On the linear model of swinging with square wave coefficient by an elementary geometric method

L. Csizmadia and L. Hatvani[*]

Dedicated to Professor László Leindler on his 80th birthday

December 12, 2017

Abstract

The equation

$$x''+a^2(t)x = 0,$$

$$a(t) := \begin{cases} \sqrt{\dfrac{g}{l-\varepsilon}} & \text{if } 2kT \le t < (2k+1)T, \\ \sqrt{\dfrac{g}{l+\varepsilon}} & \text{if } (2k+1)T \le t < (2k+2)T, \ (k = 0, 1, \dots) \end{cases}$$

is considered, where g and l denote the constant of gravity and the length of the pendulum, respectively; $\varepsilon > 0$ is a parameter measuring the intensity of swinging. Concepts of solutions going away from the origin and approaching to the origin are introduced. Necessary and sufficient conditions are given in terms of T and ε for the existence of solutions of these types, which yield conditions for the existence of $2T$-periodic and $4T$-periodic solutions as special cases. The domain of instability, i.e. the Arnold tongues of parametric resonance are deduced from these results.

[*]The second author was supported by the Hungarian Scientific Research (OTKA 109782) and Analysis and Stohastic Research Group of the Hungarian Academy of Sciences

Key words: second order linear differential equations, step function coefficients, periodic coefficients, impulsive effects, periodic solutions, parametric resonance, swinging.

2000 Mathematics Subject Classification: Primary: 34D20, 70J40; Secondary: 34A26, 34A37, 34C25.

1 Introduction

The mathematical model of swinging is the parametrically excited pendulum equation. As is known [1, 16], the small oscillations of the mathematical pendulum are described by the linear equation

$$x'' + \frac{g}{l}x = 0, \tag{1.1}$$

where x denotes the angle between the rod of the pendulum and the direction downward measured counter-clockwise; g and l are the gravity acceleration and the length of the rod, respectively. All solutions of this equation are periodic of the same period $2\pi\sqrt{l/g}$, and the equilibrium state $x = x' = 0$ is stable. A swing, as a physical pendulum is equivalent to a mathematical pendulum of an appropriate length [12], provided that the swinger is motionless. However, the swinger wants to destabilize the pendulum, so he/she moves squatting and raising consecutively. As a result, the distance of the center of gravity of the physical pendulum from the suspension point (i.e, the length of the mathematical pendulum) changes in time periodically, and motions are described by the equation

$$\begin{cases} x'' + a^2(t)x = 0, \\ a(t) := \begin{cases} a_1 := \sqrt{\dfrac{g}{l-\varepsilon}} & \text{if } 2kT \le t < (2k+1)T, \\ a_2 := \sqrt{\dfrac{g}{l+\varepsilon}} & \text{if } (2k+1)T \le t < (2k+2)T, \end{cases} \qquad (k = 0, 1, \dots). \end{cases} \tag{1.2}$$

This is a linear equation with periodic coefficient (Hill's equation [10, 14]) whose coefficient is a piecewise constant function (Meissner's equation [15, 17]). The problem of swinging is to find the instability domain on the parametric plane (T, ε) to the excited equation (1.2) (problem of *parametric resonance*). By the use of the Floque Theory it has been proved [1, 2, 16] that the stability domain has separate components ("Arnold tongues", see Figure 7), and the two boundary curves of type $T = f(\varepsilon)$, $T = g(\varepsilon)$ of every tongue consist of points of the plane for which (1.2) has either $2T$-periodic or $4T$-periodic solutions, and $f(\varepsilon)$, $g(\varepsilon)$ go to one of the points $((k/2)(\pi\sqrt{l/g}), 0)$ $(k \in \mathbb{N})$ as $\varepsilon \to 0$. This result harmonizes with experiences: for small $\varepsilon > 0$, period of swinging has to be equal to a multiple of the half of the own period of the unexcited pendulum (1.1) [3].

In this paper we show that the problem of swinging can be solved by the use of an elementary geometric method. At first we find the conditions guaranteeing that the trajectory $t \mapsto (x(t; x_0, x_0'), x'(t; x_0, x_0'))$ of (1.2) starting from a point $P(x_0, x_0')$ of the (x, x') plane returns to a point of the line L through the origin $(0, 0)$ and P in the sense that $(x(2T; x_0, x_0'), x'(2T; x_0, x_0')) \in L$ (see the *angle-periodic solutions* later). Such a solution is either going away from the origin, or approaching to the origin, or periodic of either $2T$ or $4T$. For every one of these four properties we determine the set on the parameter plane (T, ε) whose points represent equations having solutions with the given property. It will be pointed out that the set of going away and that of approaching coincide, and they are bounded by the curves of periodicity in accordance with the conclusions of the Floque Theory. It is worth emphasizing that, besides the method is constructive, it can be used also in the case when the coefficient is not exactly periodic [4], even also for nonlinear equations [7], to which cases the Floquet Theory cannot be applied.

The paper is organized as follows. In Section 2, to make the paper self-contained, we review the method. In Section 3 we construct the sets of going away and approaching, from which we will get the set of periodicity as a special case. In Section 4 we deduce stability conclusions.

2 The method

Equation (1.2) can be handled by the method of the investigation of equations with piecewise coefficients established in [5] and developed in [6]. Given two sequences $\{a_k\}_{k=1}^{\infty}$, $\{t_k\}_{k=0}^{\infty}$ of positive numbers ($\lim_{k\to\infty} t_k = \infty$) with $t_0 := 0$, consider the equation

$$x'' + a^2(t)x = 0, \qquad a(t) := a_k \text{ if } t_{k-1} \le t < t_k \quad (k \in \mathbb{N}). \tag{2.1}$$

A function $x : [0, \infty) \to \mathbb{R}$ is a solution of (2.1) if it is continuously differentiable on $[0, \infty)$, the restriction $x|_{[t_{k-1}, t_k)}$ is twice differentiable and solves the equation for $k \in \mathbb{N}$. With the new state variable $y := x'/a_k$ we can write (2.1) in the form of the 2-dimensional system

$$x' = a_k y, \quad y' = -a_k x \qquad (t_{k-1} \le t < t_k,\ k \in \mathbb{N}). \tag{2.2}$$

We must guarantee that system (2.2) is equivalent to equation (2.1). To this end it is enough to require that $x(t_k) = x(t_k - 0)$, $x'(t_k) = x'(t_k - 0)$ ($k \in \mathbb{N}$), where $f(t-0)$ denotes the left-hand side limit of function f at t. This implies some additional "connectivity" conditions for solutions of (2.2) as follows. Let the first equality be required as an initial condition on the interval $[t_k, t_{k+1})$. The second one says $a_{k+1}y(t_k) = a_k y(t_k - 0)$ for every $k \in \mathbb{N}$, which yields the other initial condition on $[t_k, t_{k+1})$. This means that (2.1) is equivalent to the system of first order differential equations with impulses

$$\begin{cases} x' = a_k y, \quad y' = -a_k x \qquad (t_{k-1} \le t < t_k), \\ x(t_k) = x(t_k - 0), \quad y(t_k) = \dfrac{a_k}{a_{k+1}} y(t_k - 0) \qquad (k \in \mathbb{N}). \end{cases} \tag{2.3}$$

If a pair x_0, y_0 are given we can construct the solution of (2.3) on $[0, \infty)$ satisfying the initial condition $x(t_0) = x_0, y(t_0) = y_0$ in the following way. We solve the equation (2.2) with these initial conditions in $[t_0, t_1)$ and have the solution $(x_1, y_1) : [t_0, t_1) \to \mathbb{R}^2$. Then we define $x_2(t_1) := x_1(t_1 - 0)$, $y_2(t_1) := (a_1/a_2)y_1(t_1 - 0)$ and solve equation (2.2) with these initial conditions in $[t_1, t_2)$, and so on. Thus we get the solution of (2.3) on $[t_0, \infty)$ by the definition

$$(x(t), y(t)) := (x_k(t), y_k(t)) \quad \text{if } t_{k-1} \le t < t_k, \qquad (k \in \mathbb{N}).$$

Since $(x^2(t) + y^2(t))' \equiv 0$ on the intervals $[t_{k-1}, t_k)$ ($k \in \mathbb{N}$), pieces of the trajectory of this solution are located on circles around the origin, and at $t = t_k$ the trajectory makes a jump parallel with the y-axis.

