The asymptotic number of ways to intersect two composition series

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\[ \tilde{H} = \{1 = H_0 \triangleleft H_1 \triangleleft \cdots \triangleleft H_n = G\} \]
\[ \tilde{K} = \{1 = K_0 \triangleleft K_1 \triangleleft \cdots \triangleleft K_m = G\} \]
The Jordan-Hölder theorem

C. Jordan (1870), O. Hölder (1889).

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G. Grätzer, J.B. Nation (2010): \( \exists \pi, \, H_i/H_{i-1} \not\cong K_{\pi(i)}/K_{\pi(i)-1} \)
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\text{subnormal} = \triangleleft^* \text{. SNSub}(G), \text{ a poset}

H. Wielandt 1939: if \( \exists \mathcal{H} \), then SNSub(G) is a sublattice of Sub(G). Not hard: then SNSub(G) is lower semimodular.
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Define \( \text{CSL}_G(\vec{H}, \vec{K}) := \{H_i \cap K_j : i, j \in \{0, \ldots, n\}\}; \subseteq \)
Fact: $\text{CSL}_G(\vec{H}, \vec{K})$ is a $\cap$-subsemilattice of $\text{NSub}(G)$, whence lower semimodular.
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O: \( \text{CSL}_G(\vec{H}, \vec{K}) \) is lower semimodular and meet-generated by two chains.
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O: \( \text{CSL}_G(\vec{H}, \vec{K}) \) is lower semimodular and meet-generated by two chains.

Prop: Assume \( k = p_1 \ldots p_n \) and \( L \) is lower semimodular, meet-generated by two chains, and \( \text{length}(L) = n \). Then the cyclic \( C_k \) group of order \( k \) has \( \vec{H}, \vec{K} \) with \( L \cong \text{CSL}_{C_k}(\vec{H}, \vec{K}) \).
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Describes what we want to count.

By duality,
Fact: $\text{CSL}_G(\vec{H}, \vec{K})$ is a $\cap$-subsemilattice of $\text{NSub}(G)$, whence lower semimodular.

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Describes what we want to count.

By duality, it suffices to count slim (= join-generated by two chains) semimodular lattices of length $n$, asymptotically.

*http://www.math.u-szeged.hu/~czedli/
1. Describe these lattices (Cz-Sch) → permutations!

2. Count permutations (Cz-O-U).

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Part I: description by permutations.

**Thm** (Cz-Sch): Slim semimodular (planar) diagrams ↔ permutations.

Need: a pair of reciprocal bijections.

*http://www.math.u-szeged.hu/~cedli/*
$D \mapsto \pi$ by a **locomotive**. $\pi \mapsto D$ by quotient join-semilattice.

\[
\begin{bmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
1 & 2 & 5 & 6 & 4 & 3 & 8 & 7
\end{bmatrix}
\]

*http://www.math.u-szeged.hu/~czedli/*
Reflecting a segment $\iff$ inverting the restriction of $\pi$

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The segments of $\pi$ are $\{1\}$, $\{2\}$, $\{3, 4, 5, 6\}$, $\{7, 8\}$.

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Lemma: \( L(\pi) \cong L(\tau) \) iff \( \pi \) and \( \tau \) are “sectionally inverse or equal”, denoted by \( \pi \sim \tau \).

It suffices to determine \( |S_n/\sim| \), asymptotically.
\[ A_0(n) := \{ \pi \in S_n : \pi = \pi^{-1} \} \].

\( j \): number of transpositions (2-cycles)

Choosing the set \( \{a_1, \ldots, a_{2j}\} \) of non-fixed elements:
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\[
\frac{|A_0(n)|}{|S_n|} = \frac{1}{n!}
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\frac{|A_0(n)|}{|S_n|} = \frac{1}{n!} \sum_{j=1}^{\lfloor n/2 \rfloor} \binom{n}{2j} (2j - 1)(2j - 3)(2j - 5) \ldots =
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$$\frac{1}{n!} \sum_{j=1}^{\lfloor n/2 \rfloor} \frac{n!}{(n-2j)! \cdot (2j)!} \cdot \frac{(2j)!}{2j \cdot j!} =$$
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\frac{1}{n!} \sum_{j=1}^{\lfloor n/2 \rfloor} \frac{n!}{(n-2j)!} \cdot \frac{(2j)!}{2^{2j} \cdot j!} = \\
\sum_{j=1}^{\lfloor n/4 \rfloor} \frac{1}{(n-2j)! \cdot 2^j \cdot j!} + \sum_{j=\lfloor n/4 \rfloor+1}^{\lfloor n/2 \rfloor} \frac{1}{(n-2j)! \cdot 2^j \cdot j!} = \Sigma' + \Sigma''.
\]

