[8] MARČENKOV, S. S., *Homogeneous algebras*. Problemy Kibernetiki *39* (1982), 85–106.
[9] MARCZEWSKI, E., *Homogeneous algebras and homogeneous operations*. Fund. Math. *56* (1964), 81–103.
[10] MCKENZIE, R. N., MCNULTY, G. F. and TAYLOR, W. F., *Algebras, lattices, varieties, Vol. I.* Wadsworth & Brooks/Cole, Monterey, California, 1987.
[11] PIXLEY, A. F., *Functionally complete algebras generating distributive and permutable classes*. Math. Z. *114* (1970), 361–372.
[12] SZENDREI, Á., *Clones in universal algebra*. SMS, vol. 99. Univ. Montréal, 1986.

Teacher's Training College
Szeged
Hungary

Bolyai Institute
Szeged
Hungary

Algebra Universalis, **30** (1993) 337–344 0002–5240/93/030337–08$01.50 + 0.20/0

Varieties generated by finite homogeneous algebras*

B. CSÁKÁNY AND A. SZEDERKÉNYI

Given an algebra $\mathbf{A}$, any subvariety of $\mathscr{V}(\mathbf{A})$ is generated by the subdirectly irreducible algebras it contains. Knoebel [7] observed that this version of Birkhoff's subdirect representation theorem facilitates determining the lattice of subvarieties of $\mathscr{V}(\mathbf{A})$, and applied it to the description of the subvariety lattice of $\mathscr{V}(A)$ for finite preprimal algebras $\mathbf{A}$. Here we shall use it to determine the subvariety lattices of varieties generated by finite homogeneous algebras. We adopt the terminology and notations of [10].

An algebra $\mathbf{A}$ is called *homogeneous* if every permutation of A is an automorphism of $\mathbf{A}$. This notion was introduced by Marczewski [9], and a complete description of homogeneous algebras up to equivalence was first given by Marčenkov [8]. For a survey of homogeneous algebras, including a streamlined proof of Marčenkov's theorem, see Ágnes Szendrei [12]. These results imply a simple classification of finite homogeneous algebras which will be stated here as it is necessary for our further considerations. As a preparation, we recall the most important *homogeneous operations*, i.e. operations of homogeneous algebras: the *dual discriminator* d, the *switching function* s, and the k-ary *near-projection* l_k, defined on an arbitrary set A by

$$d(a, b, c) = c \quad \text{if } a \neq b, \qquad d(a, b, c) = a \quad \text{otherwise;}$$

$$s(a, b, c) = c \quad \text{if } a = b, \qquad s(a, b, c) = b \quad \text{if } a = c, \qquad s(a, b, c) = a \quad \text{otherwise;}$$

$$l_k(a_1, \ldots, a_k) = a_1 \quad \text{if } |\{a_1, \ldots, a_k\}| < k, \qquad l_k(a_1, \ldots, a_k) = a_k \quad \text{otherwise;}$$

further, the $(k-1)$-ary operation r_k, defined on A iff $|A| \leq k$ by

$$r_k(a_1, \ldots, a_{k-1}) = a_k \quad \text{if } |A| = k \text{ and } A = \{a_1, \ldots, a_{k-1}, a_k\},$$

$$r_k(a_1, \ldots, a_{k-1}) = a_1 \quad \text{otherwise.}$$

Presented by A. F. Pixley.
Received September 9, 1991; accepted in final form January 8, 1992.
*Research was supported by HNFSR (OTKA) grant No. 1903.

Notice that d and s are projections only if $|A| = 1$, while l_k and r_k are projections (namely they equal p_0^k) iff $|A| < k$. We usually suppose $A = \mathbf{n} = \{0, 1,, \ldots, n-1\}$.