Due to its special form, the impulsive system (2.3) can be represented as a discrete dynamical system on the plane (x, y). Introduce the polar coordinates r, φ by the formulae

$$x = r\cos\varphi, \quad y = r\sin\varphi \qquad (r > 0, -\infty < \varphi < \infty). \tag{2.4}$$

We know that $r'(t) \equiv 0$ along any solution of (2.3) in every interval $[t_{k-1}, t_k)$. Since

$$x'(t) = -r(t)\varphi'(t)\sin\varphi(t) = a_k y(t) = a_k r(t)\sin\varphi(t) \qquad (t_{k-1} \le t < t_k),$$

we have

$$\varphi'(t) = -a_k \qquad (t_{k-1} \le t < t_k). \tag{2.5}$$

So, the continuous components of the dynamics of (2.3) are uniform clockwise rotations around the origin with the angle velocity a_k. The impulsive steps of the dynamics are either contractions or dilatations in the direction of the y-axis.

The steps of dynamics of (2.3) can be described as follows. The phase point starts from (x_0, y_0) and turns clockwise around the origin by $a_1(t_1 - t_0)$, then a contraction or a dilatation of measure a_1/a_2 happens parallel with y-axis. This will be the image (x_1, y_1) of the point (x_0, y_0) after the first step. Applying the same two transformations to the new point (x_1, y_1) with the new parameters $a_2(t_2 - t_1)$ and a_2/a_3 we get the next state: (x_2, y_2). We repeat these steps ad infinitum, see Figure 1.

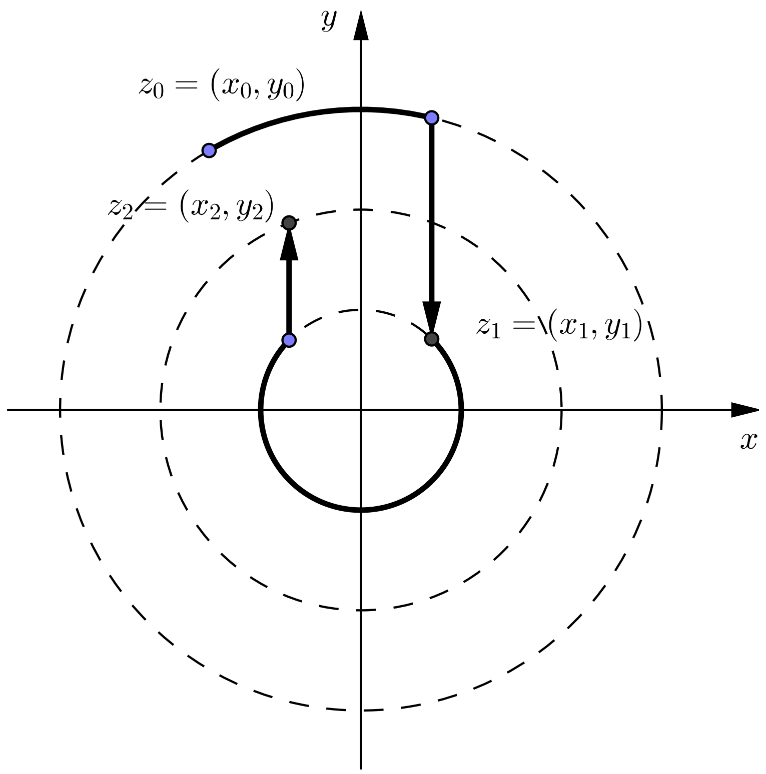

Figure 1: Steps of the dynamics of (2.3) or (2.7)

Introducing the notations $R(\theta)$ and $C(\kappa)$ for the matrices of the rotation and the contraction-dilatation, respectively, i.e.,

$$\begin{aligned} R(\theta) &= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \qquad (-\infty < \theta < \infty), \\ C(\kappa) &= \begin{pmatrix} 1 & 0 \\ 0 & \kappa \end{pmatrix} \qquad \left(\kappa = \frac{a_k}{a_{k+1}},\ 0 < \kappa < \infty\right), \end{aligned} \tag{2.6}$$

we can give the discrete dynamical system

$$\begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} = C\left(\frac{a_{k+1}}{a_{k+2}}\right) R(a_{k+1}(t_{k+1}-t_k)) \begin{pmatrix} x_k \\ y_k \end{pmatrix} \qquad (k = 0, 1, 2, \dots), \tag{2.7}$$

equivalent to (2.3).

Let us consider this system in polar coordinates. Denote by (r_R, φ_R), and $(r_C, \varphi_C) = (\rho(r, \varphi; \kappa), \phi(\varphi; \kappa))$ the image of the point (r, φ) at the rotation and the contraction-dilatation (2.6), respectively. Then, obviously, $r_R(r, \varphi) = r$, $\varphi_R(r, \varphi) = \varphi - \theta$; furthermore,

$$\rho(r, \varphi; \kappa) = \sqrt{x^2 + \kappa^2 y^2} = r\sqrt{1 + (\kappa^2 - 1)\sin^2\varphi} = f(\varphi; \kappa) r,$$

$$f(\varphi, \kappa) := \sqrt{1 + (\kappa^2 - 1)\sin^2\varphi}, \qquad (\kappa > 0, -\infty < \varphi < \infty).$$

It is easy to see that $\tan\phi(\varphi; \kappa) = \kappa y/x = \kappa \tan\varphi$ $(x \neq 0)$, so

$$\phi(\varphi; \kappa) := \begin{cases} \arctan(\kappa\tan\varphi) + \left[\dfrac{\varphi + \frac{\pi}{2}}{\pi}\right] \cdot \pi & \text{if } \varphi \neq (2k+1)\dfrac{\pi}{2}, \\ \varphi & \text{if } \varphi = (2k+1)\dfrac{\pi}{2}, \qquad (k \in \mathbb{Z}), \end{cases}$$

where $[x]$ denotes the integer part of $x \in \mathbb{R}$. Now, the system (2.3) in polar coordinates has the form

$$\begin{cases} r_{k+1} = f\left(\varphi_k - a_{k+1}(t_{k+1} - t_k); \dfrac{a_{k+1}}{a_{k+2}}\right) r_k, \\ \varphi_{k+1} = \phi\left(\varphi_k - a_{k+1}(t_{k+1} - t_k); \dfrac{a_{k+1}}{a_{k+2}}\right), \end{cases} \qquad (k = 0, 1, 2, \dots). \tag{2.8}$$

The following lemma summarizes the properties of functions f and ϕ. The very simple proof can be found in [6].

Lemma 2.1. *1. For every $\kappa > 0$ the function $f(\cdot; \kappa) : \mathbb{R} \to (0, \infty)$ is even and π-periodic; furthermore,*

$$f\left(\phi(\varphi; \kappa); \frac{1}{\kappa}\right) = \frac{1}{f(\varphi; \kappa)} \qquad (\varphi \in \mathbb{R})$$

(see Figure 2).

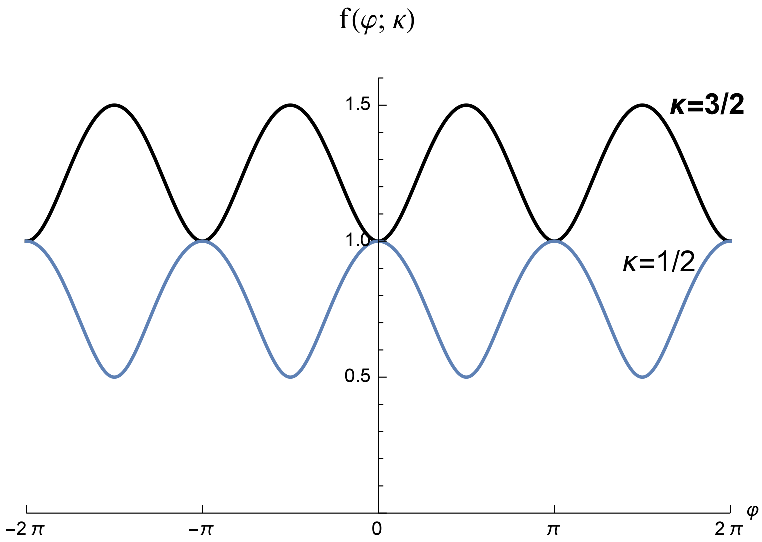

Figure 2: Graph of f

2. For every $\kappa > 0$ the function $\phi(\cdot;\kappa)$ and $\phi(\cdot+\pi/2;\kappa)-\pi/2$ are odd, $\phi(\cdot+k\pi;\kappa) = \phi(\cdot;\kappa) + k\pi$ $(k \in \mathbb{Z})$; furthermore,

$$\phi\left(\phi(\varphi;\kappa); \frac{1}{\kappa}\right) = \varphi \qquad (\varphi \in \mathbb{R}).$$

3. If $0 < \kappa < 1$, then for all $k \in \mathbb{Z}$ we have

$$\phi(\varphi;\kappa) < \varphi \quad \textit{if } 2k\frac{\pi}{2} < \varphi < (2k+1)\frac{\pi}{2},$$

$$\phi(\varphi;\kappa) > \varphi \quad \textit{if } (2k+1)\frac{\pi}{2} < \varphi < 2(k+1)\frac{\pi}{2}.$$

4. If $\kappa > 1$, then the inequalities between $\phi(\varphi;\kappa)$ and φ are of the opposite directions (see Figure 3).

