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VARIETIES WHOSE ALGEBRAS HAVE NO IDEMPOTENT ELEMENTS

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Let $\mathfrak{A} = \langle A; F\rangle$ be an algebra. The element $a \in A$ is called *idempotent* if $\langle\{a\}; F\rangle$ is a subalgebra of $\mathfrak{A}$. The aim of this note is to characterize the varieties with the property stated in the title by means of a *Mal'cev-type condition* (i.e., in terms of polynomial symbols and identities; see [2]) Lausch and Nöbauer [4] call such varieties *semidegenerate* and ask whether the existence of nullary operations is a necessary condition for semidegeneracy. The following result solves this problem in the negative. (Our notation is adopted from [1].)

THEOREM. *For any variety V, the following assertions are equivalent:*

(1) *None of the algebras in V, consisting of at least two elements, have idempotent elements.*

(2) *No algebra in V has a proper subalgebra whose carrier is a class of some congruences on that algebra.*

(3) *There exist a natural number n and ternary polynomial symbols $\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n$ as well as unary polynomial symbols $\boldsymbol{f}_1, \ldots, \boldsymbol{f}_n, \boldsymbol{g}_1, \ldots, \boldsymbol{g}_n$ such that the following identities hold in V:*

$$(*)\qquad \begin{aligned} &p_1(f_1(\boldsymbol{x}), \boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{x},\\ &p_i(g_i(\boldsymbol{x}), \boldsymbol{x}, \boldsymbol{y}) = p_{i+1}(f_{i+1}(\boldsymbol{x}), \boldsymbol{x}, \boldsymbol{y}) \qquad (i = 1, \ldots, n-1),\\ &p_n(g_n(\boldsymbol{x}), \boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{y}. \end{aligned}$$

Proof. (1) $\to$ (2). Let $\mathfrak{A} \in V$ and let $\mathfrak{B} = \langle B; F\rangle$ be a proper subalgebra of $\mathfrak{A}$. If B is a class of some congruence Θ on $\mathfrak{A}$, then the quotient algebra $\mathfrak{A}/\Theta$ has at least two elements, and B is an idempotent element of $\mathfrak{A}/\Theta$ ($\mathfrak{A}/\Theta \in V$).

(2) $\to$ (3). Denote by $\mathfrak{F}$ the free algebra over V, with the free generating set $\{x, y\}$. Consider the subalgebra $\langle[\{x\}]; F\rangle$ of $\mathfrak{F}$. Let ψ be the smallest congruence relation on $\mathfrak{F}$ under which any two elements of $[\{x\}]$ are congruent. The class of ψ containing $[\{x\}]$ is also a subalgebra of $\mathfrak{F}$.

In virtue of (2), this subalgebra cannot be proper, whence ψ is the complete relation on the carrier of $\mathfrak{F}$. Thus $x \equiv y\ (\psi)$ holds.

Using the explicit description of ψ (see, e.g., [1], Section 10) we infer that there exist a natural number n, a sequence $x = z_0, z_1, \ldots, z_n = y$ and pairs of elements $\langle u_i, v_i \rangle \in [\{x\}]^2$ as well as unary algebraic functions π_i $(i = 1, \ldots, n)$ such that

$$\langle \pi_i(u_i), \pi_i(v_i) \rangle = \langle z_{i-1}, z_i \rangle \quad \text{for } i = 1, \ldots, n.$$

Furthermore, there exist ternary polynomial symbols $\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n$ with $\pi_i(w) = p_i(w, x, y)$ $(w \in \mathfrak{F})$ and unary polynomial symbols $\boldsymbol{f}_i$ and $\boldsymbol{g}_i$ with $u_i = f_i(x)$ and $v_i = g_i(x)$ for $i = 1, \ldots, n$. This means that

$$\begin{aligned} &p_1(f_1(x), x, y) = x, \\ &p_i(g_i(x), x, y) = p_{i+1}(f_{i+1}(x), x, y) \qquad (i = 1, \ldots, n-1), \\ &p_n(g_n(x), x, y) = y \end{aligned}$$

are valid in $\mathfrak{F}$. Since $\mathfrak{F}$ is freely generated by x and y over V, identities $(*)$ hold in V.

(3) → (1). Suppose that the algebra $\mathfrak{A} \in V$ has an idempotent element a. Let b be any element of $\mathfrak{A}$. Substituting $x = a$ and $y = b$ in $(*)$, we get

$$\begin{aligned} a &= p_1(f_1(a), a, b) = p_1(a, a, b) = p_1(g_1(a), a, b) \\ &= p_2(f_2(a), a, b) = p_2(a, a, b) = p_3(a, a, b) = \ldots \\ &= p_n(a, a, b) = p_n(g_n(a), a, b) = b. \end{aligned}$$

Hence the carrier of $\mathfrak{A}$ consists of the single element a, and the proof is completed.

As an easy application, we infer that, in any variety of semigroups, there exist non-trivial semigroups with idempotent elements. (This fact follows also from the well-known description of equationally complete varieties of semigroups, due to Kalicki and Scott; see [3].)

Using our theorem we can also find examples of varieties whose algebras have no idempotent elements. Let us consider, for instance, an arbitrary Boolean algebra and form its reduct by taking the median $(xy \cup xz \cup yz)$ and complement (x') as fundamental operations. This algebra generates a variety with the studied property. Indeed, (3) is satisfied with

$$n = 1, \quad p_1(x, y, z) = xy \cup xz \cup yz, \quad f_1(x) = x \quad \text{and} \quad g_1(x) = x'.$$

More generally, the reduct of any ring with unity 1, determined by the polynomials $p_1(x, y, z) = (x-y)(z-y)+y$ and $g_1(x) = x+1$, also generates such a variety; the validity of $(*)$, with $n = 1$ and $f_1(x) = x$, may be verified immediately.

REFERENCES

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[4] H. Lausch and W. Nöbauer, *Algebra of polynomials*, North Holland, American Elsevier, 1973.

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