The classification of finite homogeneous algebras we shall need is the following (cf. [8], [12]):

PROPOSITION 1. *Any finite homogeneous algebra with universe* $\mathbf{n}$ *is equivalent to a unique member of the following six disjoint families of homogeneous algebras*:

(1) *the* dual discriminator algebras, *i.e. those algebras in which* d *is a term operation*;
(2) *the algebras* $\langle \mathbf{n}; s\rangle$ *with* $n \geq 2$, *and* $\langle \mathbf{n}; s, r_n\rangle$ *with* $n = 2$ *or* $n \geq 4$; *we call them* switching algebras (*note that there are also dual discriminator algebras with s as a term operation*);
(3) *the* near-trivial algebras $\langle \mathbf{n}; l_k\rangle$ *with* $n \geq 3$ *and* $3 \leq k \leq n$;
(4) *the* trivial algebras *whose basic operations are projections*; *for simplicity, we denote them by* $\mathbf{n}$;
(5) *the algebras* $\langle \mathbf{n}; r_n\rangle$ *with* $n \geq 2$, *and* $\langle \mathbf{n}; r_n, l_k\rangle$ *with* $n \geq 5$ *and* $3 \leq k \leq n-2$;
(6) $\langle \mathbf{2}; s\rangle^2$.

We shall also use the following fact (Ganter, Płonka, Werner [5]):

PROPOSITION 2. *All homogeneous algebras but* $\langle \mathbf{2}; s\rangle^2$ *and* $\mathbf{n}$ $(n > 2)$ *are simple.*

Our purpose is to prove the following:

THEOREM. *The subvarieties of a variety generated by a finite homogeneous algebra form a chain under inclusion.*

After Clark and Krauss [2], an arbitrary finite algebra $\mathbf{A}$ is called a *direct Stone generator* if every finite member of $\mathscr{V}(\mathbf{A})$ belongs to $\mathscr{P}\mathscr{S}(\mathbf{A})$; an algebra $\mathbf{A}$ is a *subdirect Stone generator* if $\mathscr{V}(A) = \mathscr{P}_s\mathscr{S}(\mathbf{A})(=\mathscr{S}\mathscr{P}(\mathbf{A}))$. Recall that a direct Stone generator is always a subdirect Stone generator (see Astromoff [1] and Pixley [11]).

The key of the proof of our theorem is the following.

LEMMA. *Every finite homogeneous algebra but* $\langle \mathbf{2}; r_2\rangle$ *is a subdirect Stone generator.*

We prove this lemma for the families (1)–(6) in Proposition 1.

CASE (1). If $\mathbf{A}$ is a finite dual discriminator algebra, then d is a majority term on $\mathscr{V}(\mathbf{A})$, hence $\mathscr{V}(\mathbf{A})$ is congruence distributive and $\mathscr{V}(\mathbf{A}) = \mathscr{P}_s\mathscr{H}\mathscr{S}(A)$ ([6], [4]). By Proposition 2, we can omit $\mathscr{H}$ here; hence $\mathscr{V}(A) = \mathscr{P}_s\mathscr{S}(A)$, as asserted.

CASE (2). In switching algebras, s is a Maltsev term operation. Observe that subalgebras of switching algebras are switching algebras, too; hence by Proposition 2 they are simple. Thus, switching algebras are para primal, and taking into account that they have one-element subalgebras, they are (direct, and hence) subdirect Stone generators ([2], Theorem 2.3.), which was needed.

CASE (3). Let $\mathbf{L}_{n,k}$ stand for $\langle \mathbf{n}; l_k\rangle$. The subalgebras of $\mathbf{L}_{n,k}$ are, up to isomorphism, $\mathbf{L}_{n,k}, \mathbf{L}_{n-1,k}, \ldots, \mathbf{L}_{k,k}, \mathbf{k-1}, \ldots, \mathbf{2}, \mathbf{1}$. Denote by $\mathscr{P}_{n,k}$ the class of finite algebras isomorphic to direct products whose factors are non-trivial subalgebras (i.e. subalgebras being non-trivial algebras) of $\mathbf{L}_{n,k}$ and write $\mathscr{Q}_{n,k}$ for the class of algebras isomorphic to direct products of form $\mathbf{m} \times \mathbf{P}$ with $m \geq 1$ and $\mathbf{P} \in \mathscr{P}_{n,k}$. We shall prove that *every finite member of* $\mathscr{V}(\mathbf{L}_{n,k})$ *is in* $\mathscr{Q}_{n,k}$. As $\mathbf{m}$ is a subdirect power of $\mathbf{2}$, this implies that all finite algebras of $\mathscr{V}(\mathbf{L}_{n,k})$ are in $\mathscr{P}_s\mathscr{S}(\mathbf{L}_{n,k})$. Taking into account that finitely generated homogeneous algebras are finite, we can apply Pixley's result ([11], Theorem 2.3.) asserting that, for any finite $\mathbf{A}$, if all finite algebras of $\mathscr{V}(\mathbf{A})$ are in $\mathscr{P}_s\mathscr{H}\mathscr{S}(A)$, then $\mathscr{V}(\mathbf{A}) = \mathscr{P}_s\mathscr{H}\mathscr{S}(\mathbf{A})$. Now $\mathscr{H}\mathscr{S}(\mathbf{L}_{n,k}) = \mathscr{S}(\mathbf{L}_{n,k})$ by Proposition 2. Hence $\mathscr{V}(\mathbf{L}_{n,k}) = \mathscr{P}_s\mathscr{S}(\mathbf{L}_{n,k})$, i.e. $\mathbf{L}_{n,k}$ is a subdirect Stone generator.