3 Periodic, approaching and going away solutions

To transform (1.2) into an equation containing the period of excitation explicitly as a parameter, introduce the so called non-dimensional time $\tau = (\pi/T)t$ and the

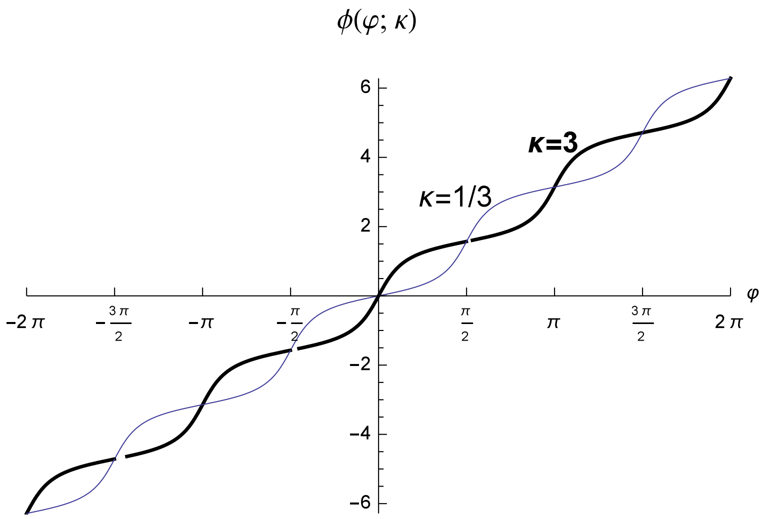

Figure 3: Graph of ϕ

new variable $z(\tau) = x((T/\pi)\tau)$. Then (1.2) has the form

$$\ddot{z} + A^2(\tau)z = 0, \qquad \dot{(\;)} = \frac{\mathrm{d}}{\mathrm{d}\tau}(\;), \tag{3.1}$$

where

$$A(\tau) = \frac{T}{\pi}a\left(\frac{T}{\pi}\tau\right) = \begin{cases} \dfrac{T}{\pi}\sqrt{\dfrac{g}{l-\varepsilon}} & \text{if } 2k\pi \le \tau < (2k+1)\pi, \\ \dfrac{T}{\pi}\sqrt{\dfrac{g}{l+\varepsilon}} & \text{if } (2k+1)\pi \le \tau < (2k+2)\pi \qquad (k \in \mathbb{Z}_+ := \{0, 1, \ldots\}). \end{cases}$$

For the sake of habit we use t instead of τ and x, x' instead of z, $\dot{z}$. If we introduce the parameter $\lambda := T/\pi$, then (3.1) takes the form

$$x'' + \lambda^2 Q(t)x = 0,$$
$$Q(t) = \begin{cases} a_1^2 = a_1^2(\varepsilon) := \dfrac{g}{l-\varepsilon} & \text{if } 2k\pi \le t < (2k+1)\pi, \\ a_2^2 = a_2^2(\varepsilon) := \dfrac{g}{l+\varepsilon} & \text{if } (2k+1)\pi \le t < (2k+2)\pi \qquad (k \in \mathbb{Z}_+). \end{cases} \tag{3.2}$$

Setting

$$t_k := k\pi, \qquad a_{2k+1} := \lambda a_1,\ a_{2k+2} := \lambda a_2 \qquad (k \in \mathbb{Z}_+),$$
$$D := \frac{a_1}{a_2}, \quad d := \frac{a_2}{a_1} \tag{3.3}$$

our equation (3.2) is of the form (2.1). The corresponding system (2.7) with the old-new variables x, $y := x_k'/a_k$ reads as follows:

$$\begin{cases} \begin{pmatrix} x_{2\ell+1} \\ y_{2\ell+1} \end{pmatrix} = C(D)R(\lambda a_1\pi)\begin{pmatrix} x_{2\ell} \\ y_{2\ell} \end{pmatrix}, \\ \begin{pmatrix} x_{2\ell+2} \\ y_{2\ell+2} \end{pmatrix} = C(d)R(\lambda a_2\pi)\begin{pmatrix} x_{2\ell+1} \\ y_{2\ell+1} \end{pmatrix} \quad (l \in \mathbb{Z}_+). \end{cases} \tag{3.4}$$

After turning to polar coordinates (2.4), let us start a trajectory from r_0, φ_0 at $t_0 = 0$. For the first five notable points of the trajectory we introduce the notations

$$\begin{array}{ll} r_0 := r(0), & \varphi_0 :\equiv \varphi(0) \pmod{2\pi},\ -\pi \le \varphi_0 < \pi; \\ r_1 := r(\pi - 0)(= r_0), & \varphi_1 := \varphi(\pi - 0); \\ r_2 := r(\pi) = f\,(\varphi_1; D)\,r_1, & \varphi_2 := \varphi(\pi) = \phi\,(\varphi_1; D)\,; \\ r_3 := r(2\pi - 0)(= r_2), & \varphi_3 := \varphi(2\pi - 0); \\ r_4 := r(2\pi) = f(\varphi_3; d)r_3, & \varphi_4 := \varphi(2\pi) = \phi(\varphi_3; d), \end{array} \tag{3.5}$$

(indexing differs from that in (2.8) and (3.4)!, see Figure 4).

Definition 3.1. A solution of system (3.4) is called *angle-periodic of period* 2π (respectively, 4π) if

$$\varphi_4 \equiv \varphi_0 \pmod{2\pi} \qquad \text{(respectively, } \varphi_4 \equiv \varphi_0 - \pi \pmod{2\pi}\text{)}.$$

Definition 3.2. An angle-periodic solution of system (3.4) with period 2π or 4π is called *approaching (to the origin)* (respectively, *going away (from the origin)*) if

$$r_4 < r_0 \quad \text{(respectively,} \quad r_4 > r_0\text{)}.$$

Taking into account properties of functions f and ϕ (see Lemma 2.1), it is easy to prove that if a solution is approaching (respectively, going away), then

$$\varphi(2n\pi) \equiv \varphi(0) \pmod{2\pi} \qquad \text{(respectively, } \varphi(4n\pi) \equiv \varphi(0) \pmod{2\pi}\text{)},$$

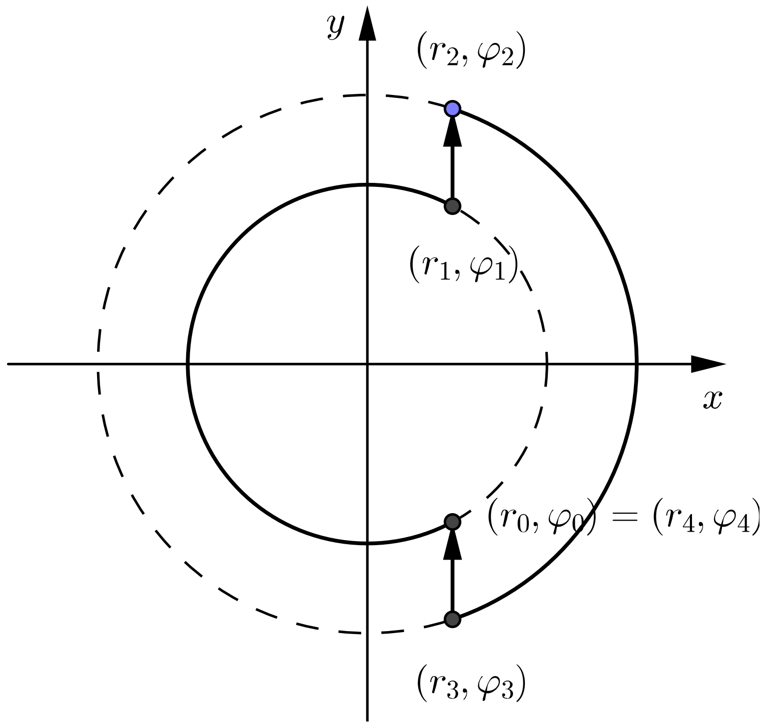

Figure 4: The first four steps of the dynamics of (3.4)

and

$$r(2n\pi) = \left(\frac{r_4}{r(0)}\right)^n r_0 \qquad (n \in \mathbb{N}).$$

Furthermore, a solution of system (3.4) is 2π-periodic (respectively, 4π-periodic) if and only if it is angle-periodic with period 2π (respectively, 4π) and $r_4 = r_0$.