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\[ \frac{|A_0(n)|}{n!} = \sum_{j=1}^{\lfloor n/4 \rfloor} \frac{1}{(n-2j)! \cdot 2^j \cdot j!} + \sum_{j=\lfloor n/4 \rfloor + 1}^{\lfloor n/2 \rfloor} \frac{1}{(n-2j)! \cdot 2^j \cdot j!} = \sum' + \sum''. \]

In \( \sum' \),
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In \( \sum' \), each denominator is at least \((n - 2\lfloor n/4 \rfloor)! \geq \lfloor n/2 \rfloor!\), and there are fewer than \( n \) summands. Hence \( \sum' \leq n \cdot (\lfloor n/2 \rfloor!)^{-1} \to 0 \).

In \( \sum'' \),
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In \( \sum'' \), each denominator is at least \( 2^{n/4} \) and there are fewer than \( n \) summands, so \( \sum'' \leq n \cdot 2^{-n/4} \to 0 \).
\[
\frac{|A_0(n)|}{n!} = \sum_{j=1}^{\lfloor n/4 \rfloor} \frac{1}{(n-2j)!2^j j!} + \sum_{j=\lfloor n/4 \rfloor+1}^{\lfloor n/2 \rfloor} \frac{1}{(n-2j)!2^j j!} = \Sigma' + \Sigma''.
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In \(\Sigma'\), each denominator is at least \((n - 2\lfloor n/4 \rfloor)! \geq \lfloor n/2 \rfloor!\), and there are fewer than \(n\) summands. Hence \(\Sigma' \leq n \cdot (\lfloor n/2 \rfloor!)^{-1} \to 0\).

In \(\Sigma''\), each denominator is at least \(2^{n/4}\) and there are fewer than \(n\) summands, so \(\Sigma'' \leq n \cdot 2^{-n/4} \to 0\). Thus,

\[
\lim_{n \to \infty} \frac{|A_0(n)|}{n!} = 0.
\]

So, involutions can be disregarded.
\[
\frac{|A_0(n)|}{n!} = \sum_{j=1}^{\lfloor n/4 \rfloor} \frac{1}{(n-2j)! \cdot 2^j \cdot j!} + \sum_{j=\lfloor n/4 \rfloor + 1}^{\lfloor n/2 \rfloor} \frac{1}{(n-2j)! \cdot 2^j \cdot j!} = \sum' + \sum''.
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In \(\sum'\), each denominator is at least \((n - 2\lfloor n/4 \rfloor)! \geq \lfloor n/2 \rfloor!\), and there are fewer than \(n\) summands. Hence \(\sum' \leq n \cdot (\lfloor n/2 \rfloor!)^{-1} \to 0\).

In \(\sum''\), each denominator is at least \(2^{n/4}\) and there are fewer than \(n\) summands, so \(\sum'' \leq n \cdot 2^{-n/4} \to 0\). Thus,

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\lim_{n \to \infty} n! \frac{|A_0(n)|}{n!} = 0.
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Large segment: consists of at least 3 elements.

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If $\pi$ has exactly $k$ segments onto which the restriction of $\pi$ is NOT an involution, then the $\sim$-block of $\pi$ is $2^k$-element. Let $A_k(n)$ be the set of all these $\pi$. 
If $\pi$ has exactly $k$ segments onto which the restriction of $\pi$ is NOT an involution, then the $\sim$-block of $\pi$ is $2^k$-element. Let $A_k(n)$ be the set of all these $\pi$. $A_0(n)$ is as before.