We split the assertion to be proved into three claims:

CLAIM 1. *If* $\mathbf{P}$ *is a finite subdirect product whose factors are non-trivial subalgebras of* $\mathbf{L}_{n,k}$, *then* $\mathbf{P} \in \mathscr{P}_{n,k}$.

Let $\mathbf{P}$ be a subdirect product of $\langle \mathbf{r}_i; l_k\rangle$ $(i = 1, \ldots, t; k \leq r_1 \leq r_2 \leq \cdots \leq r_t \leq n)$. Without loss of generality, we can suppose that $\mathbf{P}$ is *irredundant*, i.e., for $1 \leq i < j \leq t$, $pr_{i,j}P$ is not a bijection between $\langle \mathbf{r}_i; l_k\rangle$ and $\langle \mathbf{r}_j; l_k\rangle$, else we can omit the subdirect factor $\langle \mathbf{r}_j; l_k\rangle$, obtaining a subdirect product $\mathbf{P}'$ with $\mathbf{P}' \simeq \mathbf{P}$. First put $t = 2$. Fix $a_1 \in \mathbf{r}_1$, let $\mathbf{r}_2 = \{b_1, \ldots, b_{r_2}\}$, and suppose $\langle a_1, b_1\rangle \in P$. It is enough to show that $\langle a_1, b_i\rangle \in P$ for $i = 2, \ldots, r_2$. Choose $a_2, \ldots, a_{r_2} \in \mathbf{r}_1$ with

$$\langle a_2, b_2\rangle, \ldots, \langle a_{r_2}, b_{r_2}\rangle \in P. \qquad (*)$$

If $a_1, a_2, \ldots, a_{r_2}$ are not pairwise distinct, we have two possibilities:

(1) There is a j $(\neq i)$ such that $a_j = a_i$. If $j = 1$, we are done: otherwise, choose $b_2', \ldots, b_{k-2}' \in \mathbf{r}_2$ and $a_2', \ldots, a_{k-2}' \in \mathbf{r}_1$ so that $b_1, b_2', \ldots, b_{k-2}', b_i, b_j$ are distinct, and $\langle a_2', b_2'\rangle, \ldots, \langle a_{k-2}', b_{k-2}'\rangle \in P$. Then

$$l_k(\langle a_1, b_1\rangle, \langle a_2', b_2'\rangle, \ldots, \langle a_{k-2}', b_{k-2}'\rangle, \langle a_j, b_j\rangle, \langle a_i, b_i\rangle) = \langle a_1, b_i\rangle \in P.$$

(2) Such a j does not exist. Then there are $u, v(\neq i)$ with $1 \le u < v \le r_2$ such that $a_u = a_v$. Choose again $b_2', \ldots, b_{k-2}' \in \mathbf{r}_2$ and $a_2', \ldots, a_{k-2}' \in \mathbf{r}_1$ so that $b_i, b_2', \ldots, b_{k-2}', b_u, b_v$ are distinct and $\langle a_2', b_2'\rangle, \ldots, \langle a_{k-2}', b_{k-2}'\rangle \in P$. Then

$$l_k(\langle a_u, b_u\rangle, \langle a_v, b_v\rangle, \langle a_2', b_2'\rangle, \ldots, \langle a_{k-2}', b_{k-2}'\rangle, \langle a_i, b_i\rangle) = \langle a_u, b_i\rangle \in P.$$

This means that if we replace a_i by a_u in $(*)$ then (1) holds and thus (2) can be avoided.