Lemma 3.3. *1. An angle periodic solution of* (3.4) *is approaching (respectively, going away) if and only if*

$$f(\varphi_1; D) < f(\varphi_0; D) \qquad (\textit{respectively, } f(\varphi_1; D) > f(\varphi_0; D)).$$

2. An angle-periodic solution of (3.4) *with period* 2π *or* 4π *is periodic with the same period if and only if*

$$f(\varphi_1; D) = f(\varphi_0; D).$$

Proof. Let us consider an angle-periodic solution of (3.4) with period 2π. Then $\varphi_4 = \varphi_0 - 2(p+1)\pi$ with some $p \in \mathbb{Z}_+$, so using Lemma 2.1 we have

$$\varphi_3 = \phi^{-1}(\varphi_4; d) = \phi(\varphi_0 - 2(p+1)\pi; D) = \phi(\varphi_0; D) - 2(p+1)\pi \qquad (3.6)$$

and

$$r_4 = f(\varphi_3; d) r_3 = f(\phi(\varphi_0; D); d) r_3 = \frac{1}{f(\varphi_0; D)} r_3 = \frac{f(\varphi_1; D)}{f(\varphi_0; D)} r_0,$$

whence we get all the three statements regarding the 2π-periodic solutions.

In the 4π-periodic case $\varphi_4 = \varphi_0 - (2p+1)\pi$ with some integer $p \in \mathbb{Z}_+$, but this difference does not play any role in the proof. □

Lemma 3.3 says that φ_0 and φ_1 uniquely determine that the distance from the origin along an angle-periodic trajectory either tends to zero, or diverges to the infinity, or changes periodically. Now we classify the points of the stripe $(0 \leq \varphi_0 < \pi; \varphi_1 < \varphi_0)$ in the plane (φ_0, φ_1) according to these three properties. Using Lemma 2.1 (see also Figure 2) one can see that the solutions of the equation $f(\varphi_1; D) = f(\varphi_0; D)$ are all the points of the lines

$$\varphi_1 = \varphi_0 + j\pi, \qquad \varphi_1 = (\pi - \varphi_0) + j\pi \qquad (j \in \mathbb{Z}).$$

The solution set of the inequality $f(\varphi_1; D) > f(\varphi_0; D)$ in the stripe consists of components of two types:

(a) $0 \leq \varphi_0 < \dfrac{\pi}{2}$:

$$\varphi_0 - (j+1)\pi < \varphi_1 < -\varphi_0 - j\pi \quad (j \in \mathbb{Z}_+),$$

(b) $\dfrac{\pi}{2} \leq \varphi_0 < \pi$:

$$-\varphi_0 - (j-1)\pi < \varphi_1 < \varphi_0 < \varphi_0 - j\pi \quad (j \in \mathbb{Z}_+)$$

(see the shaded regions on Figure 5). Similarly, the solution set of the inequality $f(\varphi_1; D) < f(\varphi_0; D)$ is determined by

(a) $0 \leq \varphi_0 < \dfrac{\pi}{2}$:

$$-\varphi_0 - j\pi < \varphi_1 < \varphi_0 - j\pi \quad (j \in \mathbb{Z}_+),$$

(b) $\dfrac{\pi}{2} \leq \varphi_0 < \pi$:

$$\varphi_0 - (j+1)\pi < \varphi_1 < -\varphi_0 - (j-1)\pi \quad (j \in \mathbb{Z}_+).$$

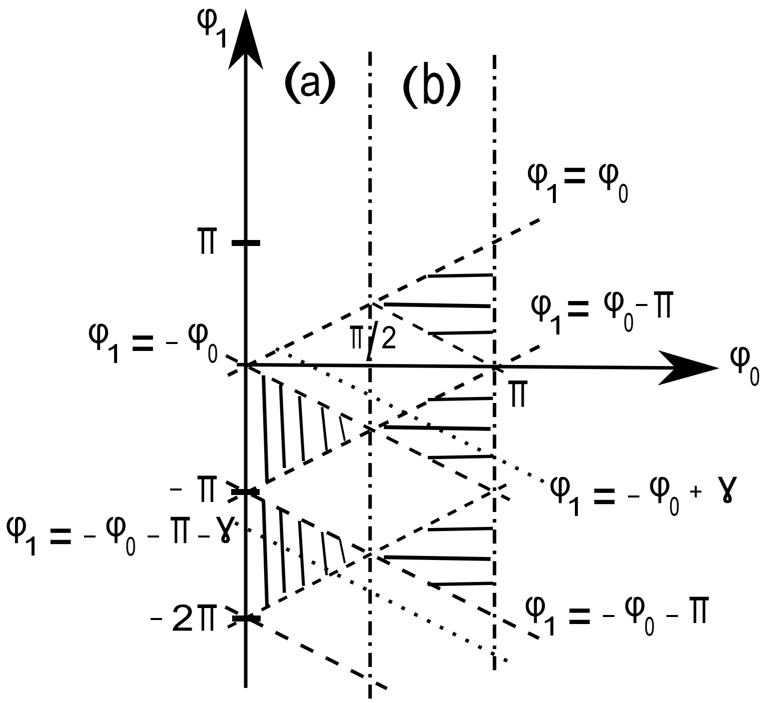

Figure 5: The solution set of the inequality $f(\varphi_1; D) > f(\varphi_0; D)$

By equation (2.5) of the dynamics we have

$$\varphi_1 - \varphi_0 = -a_1 \lambda \pi, \qquad \varphi_3 - \varphi_2 = -a_2 \lambda \pi. \tag{3.7}$$

The first equality together with Lemma 3.3 makes unique the initial angle φ_0 of periodic solutions as a function of a_1 and λ: the solution is 2π-periodic or 4π-periodic only if either

(α) $a_1\lambda$ is an integer and φ_0 is arbitrary,

or

(β) $a_1\lambda$ is not integer and either

$$\begin{gathered}\text{(a) } \varphi_0 = \{a_1\lambda\}\frac{\pi}{2} \\ \text{or} \\ \text{(b) } \varphi_0 = (\{a_1\lambda\} + 1)\frac{\pi}{2},\end{gathered} \tag{3.8}$$

where $\{\xi\}$ denotes the fractional part of the real number ξ.

Lemma 3.4. *1. The solution is angle-periodic with period 2π if and only if there exists an integer $p \in \mathbb{Z}_+$ such that*

$$\phi(\varphi_0; D) - \phi(\varphi_0 - a_1\lambda\pi; D) - 2(p+1)\pi = -a_2\lambda\pi. \tag{3.9}$$

2. The solution is angle-periodic with period 4π if and only if there exists an integer $p \in \mathbb{Z}_+$ such that

$$\phi(\varphi_0; D) - \phi(\varphi_0 - a_1\lambda\pi; D) - (2p+1)\pi = -a_2\lambda\pi. \tag{3.10}$$

Proof. Necessity. 1. The dynamics and (3.6) give

$$\varphi_2 = \phi(\varphi_1; D), \qquad \varphi_3 = \phi(\varphi_0; D) - 2(p+1)\pi. \tag{3.11}$$

Applying (3.7) we obtain (3.9).

2. As we have already mentioned in the proof of Lemma 3.3, in the 4π-periodic case we have

$$\varphi_2 = \phi(\varphi_1; D), \qquad \varphi_3 = \phi(\varphi_0; D) - (2p+1)\pi$$

instead of (3.11). Then (3.7) yields (3.10).

