REFERENCES

[1] BAKER, K. A. and PIXLEY, A. F., Polynomial interpolation and the Chinese Remainder Theorem for algebraic systems, *Math. Z.* **143** (1975), 165—174. *MR* **51** #7999.
[2] BIRKHOFF, G., Some applications of universal algebra, *Universal Algebra* (Proc. Conf., Esztergom, Hungary, 1977), Colloq. Math. Soc. J. Bolyai, **29**, North-Holland Publishing Co., Amsterdam—Oxford—New York, 1981, 107—128.
[3] BREDIHIN, D. A., Inverse semigroups of local automorphisms of universal algebras, *Sibirsk. Mat. Ž.* **17** (1976), 499—507, 715 (in Russian). *MR* **54** #2562.
[4] CLARK, D. M. and KRAUSS, P. H., Plain para primal algebras, *Algebra Universalis* **11** (1980), 365—388.
[5] CSÁKÁNY, B., Homogeneous algebras are functionally complete, *Algebra Universalis* **11** (1980), 149—158.
[6] CSÁKÁNY, B. and GAVALCOVÁ, T., Finite homogeneous algebras I, *Acta Sci. Math. (Szeged)* **42** (1980), 57—65. *MR* **81f:** 08005.
[7] DANIL'ČENKO, A. F., Parametric expressibility of functions of three-valued logic, *Algebra i Logika* **16** (1977), 397—416, 493 (in Russian). *MR* **80c:** 03024.
[8] DEMETROVICS, J., HANNÁK, L. and MARČENKOV, S. S., Some remarks on the structure of P_3, *C. R. Math. Rep. Acad. Sci. Canada* **2** (1980), 215—219. *MR* **81m:** 03028.
[9] FRIED, E. and PIXLEY, A. F., The dual discriminator function in universal algebra, *Acta Sci. Math. (Szeged)* **41** (1979), 83—100. *MR* **80g:** 08007.
[10] GRÄTZER, G., *Universal algebra,* D. Van Nostrand Co., Inc., Princeton, N. J.—Toronto, Ont.—London, 1968. *MR* **40** #1320.
[11] JABLONSKI, S. W., GAWRILOW, G. P. and KUDRJAWZEW, W. B., *Boolesche Funktionen und Postsche Klassen,* Akademie-Verlag, Berlin, 1970. *MR* **42** #2868.
[12] LAU, D., Submaximale Klassen von P_3, *Elektron. Informationsverarb. Kybernet.* (to appear).
[13] MACHIDA, H., On closed sets of three-valued monotone logical functions, *Finite algebra and multiple-valued logic* (Proc. Conf.; Szeged, Hungary, 1979), Colloq. Math. Soc. J. Bolyai, **28**, North-Holland Publishing, Co. Amsterdam—Oxford—New York, 1981, 441—467.
[14] MARČENKOV, S. S., Closed classes of self-dual functions of many-valued logic, *Problemy Kibernet.* **36** (1979), 5—22, 279 (in Russian). *MR* **81d:** 03064.
[15] MARČENKOV, S. S., On homogeneous algebras, *Dokl. Akad. Nauk SSSR* **256** (1981), 787—790 (in Russian).
[16] MARČENKOV, S. S., DEMETROVICS, J. and HANNÁK, L., On closed classes of self-dual functions in P_3, *Diskret. Analiz* **34** (1980), 38—73 (in Russian).
[17] McKENZIE, R., Para primal varieties: A study of finite axiomatizability and definable principal congruences in locally finite varieties, *Algebra Universalis* **8** (1978), 336—348. *MR* **57** #9634.
[18] MIYAKAWA, M., On enumerations of bases of three-valued logical functions, *Finite algebra and multiple-valued logic* (Proc. Conf., Szeged, 1979), Colloq. Math. Soc. J. Bolyai, **28**, North-Holland Publishing Co., Amsterdam—Oxford—New York, 1981, 469—487.
[19] PIXLEY, A. F., The ternary discriminator function in universal algebra, *Math. Ann.* **191** (1971), 167—180. *MR* **45** #1820.
[20] QUACKENBUSH, R. W., Some classes of idempotent functions and their compositions, *Colloq. Math.* **29** (1974), 71—81. *MR* **49** #191.
[21] SZABÓ, L., Characterization of some related semigroups of universal algebras, *Acta Sci. Math. (Szeged)* **37** (1975), 143—147. *MR* **53** #7914.
[22] WERNER, H., *Discriminator algebras.* Algebraic representation and model theoretic properties, Studien zur Algebra und Ihre Anwendungen, 6, Akademie-Verlag, Berlin, 1978. *MR* **80f:** 08009.

(Received January 15, 1982)

JÓZSEF ATTILA TUDOMÁNYEGYETEM
BOLYAI INTÉZETE
ARADI VÉRTANÚK TERE 1.
H—6720 SZEGED
HUNGARY

KATEDRA MATEMATIKY
VYSOKÁ ŠKOLA TECHNICKÁ
ŠVERMOVA 9
ČS—041 87 KOŠICE
CZECHOSLOVAKIA

82-2992 — Szegedi Nyomda — F. v.: Dobó József igazgató

Studia Scientiarum Mathematicarum Hungarica 16 (1981), 237—248

THREE-ELEMENT QUASI-PRIMAL ALGEBRAS

BÉLA CSÁKÁNY and TAŤJANA GAVALCOVÁ

To the memory of L. Rédei

1. Preliminaries

In the last years, a considerable amount of work was done in the area of three-element algebras (see, e.g., [7], [8], [12], [13], [14], [15], [16], [18]). On the other hand, quasi-primal algebras were thoroughly studied in the seventies (see [22]). In this paper we make an observation concerning clones of term functions of quasi-primal algebras and apply it to determining all three-element quasi-primal algebras up to cryptoisomorphism.