Now let $(*)$ imply that $a_1, a_2, \ldots, a_{r_2}$ are distinct. Then there exists $a_j \in \mathbf{r}_1$ with $a_j \neq a_1$ and $\langle a_j, b_1\rangle \in P$ otherwise $pr_{1,2}P$ is a bijection, contradicting the assumption. If $j \neq i$, take, for a_1, a_i, a_j, the elements $a_2', \ldots, a_{k-2}', b_2', \ldots, b_{k-2}'$ as in the case (1); then $l_k(\langle a_i, b_i\rangle, \langle a_2', b_2'\rangle, \ldots, \langle a_{k-2}', b_{k-2}'\rangle, \langle a_j, b_1\rangle, \langle a_1, b_1\rangle) = \langle a_1, b_i\rangle \in P$, and, finally,

$$\langle a_1, b_j\rangle = l_k(\langle a_j, b_j\rangle, \langle a_2', b_2'\rangle, \ldots, \langle a_{k-2}', b_{k-2}'\rangle, \langle a_i, b_i\rangle, \langle a_1, b_i\rangle) \in P.$$

Let $t > 2$ and suppose that subdirect products of less than t factors of form $\langle \mathbf{r}_i; l_k\rangle$ are direct. For $j = 1, \ldots, t-1$, let $a_j \in \mathbf{r}_j$. As $pr_{1,\ldots,t-1}P$ is a direct product, there exists $b_1 \in \mathbf{r}_t$ with $\langle a_1, \ldots, a_{t-1}, b_1\rangle \in P$. We have to prove $\langle a_1, \ldots, a_{t-1}, b\rangle \in P$, whenever $b \in \mathbf{r}_t, b \neq b_1$. Let $b_1, b_2', \ldots, b_{k-2}', b$ be k distinct elements of $\mathbf{r}_t$. As we have seen, $pr_{t-1,t}P$ is a direct product, hence there are $b_1' \in \mathbf{r}_1$ and $b_i \in \mathbf{r}_i$ $(i = 1, \ldots, t-2)$ such that $\langle b_1', b_2, \ldots, b_{t-2}, a_{t-1}, b\rangle \in P$. Again by induction, $pr_{1,\ldots,t-2,t}P$ is also a direct product, which implies, for $j = 2, \ldots, k-1$, the existence of $a_j' \in \mathbf{r}_{t-1}$ with $\langle a_1, \ldots, a_{t-2}, a_j', b_j'\rangle \in P$. Now

$$\begin{aligned}\langle a_1, \ldots, a_{t-2}, a_{t-1}, b\rangle = l_k(&\langle a_1, \ldots, a_{t-2}, a_{t-1}, b_1\rangle,\\ &\langle a_1, \ldots, a_{t-2}, a_2', b_2'\rangle, \ldots,\\ &\langle a_1, \ldots, a_{t-2}, a_{k-1}', b_{k-1}'\rangle,\\ &\langle b_1', b_2, \ldots, b_{t-2}, a_{t-1}, b\rangle) \in P,\end{aligned}$$

as required.

CLAIM 2. *A subdirect product of* $\mathbf{m}$ *and* $\mathbf{P} \in \mathscr{P}_{n,k}$ *is a direct product.*

Let $\mathbf{Q}$ be a subdirect product of $\mathbf{m}$ and $\mathbf{P} = \prod_{i=1}^{t} \langle \mathbf{r}_i; l_k\rangle$ $(\in \mathscr{P}_{n,k})$. Suppose $\langle a, \langle a_1, \ldots, a_t\rangle\rangle \in Q$ $(a \in \mathbf{m}, \langle a_1, \ldots, a_t\rangle \in P)$. Take an arbitrary $\langle b_1, \ldots, b_t\rangle \in P$; we have to show $\langle a, \langle b_1, \ldots, b_t\rangle\rangle \in Q$. For $i = 1, \ldots, k-2$, we can choose $\langle c_{i1}, \ldots, c_{it}\rangle \in P$ in such a way that, for $j = 1, \ldots, t$, $c_{1j}, \ldots, c_{k-2,j}$ are distinct from each other as well as from a_j and b_j. Now there are $b, c_1, \ldots, c_{k-2} \in \mathbf{m}$ such that $\langle b, \langle b_1, \ldots, b_t\rangle\rangle, \langle c_i, \langle c_{i1}, \ldots, c_{it}\rangle\rangle \in Q$ $(i = 1, \ldots, k-2)$. We have