Sufficiency. 1. Let us suppose that (3.9) is valid. Then, taking into account the second equality of (3.7) we get

$$\varphi_3 = \varphi_2 - a_2\lambda\pi = \phi(\varphi_0 - a_1\lambda\pi; D) - a_2\lambda\pi = \phi(\varphi_0; D) - 2(p+1)\pi.$$

Consequently

$$\varphi_4 = \phi(\varphi_3; d) = \phi(\phi(\varphi_0; D); d) - 2(p+1)\pi = \varphi_0 - 2(p+1)\pi \equiv \varphi_0 \pmod{2\pi},$$

so the solution is angle-periodic with period 2π.

2. If (3.10) is satisfied, then the same computation yields

$$\varphi_3 = \phi(\varphi_0; D) - (2p+1)\pi,$$

$$\varphi_4 = \varphi_0 - (2p+1)\pi \equiv \varphi_0 - \pi \pmod{2\pi},$$

i.e., the solution is angle-periodic with period 4π. ◻

If we substituted (3.8) for φ_0 in (3.9) and (3.10), then we would get necessary and sufficient conditions for the existence of periodic solutions. However, we do not do this now. Instead we solve a more general problem: we establish existence theorems for approaching and going away solutions, which theory will contain results for periodic solutions as special cases. Since we are interested in obtaining conditions for the parametric resonance (instability), we develop in detail the theory for the going away solutions.

We have to deal with the points of the shaded set on Figure 5. Fix a value $\gamma \in (0, \pi)$ and consider the going away solutions for which points (φ_0, φ_1) belong to line sections

$$\text{(a)} \quad \varphi_1 = -\varphi_0 - \gamma - j\pi \qquad \left(0 < \varphi_0 < \frac{\pi}{2} - \frac{\gamma}{2}; j \in \mathbb{Z}_+\right). \tag{3.12}$$

Let us fix a_1, $\lambda > 0$ (i.e., $T > 0$, $\varepsilon > 0$) in system (3.4) such that $a_1\lambda \notin \mathbb{Z}$. By the first equality in (3.7), at this value only the line section with $j = [a_1\lambda]$ can contain a point determining a going away solution; namely, the abscissa of this point is

$$\varphi_0 = \{a_1\lambda\}\frac{\pi}{2} - \frac{\gamma}{2}. \tag{3.13}$$

At first let us search for going away solutions with period 2π; i.e., substitute (3.13) for φ_0 in (3.9):

$$\phi\left(\{a_1\lambda\}\frac{\pi}{2} - \frac{\gamma}{2}; D\right) - \phi\left(-\{a_1\lambda\}\frac{\pi}{2} - \frac{\gamma}{2}; D\right) + [a_1\lambda]\pi - 2(p+1)\pi = -a_2\lambda\pi.$$

One has to distinguish two cases:

(1) $[a_1\lambda] = 2m$ $(m \in \mathbb{Z})$. Then

$$\begin{aligned} &\phi\left(\{a_1\lambda\}\frac{\pi}{2} - \frac{\gamma}{2}; D\right) - \phi\left(-\{a_1\lambda\}\frac{\pi}{2} - \frac{\gamma}{2}; D\right) + [a_1\lambda]\pi \\ &= \phi\left(\{a_1\lambda\}\frac{\pi}{2} + 2m\frac{\pi}{2} - \frac{\gamma}{2}; D\right) + \phi\left(\{a_1\lambda\}\frac{\pi}{2} + 2m\frac{\pi}{2} + \frac{\gamma}{2}; D\right) = \\ &= \phi\left(a_1\lambda\frac{\pi}{2} - \frac{\gamma}{2}; D\right) + \phi\left(a_1\lambda\frac{\pi}{2} + \frac{\gamma}{2}; D\right). \end{aligned} \tag{3.14}$$

(2) $[a_1\lambda] = 2m - 1$ $(m \in \mathbb{Z})$. In this case the difference (3.14) has the form

$$\phi\left(\left(a_1\lambda\frac{\pi}{2} - \frac{\gamma}{2}\right) + \frac{\pi}{2}; D\right) + \phi\left(\left(a_1\lambda\frac{\pi}{2} + \frac{\gamma}{2}\right) + \frac{\pi}{2}; D\right) - \pi.$$

Summarizing, if there is a going away solution with period 2π belonging to some of line sections (3.12), then, with $\mu := a_1\lambda\pi/2$, the condition

$$\frac{-\phi\left(\mu - \frac{\gamma}{2}; D\right) - \phi\left(\mu + \frac{\gamma}{2}; D\right)}{2} + (p+1)\pi = \frac{a_2}{a_1}\mu \tag{3.15}$$

is satisfied, provided that $[a_1\lambda]$ is even, and

$$\frac{\left(-\phi\left(\left(\mu - \frac{\gamma}{2}\right) + \frac{\pi}{2}; D\right) + \frac{\pi}{2}\right) + \left(-\phi\left(\left(\mu + \frac{\gamma}{2}\right) + \frac{\pi}{2}; D\right) + \frac{\pi}{2}\right)}{2} + (p+1)\pi = \frac{a_2}{a_1}\mu, \tag{3.16}$$

provided that $[a_1\lambda]$ is odd.

Executing the same computations for the line sections

$$\text{(b)} \quad \varphi_1 = -\varphi_0 + \gamma - (j-1)\pi \qquad \left(\frac{\pi}{2} + \frac{\gamma}{2} < \varphi_0 < \pi; j \in \mathbb{Z}_+\right) \tag{3.17}$$

and the initial angles

$$\varphi_0 = (\{a_1\lambda\} + 1)\frac{\pi}{2} + \frac{\gamma}{2},$$

we arrive at the following result:

If there is a going away solution with period 2π belonging to some of line sections (3.17), then the condition

$$\frac{\left(-\phi\left(\left(\mu - \frac{\gamma}{2}\right) + \frac{\pi}{2}; D\right) + \frac{\pi}{2}\right) + \left(-\phi\left(\left(\mu + \frac{\gamma}{2}\right) + \frac{\pi}{2}; D\right) + \frac{\pi}{2}\right)}{2} + (p+1)\pi = \frac{a_2}{a_1}\mu, \tag{3.18}$$

is satisfied, provided that $[a_1\lambda]$ is even, and

$$\frac{-\phi\left(\mu - \frac{\gamma}{2}; D\right) - \phi\left(\mu + \frac{\gamma}{2}; D\right)}{2} + (p+1)\pi = \frac{a_2}{a_1}\mu, \tag{3.19}$$

provided that $[a_1\lambda]$ is odd.

If we are searching for periodic solutions, then we have to set $\gamma = 0$ in the results (3.15)-(3.16) and (3.18)-(3.19) above. However, in the case of periodic solutions we have to take into consideration also the points of the line sections

$$\text{(c)} \quad \varphi_1 = \varphi_0 - (j+1)\pi \qquad (0 \le \varphi_0 < \pi; j \in \mathbb{Z}_+)\,.$$

By the first equality in (3.7) $a_1\lambda = j+1$, i.e., $a_1\lambda \geq 1$ is integer and φ_0 is arbitrary; consequently, condition (3.9) in Lemma 3.4 has the form

$$-\mu + (p+1)\pi = \frac{a_2}{a_1}\mu,$$

which coincides with (3.15)-(3.16) and (3.18)-(3.19), provided that $\gamma = 0$ and $a_1\lambda \in \mathbb{Z}_+$.

Now we can formulate a necessary and sufficient condition for the existence of 2π-periodic solutions.

Lemma 3.5. *Given a_1, a_2, λ, system* (3.4) *has 2π-periodic solution if and only if either*

$$-\phi\left(a_1\lambda\frac{\pi}{2};\frac{a_1}{a_2}\right) + (p+1)\pi = a_2\lambda\frac{\pi}{2} \tag{3.20}$$

or

$$-\phi\left(a_1\lambda\frac{\pi}{2}+\frac{\pi}{2};\frac{a_1}{a_2}\right) + \frac{\pi}{2} + (p+1)\pi = a_2\lambda\frac{\pi}{2} \tag{3.21}$$

for some $p \in \mathbb{Z}_+$.