We use the standard universal algebraic terminology. A finite algebra $\mathfrak{A}$ is *quasi-primal* if the ternary discriminator t is a term function of $\mathfrak{A}$ [22]. Two algebras on the same base set are *equivalent* ([10]) or *cryptoisomorphic* ([2]) if the clones of their term functions are the same, or coincide up to a permutation of the base set, respectively. Following Marczewski, the *arity* of an algebra $\langle A; F\rangle$ is the least natural number n such that the clone $[F]$ is generated by the n-ary functions it contains; thus $\langle A; F\rangle$ is *n-ary* if it is equivalent to an algebra whose basic operations are (at most) n-ary. It is important for us that quasi-primality as well as arity of algebras are invariant under cryptoisomorphism.

A *partial permutation* of a set A is a 1-1 mapping between two subsets of A; a *partial automorphism* of an algebra $\mathfrak{A}$ is an isomorphism between two subalgebras of $\mathfrak{A}$. Partial permutations (automorphisms) of A ($\mathfrak{A}$) can be considered as a special kind of subsets (subalgebras) of A^2 ($\mathfrak{A}^2$). By the concrete characterization theorem of partial automorphisms, due to SZABÓ [21] and BREDIHIN [3], a family Σ of partial permutations of a finite set A is the inverse semigroup of all partial automorphisms of an algebra $\langle A; F\rangle$ (in notation: $\Sigma=\mathrm{Part}\langle A; F\rangle$) if and only if

(i) Σ is closed under multiplication and taking inverses,
(ii) Σ is closed under set-theoretical intersection,
(iii) Σ contains the identical map of A,
(iv) if, for $a, b\in A$, $\{\langle a, a\rangle\}$, $\{\langle b, b\rangle\}\in\Sigma$, then $\{\langle a, b\rangle\}\in\Sigma$.

We can even suppose that $\langle A; F\rangle$ is quasi-primal as adjoining the ternary discriminator t to F does not change the family $\mathrm{Part}\langle A; F\rangle$. Now, if $\langle A; F\rangle$ is quasi-primal then, by PIXLEY's theorem [19], $[F]$ consists exactly of those operations on A which preserve all members of $\mathrm{Part}\langle A; F\rangle$; hence, $\mathrm{Part}\langle A; F_1\rangle=\mathrm{Part}\langle A; F_2\rangle$ implies $[F_1]=[F_2]$ whenever $t\in[F_1], [F_2]$. Thus, we have the following

PROPOSITION 1. *Let A be a finite set, S a clone of operations on A, and $\Sigma=\mathrm{Part}\langle A; S\rangle$. Then $S\leftrightarrow\Sigma$ is a* 1-1 *inclusion-reversing correspondence between*

1980 *Mathematics Subject Classification.* Primary 08A40; Secondary 03G20.
Key words and phrases. Quasi-primal algebra, clone, cryptoisomorphy, arity.

the clones of operations on A *containing the ternary discriminator and the families of partial permutations of* A *enjoying properties* (i)—(iv).

Families of partial permutations with (i)—(iv) will be termed *algebraic families*. In virtue of Proposition 1, for determining all quasi-primal algebras on a set A up to equivalence it is sufficient to produce all algebraic families of A.

PROPOSITION 2. *Two quasi-primal algebras* $\langle A; F_1\rangle$ *and* $\langle A; F_2\rangle$ *are cryptoisomorphic iff there exists a permutation* π *of* A *which transfers* Part $\langle A; F_1\rangle$ *into* Part $\langle A; F_2\rangle$.

Indeed, in this case π transfers also $[F_1]$ into $[F_2]$. Hence for determining all quasi-primal algebras on A up to cryptoisomorphism it suffices to produce a maximal set of algebraic families of A relative to the property that they cannot be obtained from each other by a permutation of A. A set with this property will be called *permutationally independent*.

2. Determining algebraic families of the three-element set

The three-element algebras we consider have the common base set $\underline{3}=\{0, 1, 2\}$. Let us agree in the notation $\{0,1\}=\mathbf{a}$, $\{0,2\}=\mathbf{b}$, $\{1,2\}=\mathbf{c}$. Algebraic families of $\underline{3}$ will be denoted by pairs where the i-th entry symbolizes the partial permutations in the family in question with i-element domain (the full permutations will always be indicated by the context). The first entry is simply the list of those elements which are domains of partial permutations; in virtue of (i) and (iv) the set of all partial permutations with one-element domain is uniquely determined by this list. In the second entry we use symbols a, b, c for the identical mappings of $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$, respectively; further, we apply bars and arrows in the following manner: x in the second entry for Σ means that the non-identical permutation of $\mathbf{x}$ belongs to Σ, and $x\to y$ $(x\leftarrow y)$ means that the monotone (antitone) 1-1 mapping of $\mathbf{x}$ onto $\mathbf{y}$ belongs to Σ (we write a two-headed arrow instead of two arrows between the same symbols). Using these notations, we list a subfamily of the partial permutations with 2-element domain from which the whole family in question can be obtained by (i). Thus, e.g. $\langle 2, b\to c\rangle$ stands for an algebraic family whose non-permutation members are

$$\begin{pmatrix}2\\2\end{pmatrix}, \begin{pmatrix}0&2\\0&2\end{pmatrix}, \begin{pmatrix}1&2\\1&2\end{pmatrix}, \begin{pmatrix}0&2\\1&2\end{pmatrix}, \begin{pmatrix}1&2\\0&2\end{pmatrix}.$$

Taking into account (i), we shall indicate mapping(s) of $\mathbf{a}$ onto $\mathbf{c}$ in the second entry only if there are no mappings of $\mathbf{a}$ onto $\mathbf{b}$ and $\mathbf{b}$ onto $\mathbf{c}$. Note that the occurrence of x in the second entry of a pair means that $\mathbf{x}$ is a subalgebra of any algebra $\mathfrak{A}$ such that Part $\mathfrak{A}$ is represented by that pair.