$$\begin{aligned}&l_k(\langle a, \langle a_1, \ldots, a_t\rangle\rangle, \langle c_1, \langle c_{11}, \ldots, c_{1t}\rangle\rangle, \ldots, \langle c_{k-2}, \langle c_{k-2,1}, \ldots, c_{k-2,t}\rangle\rangle,\\ &\quad \langle b, \langle b_1, \ldots, b_t\rangle\rangle)\\ &= \langle a, \langle b_1, \ldots, b_t\rangle\rangle,\end{aligned}$$

as l_k is the first projection on $\mathbf{m}$. Hence $\langle a, \langle b_1, \ldots, b_t\rangle\rangle \in Q$, which was needed.

CLAIM 3. *The class* $\mathscr{Q}_{n,k}$ *is closed under forming homomorphic images.*

It is enough to prove that quotient algebras of members of $\mathscr{Q}_{n,k}$ under principal congruences are in $\mathscr{Q}_{n,k}$, too. Let $\mathbf{Q}$ be the same as in Claim 2, and let $a, b \in \mathbf{Q}$, $a = \langle a', a_1, \ldots, a_t\rangle$, $b = \langle b', b_1, \ldots, b_t\rangle$; $a', b' \in \mathbf{m}$, $a_i, b_i \in \mathbf{r}_i$ for $i = 1, \ldots, t$. We examine $\theta = Cg^{\mathbf{Q}}(a, b)$, in particular, we shall establish when, for an arbitrary $c \in Q$, $c \in a/\theta$ does hold.

It may happen $a_i = b_i$; we can assume that $a_i = b_i$ for $i \le q$, where $0 \le q \le t$, and $a_i \neq b_i$ for $i > q$. For $c = \langle c', c_1, \ldots, c_q, \ldots, c_t\rangle$ from $c \in a/\theta$ we infer $c_i = a_i$ whenever $i \le q$, and we can admit $c_i = a_i$ for $q < i \le r$, $c_i = b_i$ for $r < i \le s$, and $c_i \neq a_i, b_i$ for $s < i \le t$ with $q \le r \le s \le t$. Also, as the direct factors of $\mathbf{Q}$ are homogeneous, we do not violate generality by supposing $a' = a_1 = \cdots = a_t = 0$ and $b_{q+1} = \cdots = b_t = 1$. Further, by the same reason, we can put $c = \langle c', 0, \ldots, 0, 0, \ldots, 0, 1, \ldots, 1, 2, \ldots, 2\rangle$ with $w - r$ 1's and $t - s$ 2's. Finally, in order to simplify our notations, we can suppose $q = 1$, $r = 2$, $s = 3$, $t = 4$. Thus, $a = \langle 0, 0, 0, 0, 0\rangle$, $b = \langle b', 0, 1, 1, 1\rangle$, $c = \langle c', 0, 0, 1, 2\rangle$. We show that $c \in a/\theta$ if and only if $c' = 0$ or $c' = b'$. As l_k is trivial on $\mathbf{m}$, the only if part is clear. For the converse, put $d_i = \langle 0, 0, 0, i, i\rangle$ $(i = 3, \ldots, k-1)$, $d_i' = \langle 0, 0, 0, i, 0\rangle$ $(i = 1, 3, \ldots, k-1)$, $d_i'' = \langle 0, 0, i, 0, 0\rangle$ $(i = 2, \ldots, k-1)$, and $d_2 = \langle c', 0, 0, 2, 2\rangle$, $d_2' = \langle c', 0, 1, 2, 2\rangle$, $d_2^* = \langle c', 0, 1, 1, 2\rangle$. Now, if $c' = 0$, then

$$d_2 = l_k(a, b, d_{k-1}, \ldots, d_3, d_2) \equiv l_k(a, a, d_{k-1}, \ldots, d_3, d_2) = a \quad (\theta)$$

