Fractal lattices I.*

by Gábor CZÉDLI (Szeged)

to honour the 70th birthday of Tibor Katriňák
at the conference in Tale (September, 2007)


*http://www.math.u-szeged.hu/~cedli/
Happy birthday!
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- the atomless countable boolean lattice.
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**Open problems:**
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Open problems: Are there more than countably many fractal generated varieties? Are they all modular? {all lattices} ?
There are continuously many lattice varieties such that each of them is not semifractal generated.

Key idea of the proof.
Theorem 2. There are continuously many lattice varieties such that each of them is not semifractal generated.

Key idea of the proof. Suppose $L$ is a nondistributive semifractal and $\mathcal{V} = \text{HSP}\{L\}$. Then $N_5$ of $M_3$ is a sublattice of $L$. Suppose $N_5 \leq L$. ($M_3 \leq L$ would be more complicated).
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S.i. length not bounded. \( 2^{\aleph_0} \) many varieties fails this. Q.e.d.
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An application of quasifractals
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Now, if $\mathcal{V}$ is a convexity and some $L \in \mathcal{V}$ has a nontrivial distributive interval (e.g., $a \prec b$) then $\text{HCP}\{2\}$ is included in $\mathcal{V}$. 
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Now, if \( \mathcal{V} \) is a *convexity* and some \( L \in \mathcal{V} \) has a nontrivial distributive interval (e.g., \( a \prec b \)) then \( \text{HCP}\{2\} \) is included in \( \mathcal{V} \).

J. Lihová: \( \text{HCP}\{2\} \subseteq \mathcal{V} \) in many other cases.
Theorem 4. (Partial answer to Jakubík’s problem.) If $L$ is an $M_3$-simple quasifractal then the convexity $\mathcal{V} := \text{HCP}\{L\}$ includes no minimal subconvexity.

Note that such an $L$ exists by Thm. 3.

Proof.
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Finally, if $L' \in \text{HCP}\{M\}$ then Lemma (with $M$ in place of $L$) $\Rightarrow$ $|L'| \geq |M|$, a contradiction. Hence $\text{HCP}\{M\} \subset \mathcal{W}$.
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Proof of the Lemma. \( L' \in \text{HCP}\{L\} = \text{Lihová PsHCP}_u\{L\} \). So \( L' \in \text{HCP}_u\{L\} \). Now \( P_u \) preserves the red properties (by Ľoš’ thm.), so does \( C \) (by def’s), and \( H \) does nothing. Q.e.d.

Proof. Suppose \( \mathcal{W} \subseteq \mathcal{V} \) is minimal. Choose a subd. irr. \( L' \in \mathcal{W} \) and an ultrapower \( M \) of \( L' \) with \( |M| > |L'| \). Lemma \( \Rightarrow \) \( L' \) has the red properties, the ultrapower preserves the red properties, so \( M \) is an \( M_3\)-simple quasifractal, \( |M| > |L'| \) and \( M \in \mathcal{W} \).

Finally, if \( L' \in \text{HCP}\{M\} \) then Lemma (with \( M \) in place of \( L \)) \( \Rightarrow \) \( |L'| \geq |M| \), a contradiction. Hence \( \text{HCP}\{M\} \subset \mathcal{W} \) shows that \( \mathcal{W} \) is not minimal. Q.e.d.
\( F \cong [a, b]; \ QF \xrightarrow{01} [a, b] \xrightarrow{01} QF; \ SF \rightarrow [a, b]; \ HCP \) 

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Thank you, Miroslav, for organizing this Summer School so well!

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