Representing algebraic families in such a way has the advantage that it mirrors the inclusion between families, i.e. a larger family is always denoted by a graphically richer pair. Hence in order to draw the lattice of algebraic families of $\underline{3}$ with the same full permutation part (which is an interval in the lattice of all algebraic families of $\underline{3}$) we have to draw the lattice of their representing pairs under "graphical inclusion".

The properties defining algebraic families impose the following restrictions on the form of representing pairs:

By (ii), if the second entry contains a and b (or a and c, or b and c) then the first entry contains 0 (or 1, or 2, respectively).

By (i), if x is at an endpoint of an arrow then this arrow is double if and only if x has a bar.

By (i), if the second entry indicates the presence of a partial permutation π whose domain contains $n(\in\underline{3})$ then the first entry contains either both n and $\pi(n)$ or none of them.

There are 114 pairs respecting these restrictions, and each of them with the identical permutation ι added represents an algebraic family. A maximal permutationally independent set of these families is the following:

(1) $\langle\ ,\ \rangle$

(2) $\langle 2,\ \rangle$

(3) $\langle 01,\ \rangle$

(4) $\langle 012,\ \rangle$

(5) $\langle\ , a\rangle$

(6) $\langle 2, a\rangle$

(7) $\langle 1, a\rangle$

(8) $\langle 02, a\rangle$

(9) $\langle 01, a\rangle$

(10) $\langle 012, a\rangle$

(11) $\langle\ , \bar a\rangle$

(12) $\langle 2, \bar a\rangle$

(13) $\langle 01, \bar a\rangle$

(14) $\langle 012, \bar a\rangle$

(15) $\langle 2, bc\rangle$

(16) $\langle 12, bc\rangle$

(17) $\langle 012, bc\rangle$

(18) $\langle 2, b\to c\rangle$

(19) $\langle 012, b\leftarrow c\rangle$

(20) $\langle 012, b\to c\rangle$

(21) $\langle 12, b\bar c\rangle$

(22) $\langle 012, b\bar c\rangle$

(23) $\langle 012, \bar b\bar c\rangle$

(24) $\langle 012, \bar b\leftrightarrow \bar c\rangle$

(25) $\langle 012, abc\rangle$

(26) $\langle 012, ab\leftarrow c\rangle$

(27) $\langle 012, ab\to c\rangle$

(28) $\langle 012, \bar a bc\rangle$

(29) $\langle 012, a\leftarrow b\leftarrow c\rangle$

(30) $\langle 012, a\to b\leftarrow c\rangle$

(31) $\langle 012, \bar a b\leftarrow c\rangle$

(32) $\langle 012, \bar a b\to c\rangle$

(33) $\langle 012, a\bar b\bar c\rangle$

(34) $\langle 012, \bar a\bar b\bar c\rangle$

(35) $\langle 012, a\bar b\leftrightarrow \bar c\rangle$

(36) $\langle 012, \bar a\bar b\leftrightarrow \bar c\rangle$

(37) $\langle 012, \bar a\leftrightarrow \bar b\leftrightarrow \bar c\rangle$.

Here each of (7), (8), (16), (21) and (22) produces five further algebraic families by permutations, while (1), (4), (25), (29), (34) and (37) produce no further one. The remaining families produce two additional ones each. As an illustration, we show the lattice of algebraic families arising from (25)—(37) by permutations (it is dual to the lattice of all clones C on $\underline{3}$ with the property that $\langle \underline{3}; C\rangle$ is quasi-primal, rigid, and every subset of $\underline{3}$ forms a subalgebra of $\langle \underline{3}; C\rangle$); see Fig. 1, where (n') and (n'') represent the families arising from (n) by the transpositions (02) and (12), respectively.

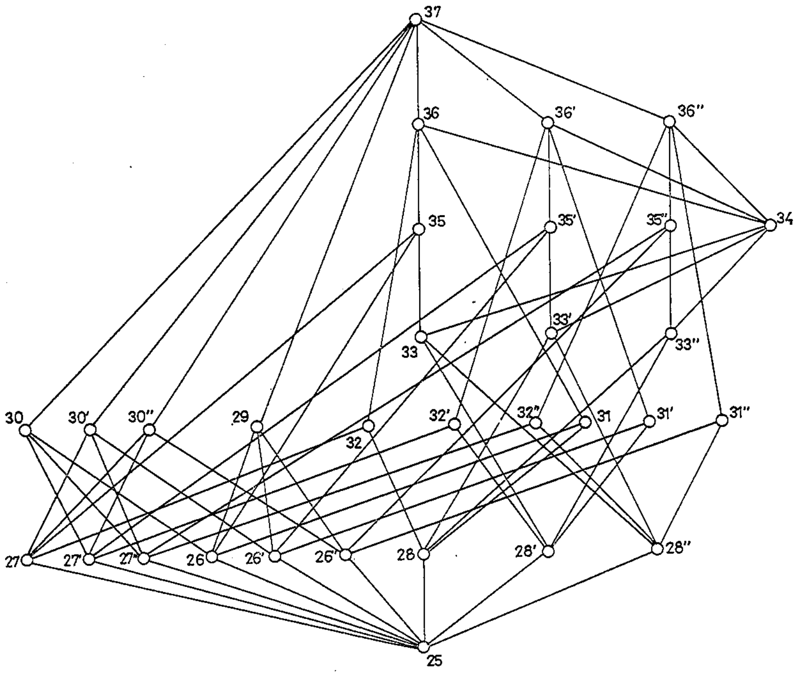

Fig. 1

Next we deal with algebraic families Σ including a unique transposition of $\underline{3}$. The presence of a transposition in Σ involves further restrictions on representing pairs. E.g., $(01)\in\Sigma$ implies:

By (i), if a appears in the second entry then it has a bar.
By (i), if the second entry contains b or c then it contains $b\to c$.
By (ii) and (iii), the first entry contains 2.