and

$$c = l_k(d_2, a, d_{k-1}', \ldots, d_3', d_1') \equiv l_k(a, a, d_{k-1}', \ldots, d_3', d_1') = a \quad (\theta),$$

i.e. $c \in a/\theta$. In the case $c' = b' \neq 0$ we have

$$d_2' = l_k(b, a, d_{k-1}, \ldots, d_3, d_2) \equiv l_k(a, a, d_{k-1}, \ldots, d_3, d_2) = a \quad (\theta),$$

$$d_2^* = l_k(d_2', a, d_{k-1}', \ldots, d_3', d_1') = l_k(a, a, d_{k-1}', \ldots, d_3', d_1') = a \quad (\theta),$$

and

$$c = l_k(d_2^*, d_{k-1}'', \ldots, d_2'', a) = l_k(a, d_{k-1}'', \ldots, d_2'', a) = a \quad (\theta),$$

whence $c \in a/\theta$.

It follows $\mathbf{Q}/\theta \simeq \mathbf{m} \times \prod_{i=1}^{q} \langle \mathbf{r}_i; l_k \rangle$ or $\mathbf{Q}/\theta \simeq \mathbf{m} - \mathbf{1} \times \prod_{i=1}^{q} \langle \mathbf{r}_i; l_k \rangle$ according to c_1 equals 0 or b_1.

Consequently, $\mathbf{Q}/\theta \in \mathcal{Q}_{n,k}$. Case (3) is settled.

CASE (4) is trivial.

CASE (5). $\mathbf{2} \in \mathscr{V}(\langle \mathbf{2}; r_2 \rangle)$ but $\mathbf{2} \notin \mathscr{S}\mathscr{P}(\langle \mathbf{2}; r_2 \rangle)$, hence $\langle \mathbf{2}; r_2 \rangle$ is not a subdirect Stone generator. $\langle \mathbf{3}; r_3 \rangle$ is the one-dimensional (free) affine space over $GF(3)$, and all finite members of $\mathscr{V}(\langle \mathbf{3}; r_3 \rangle)$ are direct powers of $\langle \mathbf{3}; r_3 \rangle$.

For $n \geq 4$, first consider an algebra of form $\mathbf{A}_{n,k} = \langle \mathbf{n}; r_n, l_k \rangle$ ($n \geq 5, 3 \leq k \leq n-2$). It has the following non-isomorphic subalgebras: $\mathbf{A}_{n,k} = \langle \mathbf{n}; r_n, l_k \rangle$, $\mathbf{A}_{n-2,k} = \langle \mathbf{n-2}; p_0^n, l_k \rangle, \ldots, \mathbf{A}_{k,k} = \langle \mathbf{k}; p_0^n, l_k \rangle$, $\mathbf{k-1}, \ldots, \mathbf{2}, \mathbf{1}$. Thus, for $k \leq j \leq n$, $\langle \mathbf{j}; l_k \rangle$ is a *reduct* of $\mathbf{A}_{j,k}$ in the sense that the clone of term operations of $\mathbf{A}_{j,k}$ contains that of $\langle \mathbf{j}; l_k \rangle$. Take into account the following fact: if, for $i = 1, \ldots, t$, $\langle A_i; G \rangle$ is a reduct of $\langle A_i; F \rangle$, and any irredundant subdirect product $\langle B; G \rangle$ of $\langle \langle A_i; G \rangle : 1 \leq i \leq t \rangle$ is a direct product, then any irredundant subdirect product $\langle B; F \rangle$ of $\langle \langle A_i; F \rangle : 1 \leq i \leq t \rangle$ is direct, too. Together with Claim 1 of Case (3), this implies: if $\mathbf{P}$ is a finite subdirect product whose factors are non-trivial subalgebras of $\mathbf{A}_{n,k}$, then $\mathbf{P}$ is a direct product of non-trivial subalgebras of $\mathbf{A}_{n,k}$. Also, from Claim 2 we infer that a subdirect product of a trivial algebra and a direct product of non-trivial subalgebras of $\mathbf{A}_{n,k}$ is always a direct product. Further, a congruence of an algebra is a congruence of any reduct of that algebra, implying together with Claim 3 that the class of all finite direct product of a trivial algebra and non-trivial subalgebras of $\mathbf{A}_{n,k}$ is closed under forming homomorphic images. These versions of Claims 1–3 enable us to repeat the consideration concerning Case (3) in order to prove that $\mathbf{A}_{n,k}$ is a subdirect Stone generator.

