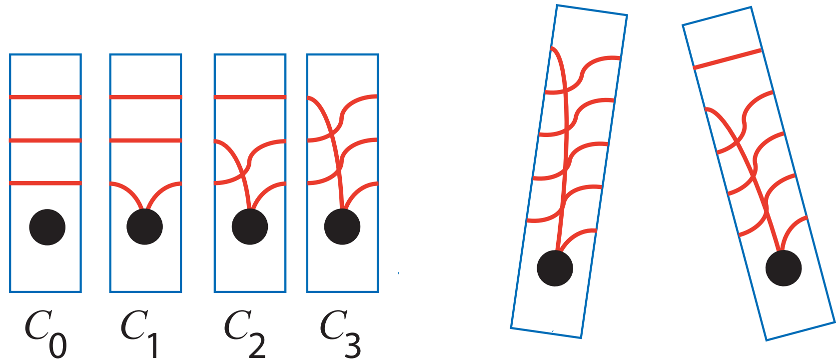
Normalization: like the aeroplanes around an airport, only some fixed heights are permitted; as many heights as the number of balls, and only very rapid transitions (all balls changing height do it at the same, very short, fixed time).





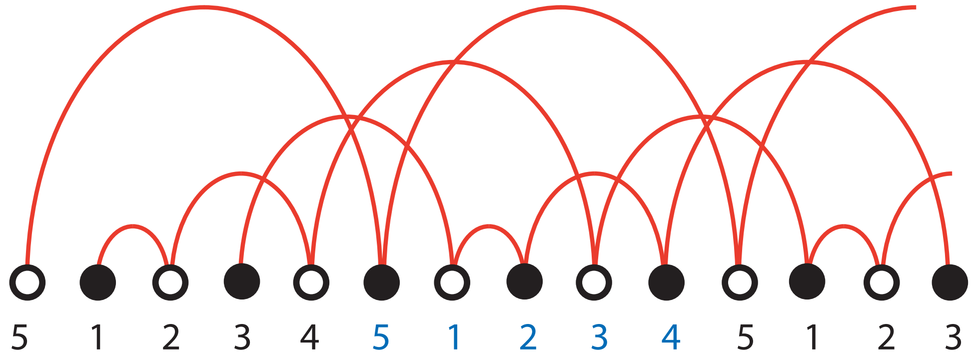
The original figure and the normalized figure of 51234

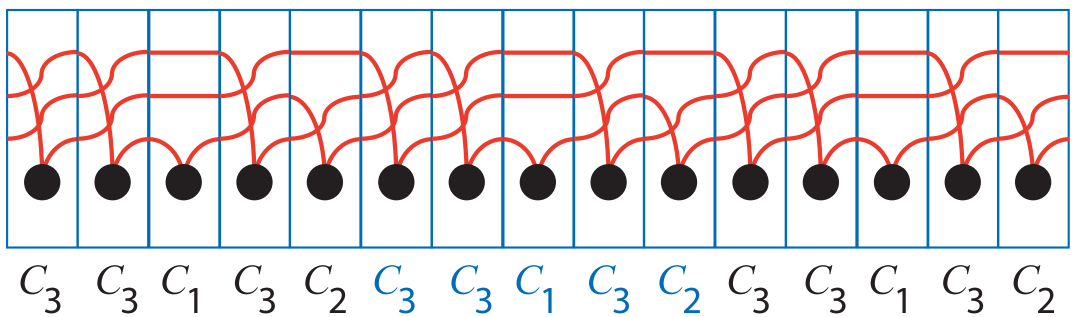
Cutting the normalized figure along the blue lines we obtain the **three-ball juggling cards**. (The b-ball case is analogous.)



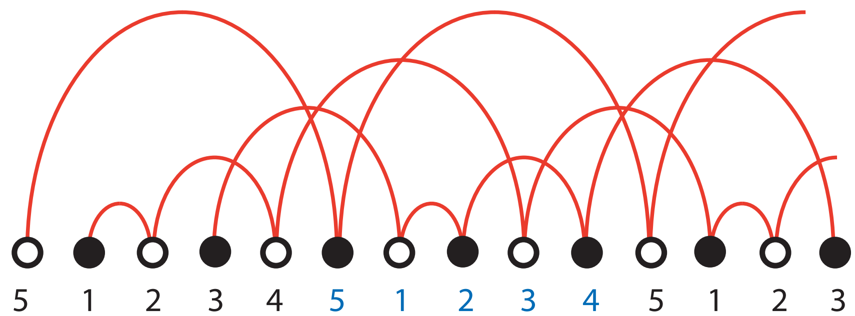
The 3-ball j-cards on the left, and two 5-balls j-cards.