Proof. Necessity was proved before the theorem. To start proving sufficiency, assume that (3.20) holds. If $a_1\lambda \notin \mathbb{Z}$ and $[a_1\lambda]$ is even, then choose

$$\varphi_0 = \{a_1\lambda\}\frac{\pi}{2}.$$

We have to prove that the sufficient condition (3.9) of the angle periodicity of period 2π and the condition $f(\varphi_1; D) = f(\varphi_0; D)$ of periodicity are satisfied. In fact, (3.20) gives

$$\begin{aligned}\phi(\varphi_0; D) - \phi(\varphi_0 - a_1\lambda\pi; D) &= \phi\left(\{a_1\lambda\}\frac{\pi}{2}; D\right) - \phi\left(\{a_1\lambda\}\frac{\pi}{2} - 2a_1\lambda\frac{\pi}{2}; D\right)\\ &= \left(\phi\left(a_1\lambda\frac{\pi}{2}; D\right) - [a_1\lambda]\,\frac{\pi}{2}\right) - \left(-\phi\left(a_1\lambda\frac{\pi}{2}; D\right) - [a_1\lambda]\,\frac{\pi}{2}\right) = 2(p+1)\pi - a_2\lambda\pi,\end{aligned}$$

so condition (3.9) in Lemma 3.4 is satisfied, which means that the solution is angle-periodic with period 2π. On the other hand,

$$\varphi_1 = \varphi_0 - a_1\lambda\pi = \{a_1\lambda\}\frac{\pi}{2} - 2a_1\lambda\frac{\pi}{2} = -\{a_1\lambda\}\frac{\pi}{2} - [a_1\lambda]\,\pi = -\varphi_0 - [a_1\lambda]\,\pi,$$

therefore, $f(\varphi_1; D) = f(\varphi_0; D)$ and the solution is 2π-periodic.

If $a_1\lambda \notin \mathbb{Z}$ and $[a_1\lambda]$ is odd, then define

$$\varphi_0 = (\{a_1\lambda\} + 1)\frac{\pi}{2}.$$

Similarly to the previous calculation, we get

$$\begin{aligned}
\phi(\varphi_0; D) - \phi(\varphi_0 - a_1\lambda\pi; D) &= \phi\left((\{a_1\lambda\} + 1)\frac{\pi}{2}; D\right) - \phi\left((\{a_1\lambda\} + 1)\frac{\pi}{2} - 2a_1\lambda\frac{\pi}{2}; D\right) \\
&= \phi\left(a_1\lambda\frac{\pi}{2}; D\right) - ([a_1\lambda] - 1)\frac{\pi}{2} + \phi\left(a_1\lambda\frac{\pi}{2}; D\right) + ([a_1\lambda] - 1)\frac{\pi}{2} \\
&= 2(p+1)\pi - a_2\lambda\pi,
\end{aligned}$$

so the solution is angle-periodic with period 2π. Furthermore,

$$\begin{aligned}
\varphi_1 &= \varphi_0 - a_1\lambda\pi = (\{a_1\lambda\} + 1)\frac{\pi}{2} - 2a_1\lambda\frac{\pi}{2} = -\{a_1\lambda\}\frac{\pi}{2} - (2[a_1\lambda] - 1)\frac{\pi}{2} \\
&= -\varphi_0 - ([a_1\lambda] - 1)\pi,
\end{aligned}$$

so Lemma 3.4 and Lemma 3.3 again guarantee 2π-periodicity.

Finally, if $a_1\lambda \in \mathbb{Z}$, then let φ_0 be arbitrary. Then

$$\phi(\varphi_0; D) - \phi(\varphi_0 - a_1\lambda\pi; D) = a_1\lambda\pi = 2(p+1)\pi - a_2\lambda\pi,$$

thus the solution is angle-periodic of period 2π. Moreover,

$$f(\varphi_1; D) = f(\varphi_0 - a_1\lambda\pi; D) = f(\varphi_0; D),$$

so the solution is 2π-periodic.

If condition (3.21) is satisfied, then the proof is similar. □

Lemma 3.6. *Given a_1, a_2, λ, system* (3.4) *has 4π-periodic solution if and only if either*

$$-\phi\left(a_1\lambda\frac{\pi}{2}; D\right) + \left(p + \frac{1}{2}\right)\pi = a_2\lambda\frac{\pi}{2} \tag{3.22}$$

or

$$-\phi\left(a_1\lambda\frac{\pi}{2} + \frac{\pi}{2}; D\right) + \frac{\pi}{2} + \left(p + \frac{1}{2}\right)\pi = a_2\lambda\frac{\pi}{2} \tag{3.23}$$

for some $p \in \mathbb{Z}_+$.

Proof. When we are searching for 4π-periodic solutions, then $\varphi_4 = \varphi_0 - (2p+1)\pi$ is supposed, therefore we have $\varphi_3 = \phi(\varphi_0; D) - (2p+1)\pi$ instead of (3.6). This is the reason that the only difference between conditions (3.9) and (3.10) is: the letter contains the member $-(2p+1)$ instead of $-2(p+1)$. Making this change in conditions of Lemma 3.5 we get conditions of Lemma 3.6. □

Now we are ready to formulate and prove the main theorem on the existence of periodic solutions to the original equation (1.2).

Theorem 3.7. *For every $\varepsilon > 0$ there are sequences $\{T_k(\varepsilon)\}_{k=1}^{\infty}$, $\{\widetilde{T}_k(\varepsilon)\}_{k=1}^{\infty}$ such that equation* (1.2) *with $T = T_k(\varepsilon)$ (respectively, with $T = \widetilde{T}_k(\varepsilon)$) has $2T_k(\varepsilon)$-periodic (respectively, $4\widetilde{T}_k(\varepsilon)$-periodic) solutions. In addition,*

$$0 < \widetilde{T}_1 \le \widetilde{T}_2 < T_1 \le T_2 < \widetilde{T}_3 \le \widetilde{T}_4 < \ldots; \quad \lim_{k\to\infty} T_k = \infty, \tag{3.24}$$

and

$$\begin{aligned} \lim_{\varepsilon\to 0+0} 2T_{2p+1}(\varepsilon) = \lim_{\varepsilon\to 0+0} 2T_{2p+2}(\varepsilon) = (2p+2)\left(\frac{1}{2}\left(2\pi\sqrt{\frac{l}{g}}\right)\right), \\ \lim_{\varepsilon\to 0+0} 2\widetilde{T}_{2p+1}(\varepsilon) = \lim_{\varepsilon\to 0+0} 2\widetilde{T}_{2p+2}(\varepsilon) = (2p+1)\left(\frac{1}{2}\left(2\pi\sqrt{\frac{l}{g}}\right)\right) \end{aligned} \tag{3.25}$$

hold for all $p \in \mathbb{Z}_+$.

Proof. Equation (1.2) has $2T$-periodic (respectively, $4T$-periodic) solutions if and only if system (3.4) has 2π-periodic (respectively, 4π-periodic) ones, so we will apply Lemmas 3.5-3.6. Introduce the notations

$$F_p(\mu) := -\phi(\mu; D) + (p+1)\pi, \; G_p(\mu) := \left(-\phi\left(\mu + \frac{\pi}{2}; D\right) + \frac{\pi}{2}\right) + (p+1)\pi \quad (p \in \mathbb{Z}_+).$$

For every p, functions F_p and G_p are strictly decreasing and vanish, so the equations $F_p(\mu) = (a_2/a_1)\mu$, $G_p(\mu) = (a_2/a_1)\mu$ each have exactly one solution in the interval $((p+1)\pi/2, (p+1)\pi)$ (see Figure 6); they will be denoted by $\mu_{2p+1} \le \mu_{2p+2}$. Similarly, equations $F_p(\mu) - \pi/2 = (a_2/a_1)\mu$, $G_p(\mu) - \pi/2 = (a_2/a_1)\mu$ have solutions $\widetilde{\mu}_{2p+1} \le \widetilde{\mu}_{2p+2}$ and

$$0 < \widetilde{\mu}_1 < \widetilde{\mu}_2 < \mu_1 \le \mu_2 < \widetilde{\mu}_3 \le \widetilde{\mu}_4 < \mu_3 \le \mu_4 < \ldots. \tag{3.26}$$