There are 26 pairs respecting these restrictions as well (or their substitutes under changing (01) into other transposition), and 24 of them represent one algebraic family (containing ι and a transposition) each, while two of them represent three such algebraic families each (as they may be combined by any transposition). Thus

we have 30 further distinct algebraic families, and a maximal permutationally independent set of them is the following:

(38)	(2) with ι and (01)	(43)	(20) with ι and (01)
(39)	(4) with ι and (01)	(44)	(24) with ι and (01)
(40)	(12) with ι and (01)	(45)	(32) with ι and (01)
(41)	(14) with ι and (01)	(46)	(36) with ι and (01)
(42)	(18) with ι and (01)	(47)	(37) with ι and (01).

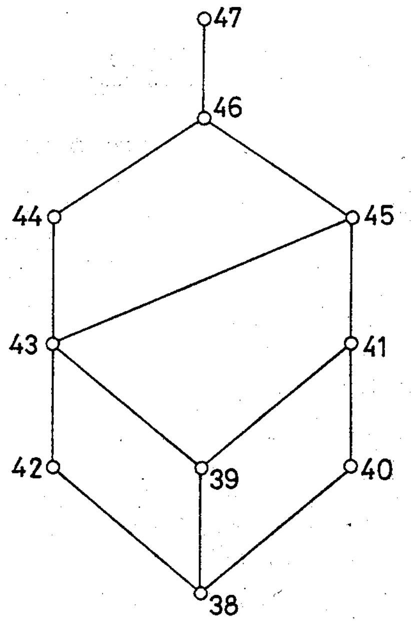

Fig. 2

Fig. 2 shows the lattice of these families. It is dual to the lattice of all clones C on $\underline{3}$ such that $\langle \underline{3}; C\rangle$ is quasi-primal, and (01) is an automorphism of it.

Finally, DEMETROVICS, HANNÁK and MARČENKOV [8], [16] determined all clones C on 3 such that $t\in C$ and the automorphism group of $\langle 3; C\rangle$ is A_3 or S_3. The corresponding algebraic families are the following:

(48)	(1) with A_3	(52)	(4) with S_3
(49)	(4) with A_3	(53)	(37) with S_3.
(50)	(29) with A_3		
(51)	(37) with A_3		

We have established that there exist 150 distinct algebraic families on 3 and a maximal permutationally independent set of them consists of 53 families. Now

Propositions 1 and 2 imply that *there exist exactly* 150 *pairwise non-equivalent quasi-primal algebras with the base set* $\underline{3}$, and *there exist exactly* 53 *pairwise non-cryptoisomorphic three-element quasi-primal algebras.* The (up to equivalence) unique three-element quasi-primal algebra whose partial automorphisms are given by (n), $n=1, \ldots, 53$, will be denoted by $\mathfrak{A}_n$.

3. Determining basic operations

In this section we show that three-element quasi-primal algebras are at most ternary; moreover, for every quasi-primal algebra we determine a set of basic operations with minimal arity. Clearly, it is enough to deal with the algebras $\mathfrak{A}_n$ $(n=1, \ldots, 53)$.

LEMMA 1. *A three-element quasi-primal algebra* $\mathfrak{A}$ *is at least ternary provided*
(α) $\mathfrak{A}$ *has a two-element subalgebra whose both elements are also subalgebras, or*
(β) $\mathfrak{A}$ *has a two-element subalgebra with a non-trivial automorphism.*

PROOF. Suppose $\mathfrak{A}=\langle \underline{3}; f_1, \ldots, f_k\rangle$ where each f_i is binary. If (α) holds and — say — **a** is the required subalgebra, then the restriction of each f_i to **a** is a binary Boolean function preserving 0 and 1, i.e., f_i is maximum or minimum function or a projection. However, as the Post diagram [11] shows, these functions do not generate the ternary discriminator t on **a**. Hence t is not a term function of $\mathfrak{A}$, i.e., $\mathfrak{A}$ is not quasi-primal, a contradiction. In the case (β), the restriction of each f_i to **a** is a self-dual Boolean function, and a similar argument works.

Lemma 1 applied to the above list (1)—(53) implies that $\mathfrak{A}_9, \ldots, \mathfrak{A}_{14}, \mathfrak{A}_{16}, \mathfrak{A}_{17}, \mathfrak{A}_{19}, \ldots, \mathfrak{A}_{37}, \mathfrak{A}_{40}, \mathfrak{A}_{41}, \mathfrak{A}_{43}, \ldots, \mathfrak{A}_{47}, \mathfrak{A}_{50}, \mathfrak{A}_{51}$, and $\mathfrak{A}_{53}$ are at least ternary. Furthermore, $\mathfrak{A}_{52}$ also is at least ternary as its term functions are homogeneous [5], and the unique binary homogeneous function on $\underline{3}$ is $x \circ y = 2x+2y$, which, however, does not generate t. (Here and in what follows, addition and multiplication will be done modulo 3.) Let T stand for the set of indices of these at least ternary algebras, and put $B=\{1, \ldots, 53\} \setminus T$.

LEMMA 2. *The algebras* $\mathfrak{A}_n$ $(n \in T)$ *are ternary.*

PROOF. For each $n \in T$, we define a ternary operation f_n on $\underline{3}$ such that Part $\langle \underline{3}; t, f_n\rangle = (n)$. Then $\langle \underline{3}; t, f_n\rangle$ is equivalent to $\mathfrak{A}_n$, and hence $\mathfrak{A}_n$ is ternary.