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$$S_n = A_0(n) \cup A_1(n) \cup A_2(n) \cup A_3(n) \cup \cdots = A_0(n) \cup A_1(n) \cup B(n).$$

(1)

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The number of $\sim$-blocks is:

$$S_n \mid \sim \mid \frac{n!}{n!} + \frac{|A_1(n)|}{2n!} + \frac{|A_2(n)|}{4n!} + \frac{|A_3(n)|}{8n!} + \cdots. \quad (2)$$

We already know that $|A_0(n)|/n! \to 0$. 
If $\pi$ has exactly $k$ segments onto which the restriction of $\pi$ is NOT an involution, then the $\sim$-block of $\pi$ is $2^k$-element. Let $A_k(n)$ be the set of all these $\pi$. $A_0(n)$ is as before.

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If $\pi$ has exactly $k$ segments onto which the restriction of $\pi$ is NOT an involution, then the $\sim$-block of $\pi$ is $2^k$-element. Let $A_k(n)$ be the set of all these $\pi$. $A_0(n)$ is as before.

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If $\pi$ has exactly $k$ segments onto which the restriction of $\pi$ is NOT an involution, then the $\sim$-block of $\pi$ is $2^k$-element. Let $A_k(n)$ be the set of all these $\pi$. $A_0(n)$ is as before.

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We already know that $|A_0(n)|/n! \to 0$. We are going to show that $|B(n)/n!| \to 0$. Then, since this majorizes the tail, tail $\to 0$. From $|A_0(n)|/n! \to 0$, we have $|B(n)/n!| \to 0$. Therefore, the number of $\sim$-blocks $S_n/|\sim|/n! \to 0$. 
If \( \pi \) has exactly \( k \) segments onto which the restriction of \( \pi \) is NOT an involution, then the \( \sim \)-block of \( \pi \) is \( 2^k \)-element. Let \( A_k(n) \) be the set of all these \( \pi \). \( A_0(n) \) is as before.

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S_n = A_0(n) \cup A_1(n) \cup A_2(n) \cup A_3(n) \cup \cdots = A_0(n) \cup A_1(n) \cup B(n). \tag{1}
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We already know that \( |A_0(n)|/n! \to 0 \). We are going to show that \( |B(n)/n!| \to 0 \). Then, since this majorizes the tail, \( \text{tail} \to 0 \). From \( |A_0(n)|/n! \to 0 \), \( |B(n)/n!| \to 0 \), and
If $\pi$ has exactly $k$ segments onto which the restriction of $\pi$ is NOT an involution, then the $\sim$-block of $\pi$ is $2^k$-element. Let $A_k(n)$ be the set of all these $\pi$. $A_0(n)$ is as before.

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(2)

We already know that $|A_0(n)|/n! \to 0$. We are going to show that $|B(n)/n!| \to 0$. Then, since this majorizes the tail, tail $\to 0$. From $|A_0(n)|/n! \to 0$, $|B(n)/n!| \to 0$, and (1) we obtain $|A_1(n)/n!| \to 1$. Hence $|A_1(n)/(2n!)| \to 1/2$. Finally, $|A_0(n)/n! \to 0$, tail $\to 0$, and $|A_1(n)/(2n!)| \to 1/2$ give the desired $|S_n/\sim|/n! \to 1/2$.

*http://www.math.u-szeged.hu/~czedli/
Suppose $\pi \in B(n)$. Then there are at least two large $\pi$-segments.
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Suppose $\pi \in B(n)$. Then there are at least two large $\pi$-segments. We define the pivot element $p(\pi)$ of $\pi$ as the greatest element of the leftmost large $\pi$-segment. Then $3 \leq p(\pi) \leq n - 3$ since there are at least two large $\pi$-segments.
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Both the intervals $[1, p(\pi)] = \{1, \ldots, p(\pi)\}$ and $[p(\pi) + 1, n]$ are unions of $\pi$-segments, whence both are closed with respect to $\pi$. Hence if we denote the restrictions of $\pi$ to these intervals by $\lambda = \pi|_{[1, p(\pi)]}$ and $\rho = \pi|_{[p(\pi) + 1, n]}$, then $\pi$ is determined by $\lambda$ and $\rho$. 
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Since $\lambda \in S_{p(\pi)}$, there are at most $p(\pi)!$ many such $\lambda$. (In fact, there are much fewer.) Similarly, there are at most $(n - p(\pi))!$ many $\rho$.