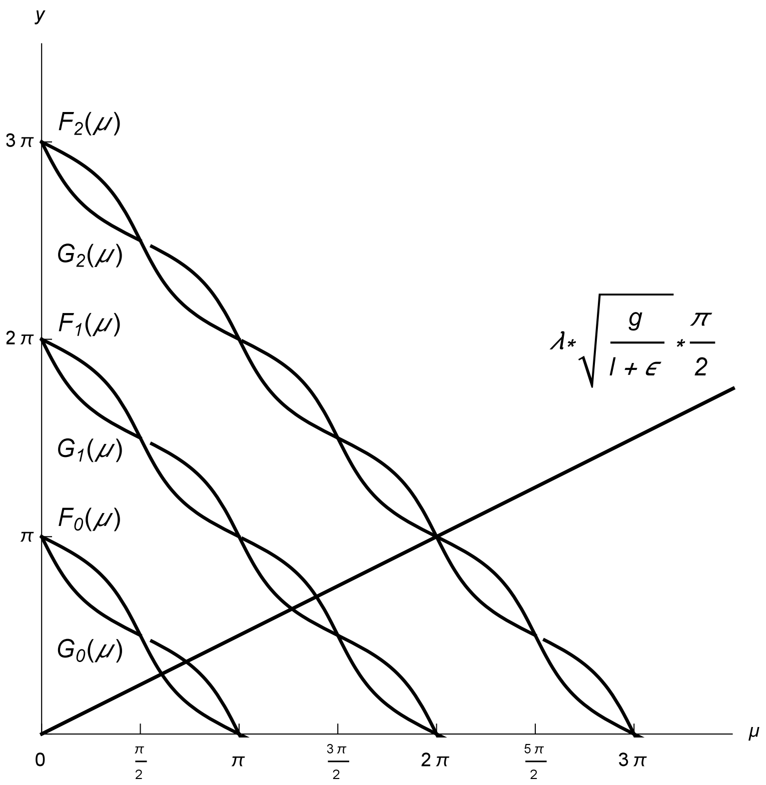

Figure 6: Graphs of F_p and G_p for $l = 2,\ \epsilon = 1.2$

Taking the limit $\varepsilon \to 0$ in equations (3.20)-(3.21) and (3.22)-(3.23) we obtain

$$\lim_{\varepsilon\to 0+0} \mu_{2p+1}(\varepsilon) = \lim_{\varepsilon\to 0+0} \mu_{2p+2}(\varepsilon) = (p+1)\left(\frac{\pi}{2}\right),$$

$$\lim_{\varepsilon\to 0+0} \widetilde{\mu}_{2p+1}(\varepsilon) = \lim_{\varepsilon\to 0+0} \widetilde{\mu}_{2p+2}(\varepsilon) = \left(p+\frac{1}{2}\right)\left(\frac{\pi}{2}\right).$$

We can complete the proof by setting

$$T_k(\varepsilon) := 2\sqrt{\frac{l-\varepsilon}{g}}\mu_k(\varepsilon), \quad \widetilde{T}_k(\varepsilon) := 2\sqrt{\frac{l-\varepsilon}{g}}\widetilde{\mu}_k(\varepsilon).$$

□

It is worth noticing that Theorem 3.7 is a special case of the classical Oscillation Theorem of the theory of Hill's equation [14] for system (3.4). The advantage of our approach is that *we have constructed the periodic solutions*, so asymptotic properties (number of periodic solutions, number of zeros) formulated in the Oscillation Theorem can be deduced directly. What is more, from Lemmas 3.5-3.6 we obtain an answer to the problem of the coexistence of periodic solutions [11] for equation (1.2). This equation has two linearly independent 2π-periodic (respectively, 4π-periodic) solutions if and only if both (3.20) and (3.21) (respectively, (3.22) and (3.23)) are satisfied for the same p and λ.

Corollary 3.8. *Given T and ε, if $a_2/a_1 = \sqrt{(l-\varepsilon)/(l+\varepsilon)}$ is rational, i.e.,*

$$\sqrt{\frac{l-\varepsilon}{l+\varepsilon}} = \frac{m}{n} \qquad (m, n \in \mathbb{N},\ (m,n) = 1),$$

then for every $\varepsilon > 0$ there are countable many values of T such that all solutions of equation (1.2) *are either $2T$-periodic or $4T$-periodic. More precisely, if*

$$\sqrt{\frac{g}{l-\varepsilon}}\frac{T}{\pi} = jn \quad \textit{and} \quad j(m+n) \textit{ is even}$$

(respectively,

$$\sqrt{\frac{g}{l-\varepsilon}}\frac{T}{\pi} = jn \quad \textit{and} \quad j(m+n) \textit{ is odd})$$

with some $j \in \mathbb{N}$, then all solutions of equation (1.2) *are $2T$-periodic (respectively, $4T$-periodic).*

Using the computation (3.12)-(3.19) and the same reasoning as in the proof of Lemma 3.5, we obtain existence results for going away solutions.

Lemma 3.9. *Given a_1, a_2, λ, system* (3.4) *has a solution going away from the origin of angle period 2π if and only if there is a $\gamma \in (0, \pi)$ such that*

$$\frac{-\phi\left(\mu - \frac{\gamma}{2}; D\right) - \phi\left(\mu + \frac{\gamma}{2}; D\right)}{2} + (p+1)\pi = \frac{a_2}{a_1}\mu$$

or

$$\frac{\left(-\phi\left((\mu - \frac{\gamma}{2}) + \frac{\pi}{2}; D\right) + \frac{\pi}{2}\right) + \left(-\phi\left((\mu + \frac{\gamma}{2}) + \frac{\pi}{2}; D\right) + \frac{\pi}{2}\right)}{2} + (p+1)\pi = \frac{a_2}{a_1}\mu$$

for some integer $p \in \mathbb{Z}_+$.

Lemma 3.10. *Given a_1, a_2, λ, system* (3.4) *has a solution going away from the origin of angle period* 4π *if and only if there is a* $\gamma \in (0, \pi)$ *such that*

$$\frac{-\phi\left(\mu - \frac{\gamma}{2}; D\right) - \phi\left(\mu + \frac{\gamma}{2}; D\right)}{2} + (p + \frac{1}{2})\pi = \frac{a_2}{a_1}\mu$$

or

$$\frac{\left(-\phi\left((\mu - \frac{\gamma}{2}) + \frac{\pi}{2}; D\right) + \frac{\pi}{2}\right) + \left(-\phi\left((\mu + \frac{\gamma}{2}) + \frac{\pi}{2}; D\right) + \frac{\pi}{2}\right)}{2} + (p + \frac{1}{2})\pi = \frac{a_2}{a_1}\mu$$

for some integer $p \in \mathbb{Z}_+$.

If we want to obtain conditions for the existence of solutions approaching to the origin, then we have to consider the unshaded domain on Figure 5. Repeating computations we can see that the necessary and sufficient conditions are the same as ones in Lemmas 3.9 – 3.10. In other words, the equations of type (3.4) possessing going away solutions and ones possessing approaching solutions are the same.

4 Stability chart

By the Floque Theory the domain of instability in the parameter plane (T, ε) is a disconnected open set, whose components are called "Arnold tongues" (see Figure 7), and the angular points of Arnold tongues on the T-axis are at the abscissas $(1/2)k\pi\sqrt{l/g}$ [1, 16]. We will show that the Arnold tongues are the domains between the graphs of functions T_{2p+1} and T_{2p+2} together with the domains between the graphs of $\widetilde{T}_{2p+1}$ and $\widetilde{T}_{2p+2}$ defined in Theorem 3.7. (This fact is in accordance with (3.25).) At first we prove that these domains are subsets of the set of instability. We use the notations

$$H := \cup_{p=0}^{\infty}(\mu_{2p+1}, \mu_{2p+2}), \qquad \widetilde{H} := \cup_{p=0}^{\infty}(\widetilde{\mu}_{2p+1}, \widetilde{\mu}_{2p+2}) \tag{4.1}$$

(for μ_j, $\widetilde{\mu}_j$ see (3.26)).