In order to define f_n we shall need the following operations on 3: the dual discriminator d ([9]; d is a majority function); the switching function s ([22]; s is a minority function); the maximum and minimum function (of arbitrary arity), denoted by $\vee$ and $\wedge$, respectively;

$$g(x, y, z) = x \wedge (y \vee z);$$

$$h(x, y, z) = g(y, x, z);$$

$$u(x, y, z) \begin{cases} \in \mathbf{a}, \neq x, & \text{if } x, y, z \in \mathbf{a}, \\ = d(x, y, z) & \text{otherwise}; \end{cases}$$

$$k_{ij}(x, y, z) = \begin{cases} i & \text{if } x = y = z, \\ j & \text{otherwise}; \end{cases} \quad (i, j \in \underline{3})$$

$$p(x, y, z) = y \circ z = 2y+2z.$$

Now we define f_n in such a way that it is compatible with every member in (n) and incompatible with every partial or full permutation outside of (n), thus guaranteeing the equality Part $\langle \underline{3}; t, f_n\rangle = (n)$. This is possible, e.g. as follows. First we fix $f_n(x, y, z)$ for pairwise distinct $x, y, z \in \underline{3}$ by the following table (these values of f_n take care of the full permutations):

	$1 \leqq n \leqq 37$	$38 \leqq n \leqq 47$	$48 \leqq n \leqq 51$	$n=52$	$n=53$
$f_n(0, 1, 2)$	0	0	0	0	2
$f_n(1, 2, 0)$	0	0	1	1	0
$f_n(2, 0, 1)$	1	0	2	2	1
$f_n(0, 2, 1)$	0	1	2	0	1
$f_n(1, 0, 2)$	0	1	0	1	2
$f_n(2, 1, 0)$	0	1	1	2	0

The table below defines the restrictions of f_n to **a**, **b**, and **c**.

n	**a**	**b**	**c**	n	**a**	**b**	**c**
9	$\vee$	k_{01}	k_{01}	25	$\vee$	g	h
11	u	k_{01}	k_{01}	26	g	$\vee$	$\wedge$
12,40	u	u	u	27	g	$\vee$	$\vee$
13	d	k_{01}	k_{01}	28	d	$\vee$	g
16	k_{20}	k_{20}	$\vee$	29,50	$\vee$	$\wedge$	$\vee$
21	k_{20}	k_{20}	d	30	$\vee$	$\vee$	$\wedge$
				31	d	$\vee$	$\wedge$
10	$\vee$	p	p	32,45	d	$\vee$	$\vee$
14,41	d	p	p	33	$\vee$	d	s
17	p	$\vee$	g	34	d	s	t
19	p	$\vee$	$\wedge$	35	$\vee$	s	t
20,43	p	$\vee$	$\vee$	36,46	d	s	s
22	p	$\vee$	d	37, 47, 51, 53	d	d	d
23	p	d	s				
24,44	p	d	d				
52	p	p	p				

In order to check that these $f_n (n \in T)$ preserve the partial permutations in (n) and violate the remaining ones take into account that t, d, s are distinct self-dual (Boolean) functions on each two-element subset of $\underline{3}$; $\vee, \wedge, g, h$ are distinct non-self-dual functions there and $\wedge$ is dual to $\vee$; p preserves the one-element subsets and violates the two-element subsets of $\underline{3}$; finally, t, d, s, and p preserve all permutations of $\underline{3}$. These remarks help to settle the cases $n=10, \ldots, 52$ and $25, \ldots, 53$, while the cases $n=9, \ldots, 21$ require separate (trivial) computation.

As for the remaining cases, the following lemmas will be crucial.

LEMMA 3. $\mathfrak{A}_8$ *is binary.*

PROOF. Consider $\mathfrak{A}=\langle \underline{3};\ \vee, \wedge, b_0, b_2\rangle$ with binary $\vee$ and $\wedge$, where b_i $(i=0, 2)$ is given by the Cayley table

	0	1	2
0	0	1	1
1	0	0	2
2	2	i	2.

We prove that $\mathfrak{A}$ is equivalent to $\mathfrak{A}_8$. For this aim first observe that Part $\mathfrak{A}=$ $=$Part $\mathfrak{A}_8$ $(=(8))$; then it is enough to prove that t is a term function of $\mathfrak{A}$.

Clearly, $m(x, y, z)=(x\wedge y)\vee(x\wedge z)\vee(y\wedge z)$ is a ternary majority function (see, e.g., [20]). Hence, by the well-known BAKER—PIXLEY theorem [1] any operation on $\underline{3}$ is a term function of $\mathfrak{A}$ iff it preserves all subalgebras of $\mathfrak{A}^2$. Thus we have to show that t preserves all of them. We do this by showing that any subalgebra of $\mathfrak{A}^2$ is either a partial automorphism of $\mathfrak{A}$ (briefly: a *string* of $\mathfrak{A}$, cf. [6]) or the product of two subalgebras of $\mathfrak{A}$ (briefly: a *block* of $\mathfrak{A}$).

Let $C\subseteq\underline{3}^2$ be a subalgebra of $\mathfrak{A}^2$. If $C_1\times C_2$ is the least block in 3 which contains C then we say that C is of *size* $|C_1|\times|C_2|$. If C is of size $1\times k$ or $k\times 1$ then C is a block. If C is of size 2×2, then $\vee C$ (the component-wise maximum of all pairs in C) equals $\langle 1, 1\rangle$, and $\wedge C=\langle 0, 0\rangle$. Now either C is the string $\{\langle 0, 0\rangle, \langle 1, 1\rangle\}$ or, e.g., $\langle 0, 1\rangle\in C$; then $\langle 1, 0\rangle=b_2(\langle 0, 1\rangle, \langle 1, 1\rangle)\in C$ and $C=\underline{2}^2$. If C is of size 3×2, then $\vee C=\langle 2, 1\rangle$, $\wedge C=\langle 0, 0\rangle$. Now we obtain $b_2(\langle 0, 0\rangle, \langle 2, 1\rangle)=\langle 1, 1\rangle$, $b_2(\langle 2, 1\rangle, \langle 1, 1\rangle)=\langle 2, 0\rangle$; $\langle 1, 1\rangle\wedge\langle 2, 0\rangle=\langle 1, 0\rangle$; $b_2(\langle 1, 0\rangle, \langle 1, 1\rangle)=\langle 0, 1\rangle$; hence $C=\underline{3}\times\underline{2}$. The case of size 2×3 can be settled in the same way.