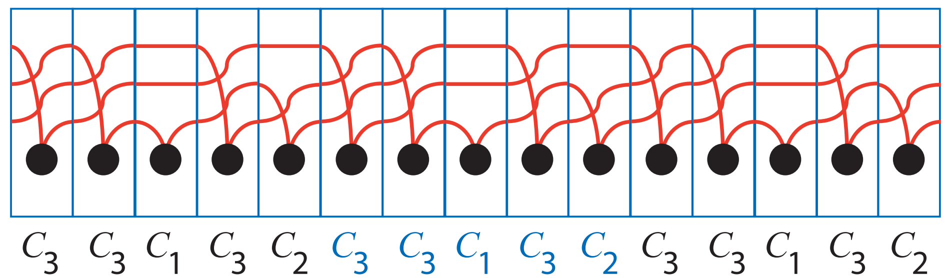
The talk is restricted to 3 balls; the general case is analogous. In a 3-ball juggling card, there are 3 „input levels” on the left, at height 1,2,3, and there are 3 ”output levels” on the right, at height 1,2,3. In case of C_0, nothing happens, the trajectories keep going horizontally.
In case of C_i, for $i \in \{1, \dots, 3\}$, the trajectory of input height i leaves at height 1, those with input height $< i$ are lifted by 1 level, and those with input height $> i$ keep their heights.
The black-filled circle represents our hands.





Observe that normalization means the following: balls try to get higher by 1, but they can only when there is a vacancy above. This happens exactly when a ball nosedives down. Otherwise, the balls keep their heights.





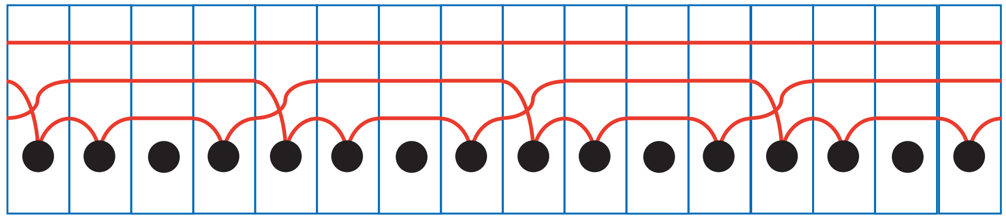
A trajectory intersects other trajectories only when it nosedives, and intersects as many, as are below. Therefore, the classical figure determines the normalized figure as follows. Take a beat; e.g., take the blue 5. Count the intersection points on the descending half of the trajectory arc arriving at 5. If it is i (now $i = 2$), then we need the card. C_{i+1} here (now it is C_3).

We have seen that the juggling sequence determines the classical figure, and the classical figure determines the normalized figure.

Conversely, the normalized figure obviously determines the classical one: just reshape the „dented" arcs of trajectories to make them regular arcs.

Clearly, this process, back and forth, keeps the period.

$\Longrightarrow$ it suffices to count the sequences $(C_{i_0}, \ldots, C_{i_{p-1}})$ of b-ball juggling cards! This number is 4^p. If we have b balls, then there are $(b+1)^p$ many length p sequences of **at most** b-ball juggling cards.



Why „**at most** b-ball" rather than „exactly b-ball"? Because if $C_3 = C_b$ is not used, then there is an imaginary ball that never drops and never was thrown, so the figure above is actually a 2-ball pattern.

In general: the number of balls = the greatest subscript in the sequence of juggling cards. It follows that:

Lemma: The number of at most b-ball juggling sequences of length p is $(b+1)^p$.

The number of b-ball juggling sequences of length p is $(b+1)^p - b^p$.

Lemma: A juggling sequence $(a_0, a_1, \ldots, a_{p-1})$ is minimal iff it has exactly p many cyclic shifts.

Examples (instead of the easy proof): 3456 is minimal and it has four shifts: 3456, 6345, 5634, 6345.
343434 is not minimal, and it has only two shifts: 343434 and 434343.

Each juggling sequence $\vec{s}$ is obtained by repeating a minimal one, whose length divides the length of $\vec{s}$. Let $m(b, d)$ denote the number of minimal b-ball sequences of length d. Then $\sum_{d|p} m(b, d) = |\{b\text{-ball j.sequences of length } p\}| = (b+1)^p - b^p$

We have proved $\sum_{d|p} m(b,d) = (b+1)^p - b^p$.

Möbius inversion formula says that if $f, g\colon \mathbf{N} \to \mathbf{C}$ are function such that $f(n) = \sum_{d|n} g(d)$, for all $n \in \mathbf{N}$, then $g(n) = \sum_{d|n} \mu(n/d) \cdot f(d)$, for all $n \in \mathbf{N}$.

Applying Möbius inversion formula for $f(n) = (b+1)^n - b^n$ and $g(n) = m(n)$, we conclude that

the number of minimal b-ball j.sequences of length $p = m(p) = \sum_{d|n} \mu(n/d) \cdot ((b+1)^n - b^n)$. Finally, we divide by p to pass from p distinct cyclic shifts of a minimal juggling sequence to a single juggling pattern. Q.e.d

The most important instruction for the practical side: do **not** practice without knowing what to practice and how. (The average time to learn the 3-ball cascade, whose sequence is 3, is less than 7 hours; one hour per days during a week. But if someone gets used to a wrong move, it may take a month to correct it.)

http://www.math.u-szeged.hu/~czedli/