Lemma 4.1. *For every $\mu \in H$ (respectively, $\mu \in \widetilde{H}$) system* (3.4) *with $a_1\lambda\pi/2 = \mu$ has a solution going away from the origin with angle period 2π (respectively, 4π).*

Proof. We prove the first statement in an arbitrarily fixed non-empty open interval (μ_{2p+1}, μ_{2p+2}). For the sake of definiteness, let us suppose that

$$2m\frac{\pi}{2} < \mu_{2p+1} < \mu_{2p+2} < (2m+1)\frac{\pi}{2} \qquad (m \in \mathbb{Z}_0);$$

then

$$F_p(\mu_{2p+1}) = \frac{a_2}{a_1}\mu_{2p+1}, \qquad G_p(\mu_{2p+2}) = \frac{a_2}{a_1}\mu_{2p+2}.$$

If $0 < \gamma_1 < \gamma_2 < \pi$, then for functions

$$F_p^{\gamma}(\mu) := \frac{F_p\left(\mu - \frac{\gamma}{2}\right) + F_p\left(\mu + \frac{\gamma}{2}\right)}{2}, \quad G_p^{\gamma}(\mu) := \frac{G_p\left(\mu - \frac{\gamma}{2}\right) + G_p\left(\mu + \frac{\gamma}{2}\right)}{2}$$

the inequalities

$$F_p(\mu) < F_p^{\gamma_1}(\mu) < F_p^{\gamma_2}(\mu) < G_p(\mu), \quad F_p(\mu) < G_p^{\gamma_2}(\mu) < G_p^{\gamma_1}(\mu) < G_p(\mu) \quad (4.2)$$

are satisfied on the interval (μ_{2p+1}, μ_{2p+2}). In fact, since F_p is convex and G_p is concave in the interval $(2m\pi/2, (2m+1)\pi/2)$, (4.2) holds for small $\gamma > 0$. For large γ when either $\mu - \gamma/2$ or $\mu + \gamma/2$ or both is out of this interval, then (4.2) is all the more true because of symmetricity properties of ϕ. On the other hand,

$$F_p^0(\mu) \equiv F_p(\mu), \qquad G_p^0 \equiv G_p(\mu),$$

$$\lim_{\gamma \to \pi - 0} F_p^{\gamma} = G_p(\mu), \qquad \lim_{\gamma \to \pi - 0} G_p^{\gamma}(\mu) = F_p(\mu),$$

which completes the proof of (4.2).

From (4.2) it follows that for every $\gamma \in (0, \pi)$ the equations

$$F_p^{\gamma}(\mu) = \frac{a_2}{a_1}\mu, \qquad G_p^{\gamma}(\mu) = \frac{a_2}{a_1}\mu$$

have solutions which fill in the interval (μ_{2p+1}, μ_{2p+2}) while γ changes from 0 to π. By Lemma 3.9, every system (3.4) with $a_1\lambda\pi/2 = \mu \in (\mu_{2p+1}, \mu_{2p+2})$ has a going away solution of angle period 2π.

The proof for $\widetilde{H}$ is analogous. □

Theorem 4.2. *The inside of the domain of instability (parametric resonance) on the parameter plane* (T, ε) *for equation* (1.2) *is*

$$\cup_{0<\varepsilon<l}(\cup_{p=0}^{\infty}(\{(T,\varepsilon) : T_{2p+1}(\varepsilon) < T < T_{2p+2}(\varepsilon)\} \cup \{(T,\varepsilon) : \widetilde{T}_{2p+1}(\varepsilon) < T < \widetilde{T}_{2p+2}(\varepsilon)\})) \quad (4.3)$$

(T_k, $\widetilde{T}_k$ were defined in (3.24)*).*

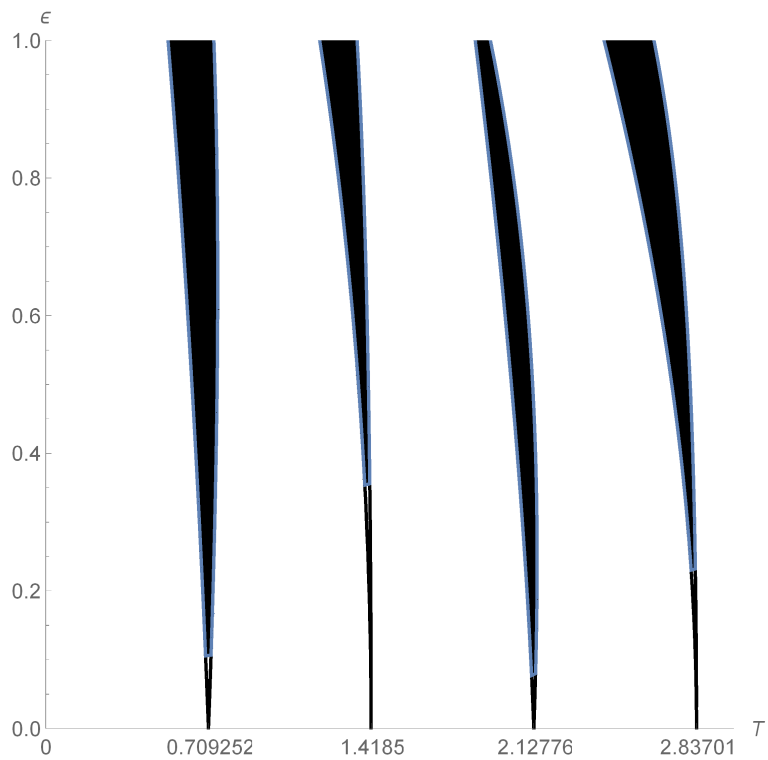

Figure 7: Arnold tongues for $l = 2$

Proof. Let us start with an observation about the stability of the system of linear difference equations (3.4) based upon some basic facts from linear algebra. The linear mapping whose matrix is $C(d)R(\lambda a_2\pi)C(D)R(\lambda a_1\pi)$ preserves the area on the phase plane (x, y), therefore the determinant of this matrix (consequently, the product of the eigenvalues of the matrix) equals 1. This implies that the boundary of the domain of instability (disregarding the points of axes T and ε) consists of the points (T, ε) for which the eigenvalues equal either 1 or -1, i.e., of the points (T, ε) for which system (3.4) has either 2π-periodic or 4π-periodic solution. By Theorem 3.7 this means that the boundary of the instability domain consists of graphs of T_k and $\widetilde{T}_k$. On the other hand, by Lemma 4.1, set (4.3) belongs to the instability domain. These together complete the proof. □

References

[1] Arnold, V. I. *Ordinary Differential Equations.* Springer-Verlag, Berlin, 2006.

[2] Butikov, E. I. Parametric resonance in a linear oscillator at square-wave modulation. *Eur. J. Phys.* **26** (2005), 157–174.

[3] Curry, S. M. How chidren swing. *Amer. J. Phys.* **44** (1976), 924–926.

[4] Csörgő, S.; Hatvani, L. Stability properties of solutions of linear second order differential equations with random coefficients. *J. Differential Equations* **248**(2010), no. 1, 21–49.

[5] Hatvani, L. On the existence of a small solution to linear second order differential equations with step function coefficients. *Dynam. Contin. Discrete Impuls. Systems* **4** (1998), 321–330.

[6] Hatvani, L. An elementary method for the study of Meissner's equation and its application to proving the Oscillation Theorem. *Acta Sci. Math.* **79**(2013), no. 1–2, 87–105.

[7] Hatvani, L. On the parametrically excited pendulum equation with a square wave coefficient (in preparation).

[8] Haupt, O. Über eine Methode zum Bewise von Oszillationstheoremen. *Math. Ann.* **76**(1914), 67–104.

[9] Haupt, O. Über lineare homogene Differentialgleichungen 2. Ordnung mit periodischen Koeffizienten. *Math. Ann.* **79**(1917), 278–285.

[10] Hill, G. W. On the part of the motion of the lunar perigee which is a function of the mean motions of the Sun and Moon. *Acta Mathematica* **8**(1886), 1–36.

[11] Hochstadt, H. A special Hill's equation with discontinuous coefficients. *Amer. Math. Monthly* **70**(1963), 18–26.

[12] Landau, L. D.; Lifshitz, E. M. *Mechanics*. Course of Theoretical Physics. Vol. 1, Elsevier, Amsterdam, 1976.

[13] Lyapunov, A. M. The general problem of the stability of motion. Lyapunov centenary issue. *Internat. J. Control* **55**(1992), no. 3, 521–790.

[14] Magnus, W.; Winkler, S. *Hill's equation.* Dover Publications, Inc., New York, 1979.

[15] Meissner, E. Über Schüttel-schwingungen in Systemen mit Periodisch Veränderlicher Elastizität. *Schweizer Bauzeitung* **72**(1918), no. 10, 95–98.

[16] D. R. Merkin, *Introduction to the Theory of Stability.* Springer-Verlag, 1997.

[17] Seyranian, A. P.; Seyranian, A. A. Breakup of resonance zones for Meissner's equation with small damping. *Moscow University Mechanics Bulletin* (2003), No. 5, 53–59.

L. Csizmadia, Department of Mathematics, College of Kecskemét, Current address: Izsáki út 10, Kecskemét, Hungary-6000, e-mail:csizmadialaszlog@gmail.com

L. Hatvani, Bolyai Institute, University of Szeged, Current address: Aradi vértanúk tere 1, Szeged, Hungary-6724, e-mail:hatvani@math.u-szeged.hu