Finally, let C be of size 3×3; then $\vee C=\langle 2, 2\rangle$, $\wedge C=\langle 0, 0\rangle$. We have $b_0(\langle 0, 0\rangle, \langle 2, 2\rangle)=\langle 1, 1\rangle$, hence $C\supseteq\{\langle i, i\rangle : i\in\underline{3}\}$. If the equality holds then C is a string. Assume $\langle 0, 2\rangle\in C$. Then

$$m(\langle 0,\ 0\rangle,\ \langle 1,\ 1\rangle,\ \langle 0,\ 2\rangle) = \langle 0,\ 1\rangle,$$

$$m(\langle 0,\ 2\rangle,\ \langle 1,\ 1\rangle,\ \langle 2,\ 2\rangle) = \langle 1,\ 2\rangle,$$

$$b_0(\langle 2,\ 2\rangle,\ \langle 0,\ 1\rangle) = \langle 2,\ 0\rangle,$$

$$m(\langle 0,\ 0\rangle,\ \langle 1,\ 1\rangle,\ \langle 2,\ 0\rangle) = \langle 1,\ 0\rangle,$$

$$m(\langle 2,\ 0\rangle,\ \langle 1,\ 1\rangle,\ \langle 2,\ 2\rangle) = \langle 2,\ 1\rangle,$$

i.e., $C=\underline{3}^2$. By symmetry, the assumption $\langle 2, 0\rangle\in C$ gives the same result. If $\langle 0, 1\rangle\in C$ then

$$b_0(\langle 2,\ 2\rangle,\ \langle 0,\ 1\rangle) = \langle 2,\ 0\rangle.$$

If $\langle 1, 2\rangle\in C$ then

$$b_0(\langle 2,\ 2\rangle,\ \langle 1,\ 2\rangle) = \langle 0,\ 2\rangle.$$

In both cases we can proceed as before, and the same procedure works for $\langle 1, 0\rangle$ and $\langle 2, 1\rangle$. Thus, $C=\underline{3}^2$ and we are done.

LEMMA 4. $\mathfrak{A}_{39}$ *and* $\mathfrak{A}_{42}$ *are binary.*

PROOF. Let us consider the following operations on $\underline{3}$:

$$x\cup y=\begin{cases} x & \text{if} \quad x=y,\\ 2 & \text{otherwise};\end{cases} \qquad x\cap y=\begin{cases} y & \text{if} \quad x=2,\\ x & \text{otherwise.}\end{cases}$$

Observe that $\cup$ and $\cap$ are compatible with (39) as well as with (42). Denote $(x\cap y)\cup(x\cap z)\cup(y\cap z)$ by $\mu(x, y, z)$; then $\mu(x, y, z)\cap\mu(z, x, y)\cap\mu(y, z, x)$ is a ternary majority function on $\underline{3}$ (the reader has to check this on $\mathbf{a}$ only, as the restriction of $\cup(\cap)$ on $\mathbf{b}$ and $\mathbf{c}$ is $\vee(\wedge)$).

Let $*$ denote the binary operation with Cayley table

	0	1	2
0	2	1	0
1	0	2	1
2	0	1	2.

We show that $\mathfrak{B}=\langle 3;\ \cup, \cap, *\rangle$ is equivalent to $\mathfrak{A}_{42}$. After checking Part $\mathfrak{B}=(42)$ we proceed as in Lemma 3: having a majority function we need only to establish that any subalgebra of $\mathfrak{B}^2$ is a string or a block. We can confine ourselves to subalgebras C with size 2×2, 2×3, or 3×3. First, let C be of size 2×2. Any two blocks of the same size of $\mathfrak{B}$ may be translated into each other by automorphisms of $\mathfrak{B}^2$, hence we can assume $C\subseteq\mathbf{c}^2$. It is enough to show that $|C|=3$ is impossible. This is the case indeed, as $\langle 1, 1\rangle*\langle 2, 1\rangle=\langle 1, 2\rangle$, $\langle 2, 1\rangle*\langle 1, 2\rangle=\langle 1, 1\rangle$, $\langle 1, 2\rangle*\langle 1, 1\rangle=$ $=\langle 2, 1\rangle$, and for any $x\in\mathfrak{B}^2$ we have $x*x=\langle 2, 2\rangle$.

Let C be of size 2×3. Again we can assume $C\subseteq\mathbf{c}\times 3$. We prove that C is a block. By assumption, $C\cap(3\times\{0\})$, $C\cap(\underline{3}\times\{1\})\neq\emptyset$; furthermore, $C\cap(\{2\}\times\underline{3})=$ $=\{2\}\times\mathbf{b}$ or $\{2\}\times\mathbf{c}$ or $\{\langle 2, 2\rangle\}$ or $\{2\}\times\underline{3}$. Accordingly, we have four possibilities:

I. $\langle 2, 0\rangle$, $\langle 1, 1\rangle$, $\langle 2, 2\rangle\in C$. In such a case, $\langle 1, 1\rangle*\langle 2, 0\rangle=\langle 1, 0\rangle\in C$, hence $C\cap(\mathbf{c}\times\mathbf{b})$ has at least 3 elements implying $C\supseteq\mathbf{c}\times\mathbf{b}$. It follows that $C\cap\mathbf{c}^2$ has at least 3 elements implying $C\supseteq\mathbf{c}^2$. Thus $C=\mathbf{c}\times\underline{3}$.

II. $\langle 1, 0\rangle$, $\langle 2, 1\rangle$, $\langle 2, 2\rangle\in C$. Now the same argument works due to the partial automorphism between $\mathbf{b}$ and $\mathbf{c}$.