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Taking the well-known fact
\[ \binom{n}{3} \leq \binom{n}{4} \leq \cdots \leq \binom{n}{\lfloor n/2 \rfloor} = \binom{n}{\lfloor n/2 \rfloor} \geq \binom{n}{\lceil n/2 \rceil + 1} \geq \cdots \geq \binom{n}{n-3} \]
at \leq^* into account and counting the permutations according to their pivot elements, we obtain:

\[ \frac{|B(n)|}{n!} \leq \frac{1}{n!} \sum_{k=3}^{n-3} k! \cdot (n-k)! = \]
Taking the well-known fact
\[
\binom{n}{3} \leq \binom{n}{4} \leq \cdots \leq \binom{n}{\lfloor n/2 \rfloor} = \binom{n}{\lfloor n/2 \rfloor} \geq \binom{n}{\lfloor n/2 \rfloor + 1} \geq \cdots \geq \binom{n}{n-3}
\]
at \leq^* \text{ into account and counting the permutations according to their pivot elements, we obtain:}

\[
\frac{|B(n)|}{n!} \leq \frac{1}{n!} \sum_{k=3}^{n-3} k! \cdot (n-k)! = \sum_{k=3}^{n-3} \frac{k! \cdot (n-k)!}{n!} =
\]
Taking the well-known fact
\[ \binom{n}{3} \leq \binom{n}{4} \leq \cdots \leq \binom{n}{\lfloor n/2 \rfloor} = \binom{n}{\lceil n/2 \rceil} \geq \binom{n}{\lceil n/2 \rceil + 1} \geq \cdots \geq \binom{n}{n-3} \]
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\]
Taking the well-known fact

\[ \binom{n}{3} \leq \binom{n}{4} \leq \cdots \leq \binom{n}{\lfloor n/2 \rfloor} = \binom{n}{\lfloor n/2 \rfloor + 1} \geq \cdots \geq \binom{n}{n-3} \]

at \( \leq^* \) into account and counting the permutations according to their pivot elements, we obtain:

\[
\frac{|B(n)|}{n!} \leq \frac{1}{n!} \sum_{k=3}^{n-3} k! \cdot (n-k)! = \sum_{k=3}^{n-3} \frac{k! \cdot (n-k)!}{n!} = \sum_{k=3}^{n-3} \left( \frac{n}{k} \right)^{-1}
\]

\[
\leq^* \sum_{k=3}^{n-3} \left( \frac{n}{3} \right)^{-1} \leq n \cdot \frac{6}{n(n-1)(n-2)} \to 0.
\]
Taking the well-known fact
\[
\binom{n}{3} \leq \binom{n}{4} \leq \cdots \leq \binom{n}{\lfloor n/2 \rfloor} = \binom{n}{\lfloor n/2 \rfloor} \geq \binom{n}{\lfloor n/2 \rfloor + 1} \geq \cdots \geq \binom{n}{n-3} = \binom{n}{\lceil n/2 \rceil} \geq \binom{n}{\lceil n/2 \rceil + 1} \geq \cdots \geq \binom{n}{n-3}
\]
at \leq^* into account and counting the permutations according to their pivot elements, we obtain:

\[
\frac{|B(n)|}{n!} \leq \frac{1}{n!} \sum_{k=3}^{n-3} \frac{k! \cdot (n-k)!}{n!} = \sum_{k=3}^{n-3} \frac{k! \cdot (n-k)!}{n!} = \sum_{k=3}^{n-3} \binom{n}{k}^{-1}
\]

\[
\leq^* \sum_{k=3}^{n-3} \binom{n}{3}^{-1} \leq n \cdot \frac{6}{n(n-1)(n-2)} \to 0. \quad \text{Q.E.D.}
\]

*http://www.math.u-szeged.hu/~cedli/