As for $\langle \mathbf{n}; r_n \rangle$ ($n \geq 4$), it has no non-trivial proper subalgebras and $\langle \mathbf{n}; b_{k-1} \rangle$ is a reduct of $\langle \mathbf{n}; r_n \rangle$ because the equation

$$r_n(x_1, \ldots, x_{n-2}, r_n(x_1, \ldots, x_{n-1})) = l_{n-1}(x_1, \ldots, x_{n-1})$$

holds identically (cf. [3], Lemma 1, (3)). Thus, the above consideration works again, showing that $\langle \mathbf{n}; r_n \rangle$ is a subdirect Stone generator.

CASE (6). $\langle \mathbf{2}; s \rangle$ is the one-dimensional affine space over $GF(2)$; it is embedded into $\langle \mathbf{2}; s \rangle^2$. Thus, $\langle \mathbf{2}; s \rangle^2$ is a direct (and hence a subdirect) Stone generator. This concludes the proof of the Lemma.

Proof of the Theorem. First observe that the subvarieties of $\mathscr{V}(\langle \mathbf{2}; r_2 \rangle)$ form a chain of length 2, as it has a unique non-trivial subvariety, namely the variety of trivial algebras. By the Lemma, any *other* finite homogeneous algebra $\mathbf{A}$ is a subdirect Stone generator, and the subalgebras of $\mathbf{A}$ are subdirect Stone generators, too, they are homogeneous and not isomorphic with $\langle \mathbf{2}; r_2 \rangle$. Note that subalgebras of $\mathbf{A}$ with universes of the same power are isomorphic by the homogeneity of $\mathbf{A}$, hence we can let $\mathbf{A}_i$ stand for any subalgebra of $\mathbf{A}$ with and i-element universe. Further, if $\mathbf{A}_i$ and $\mathbf{A}_j$ are subalgebras of $\mathbf{A}$ with $i \leq j$, then $\mathbf{A}_i$ can be embedded in $\mathbf{A}_j$, and thus we have $\mathscr{V}(\mathbf{A}_i) \subseteq \mathscr{V}(A_j)$.

Now, Birkhoff's subdirect representation theorem and our lemma together imply that $\mathscr{V}(\mathbf{A}_i)$ is a proper subvariety of $\mathscr{V}(A_j)$ if and only if $i < j$ and $\mathbf{A}_j$ is subdirectly irreducible. Let $\mathscr{W}$ be an arbitrary subvariety of $\mathscr{V}(\mathbf{A}_n)$; we have $\mathscr{W} = \mathscr{V}(\mathbf{A}_j)$ where $\mathbf{A}_j$ is the largest subdirectly irreducible subalgebra of $\mathbf{A}$ with $\mathbf{A}_j \in \mathscr{W}$. Hence the subvarieties of $\mathscr{V}(A_n)$ form a chain, which is isomorphic with the chain of all subdirectly irreducible subalgebras of $\mathbf{A}$ under embeddability. Theorem is proved.

REFERENCES

[1] ASTROMOFF, A., *Some structure theorems for primal and categorical algebras.* Math. Z. *87* (1965), 365–377.

[2] CLARK, D. M. and KRAUSS, P. H., *Para primal algebras.* Algebra Universalis *6* (1967), 165–192.

[3] CSÁKÁNY, B. and GAVALCOVÁ, T., *Finite homogeneous algebras.* Acta Sci. Math. (Szeged) *42* (1980), 57–65.

[4] FRIED, E. and PIXLEY, A. F., *The dual discriminator function in universal algebra.* Acta Sci. Math. (Szeged) *41* (1979), 83–100.

[5] GANTER, B., PLONKA, J. and WERNER, H., *Homogeneous algebras are simple.* Fund. Math. *79* (1973), 217–220.

[6] JÓNSSON, B., *Algebras whose congruence lattices are distributive.* Math. Scand. *21* (1967), 110–121.

[7] KNOEBEL, A., *The equational classes generated by single functionally precomplete algebras.* Memoirs Amer. Math. Soc. *332* (1985), 1–83.