III. $\langle 1, 0\rangle$, $\langle 1, 1\rangle$, $\langle 2, 2\rangle\in C$. Now we have $\langle 1, 0\rangle*\langle 1, 1\rangle=\langle 2, 1\rangle$, and we can go back to the case II.

IV. C includes $\{2\}\times\underline{3}$ and at least one element of form $\langle 1, i\rangle$ $(i\in\underline{3})$. If $i=0$ or 1, we have case II or I, respectively. If $i=2$, $\langle 1, 2\rangle*\langle 2, 1\rangle=\langle 1, 1\rangle\in C$, and this is case I again.

We have established that C is a block provided it has at least 3 elements and it is not of size 3×3. Finally, suppose that C is of size 3×3. This implies $C\supseteq\iota$ or $C\supseteq(01)$. In both cases, if strong inclusion holds then at least one of $C\cap(\underline{3}\times\mathbf{b})$ and $C\cap(\underline{3}\times\mathbf{c})$ contains at least 3 elements. Thus either we have $C\supseteq\underline{3}\times\mathbf{b}$ implying $|C\cap(\underline{3}\times\mathbf{c})|\geqq 3$ and $C\supseteq\underline{3}\times\mathbf{c}$, or $C\supseteq\underline{3}\times\mathbf{c}$ holds implying $C\supseteq\underline{3}\times\mathbf{b}$ analogously. In any case, $C=\underline{3}^2$. This concludes the proof concerning $\mathfrak{A}_{42}$.

As for $\mathfrak{A}_{39}$, we prove that it is equivalent to $\mathfrak{C}=\langle\underline{3};\ \cup, \cap, \circ\rangle$. Checking Part $\mathfrak{C}=(39)$ is immediate. Now we apply the following theorem of Pixley ([19]; see also [20]): t is a term function of the finite algebra $\mathfrak{C}$ if $\mathfrak{C}$ has a ternary majority term function, a Malcev term function, and every subalgebra of $\mathfrak{C}$ is simple. We have seen that $\cup$ and $\cap$ generate the majority function μ. On the other hand,

$(x\circ z)\circ y$ is a Malcev function. The subalgebras of our $\mathfrak{C}$ are obviously simple. Thus t is a term function of $\mathfrak{C}$. Lemma 4 is proved.

Now we are ready to prove

LEMMA 5. *The algebras* $\mathfrak{A}_k$ $(k\in B)$ *are binary.*

PROOF. $\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_5$, and $\mathfrak{A}_7$ are familiar: $\mathfrak{A}_1$ is (up to equivalence) P_3, the three-element *Post algebra* (i.e., the up to cryptoisomorphism unique three-element *primal* algebra). $\mathfrak{A}_2$ is cryptoisomorphic to a quasi-primal algebra $\mathfrak{A}$ with Part $\mathfrak{A}=\langle 0, \ \rangle$, and this is $GF(3)$, the three-element *Galois field.* $\mathfrak{A}_5$ is cryptoisomorphic to a quasi-primal algebra $\mathfrak{A}$ with Part $\mathfrak{A}=\langle \ , b\rangle$, and this is L_3, the three-element *Łukasiewicz-algebra.* $\mathfrak{A}_7$ is cryptoisomorphic to an $\mathfrak{A}$ with Part $\mathfrak{A}=\langle 0, b\rangle$, and this is S_3, *Bosbach's complementary semigroup* on $\underline{3}$. These observations follow immediately from Propositions 1 and 2 (for definitions, consult [22]).

In [16] it is established that $\mathfrak{A}_{48}$ is the algebra $\langle \underline{3}; \alpha, x+1\rangle$ and $\mathfrak{A}_{49}$ is $\langle \underline{3}; \alpha, \circ\rangle$, where α is given by the table

α	0	1	2
0	0	1	0
1	1	1	2
2	0	2	2.

By Proposition 1, as (4) is the meet of (39) and (49), the clone of $\mathfrak{A}_4$ is the join of those of $\mathfrak{A}_{39}$ and $\mathfrak{A}_{49}$. Hence $\mathfrak{A}_4$ is $\langle \underline{3}; \cup, \cap, \alpha, \circ\rangle$. Further, (3) differs from (4) in the lack of 2 in the first entry; hence a system of operations for $\mathfrak{A}_3$ may be got from that of $\mathfrak{A}_4$ by adding an operation which preserves 0, 1, and violates 2 — say, x^2. Hence $\mathfrak{A}_3$ is $\langle \underline{3}; \cup, \cap, \alpha, \circ, x^2\rangle$. Similarly, (38) differs from (42) in the lack of b and c in the second entry; thus in order to obtain a system of operations for $\mathfrak{A}_{38}$ it is sufficient to add the operation $\circ$ to that of $\mathfrak{A}_{42}$, as $\circ$ preserves 2 and (01), and it violates **b** and **c**. Thus, $\mathfrak{A}_{38}$ is $\langle \underline{3}; \cup, \cap, *, \circ\rangle$.

By analogous reasoning, the set of operations for $\mathfrak{A}_{42}$ augmented by $\vee$ forms a system of operations for $\mathfrak{A}_{18}$, while by $2x^2$ added it becomes a system of operations for $\mathfrak{A}_{15}$. Thus, $\mathfrak{A}_{18}$ is $\langle \underline{3}; \cup, \cap, *, \vee\rangle$, and $\mathfrak{A}_{15}$ is $\langle \underline{3}; \cup, \cap, *, 2x^2\rangle$. Finally, operations for $\mathfrak{A}_8$ together with $2x+1$ are a system of operations for $\mathfrak{A}_6$; hence, $\mathfrak{A}_6$ is $\langle \underline{3}; \vee, \wedge, b_0, b_2, 2x+1\rangle$, concluding the proof.

The result of our considerations may be summarized as follows:

THEOREM. *The following* 53 *three-element algebras are pairwise non-cryptoisomorphic and any three-element quasi-primal algebra is cryptoisomorphic to one of them:*

$$P_3;\ GF(3);\ L_3;\ S_3;$$
$$\langle \underline{3};\ \vee,\ \wedge,\ b_0,\ b_2\rangle;\ \ \langle \underline{3};\ \vee,\ \wedge,\ b_0,\ b_2,\ 2x+1\rangle;$$
$$\langle \underline{3};\ \cup,\ \cap,\ *\rangle;\ \ \langle \underline{3};\ \cup,\ \cap,\ *,\ \circ\rangle;$$
$$\langle \underline{3};\ \cup,\ \cap,\ *,\ \vee\rangle;\ \ \langle \underline{3};\ \cup,\ \cap,\ *,\ 2x^2\rangle;$$
$$\langle \underline{3};\ \alpha,\ x+1\rangle;\ \ \langle \underline{3};\ \alpha,\ \circ\rangle;$$
$$\langle \underline{3};\ \cup,\ \cap,\ \circ\rangle;\ \ \langle \underline{3};\ \cup,\ \cap,\ \alpha,\ \circ\rangle;$$
$$\langle \underline{3};\ \cup,\ \cap,\ \alpha,\ \circ,\ x^2\rangle;\ \text{and}$$
$$\langle \underline{3};\ t,\ f_n\rangle\ \text{for each}\ n\in T.$$

Remark that $\langle \underline{3}; t, f_{53}\rangle=\langle \underline{3}; t\rangle$ as the ternary discriminator generates $f_{53}=d$, and $\langle \underline{3}; t, f_{52}\rangle=;\langle \underline{3}; t, \circ\rangle$ as $f_{52}(x, y, z)=y\circ z$. The term functions of these algebras are the pattern functions and the homogeneous functions on $\underline{3}$, respectively (cf. [20], [5]). The first who established that $\langle \underline{3}$; all idempotent operations$\rangle$ $(=\mathfrak{A}_4)$ is binary was QUACKENBUSH [20]; his considerations give $\mathfrak{A}_4=\langle \underline{3}; \vee, \circ.\rangle$

4. Special quasi-primal algebras

Let Ω be a set of (finitary) relations on the finite set A, i.e. a set consisting of several subsets of finite powers of A. As usual, denote by Pol Ω the clone of all operations preserving every member of Ω. Take an arbitrary clone C of operations on A. The mapping $C\to$Pol Part $\langle A; C\rangle$ is a closure operator $\mathrm{Cl}_{\mathrm{Part}}$ on the set of all clones on A, and the 1—1 correspondence in Proposition 1 is the Galois connection related to this closure operator. Closedness of C means that it contains t, i.e. $\langle A; C\rangle$ is quasi-primal.

Replace Part by Aut (the full automorphism group) here; in such a way we obtain another closure operator $\mathrm{Cl}_{\mathrm{Aut}}$. If C is a fix-point of $\mathrm{Cl}_{\mathrm{Aut}}$ then $\langle A; C\rangle$ is called *demi-primal.* Similarly, *semi-primal* algebras arise from Sub (the subalgebra lattice), and *demi-semi-primal* algebras from Aut & Sub (cf. WERNER [22]). As operations preserving all partial automorphisms preserve all automorphisms and subalgebras a fortiori, the algebras of the listed types are quasi-primal. We show on the example of semi-primals how to select them from the set of all quasi-primals.

Every algebra $\mathfrak{A}_n$ $(n=1, ..., 53)$ may be expressed as $\langle \underline{3}; \mathrm{Pol}\,(n)\rangle$. Semi-primality of $\mathfrak{A}_n$ means that Pol (n) consists of all operations preserving the subalgebras of $\mathfrak{A}_n$. This is the case exactly when (n) does not contain non-identical (partial) automorphisms with two- or three-element domain, i.e. when (n) is automorphism-free and contains neither arrow nor bar. Hence a full list of three-element semi-primal algebras is up to cryptoisomorphism the following:

$$\mathfrak{A}_1, ..., \mathfrak{A}_{10}, \mathfrak{A}_{15}, \mathfrak{A}_{16}, \mathfrak{A}_{17}, \mathfrak{A}_{25}.$$

An analogous argument shows that there exist four three-element demi-primal algebras, namely $\mathfrak{A}_1, \mathfrak{A}_{38}, \mathfrak{A}_{48}$, and $\mathfrak{A}_{52}$. Further, there exist 26 three-element demi-semi-primal algebras: $\mathfrak{A}_1, ..., \mathfrak{A}_{10}, \mathfrak{A}_{15}, \mathfrak{A}_{16}, \mathfrak{A}_{17}, \mathfrak{A}_{25}, \mathfrak{A}_{38}, ..., \mathfrak{A}_{43}, \mathfrak{A}_{45}, \mathfrak{A}_{48}, \mathfrak{A}_{49}, \mathfrak{A}_{50}, \mathfrak{A}_{52}, \mathfrak{A}_{53}$.

Our final remark concerns *plain para primal* algebras, i.e. algebras which are simple, have no non-trivial subalgebras and generate a congruence permutable variety. CLARK and KRAUSS developed the theory of such algebras in [4]; among others they have shown that if p is a prime then there are (up to cryptoisomorphism) four p-element affine plain para primal algebras, and gave a recipe for determining the non-affine ones (by a theorem of MCKENZIE, they are quasi-primal [17]). It turns out that the non-affine plain para primal algebras are $\mathfrak{A}_1, ..., \mathfrak{A}_4, \mathfrak{A}_{38}, \mathfrak{A}_{39}, \mathfrak{A}_{48}, \mathfrak{A}_{49}$, and $\mathfrak{A}_{52}$. It is also remarkable that all the 13 three-element plain para primal algebras are binary.