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Life is functionally complete

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Life, the popular no-player game invented by J. H. Conway (see [1], Ch. 25), is played on an infinite squared board. At any time t (a non-negative integer), the state of each cell can be 1 (live) or 0 (dead). Let the states of the cells of a solid 3×3 square at time t be

$$\begin{matrix} s_8 & s_1 & s_2 \\ s_7 & s_0 & s_3 \\ s_6 & s_5 & s_4 \end{matrix}$$

($s_i \in \underline{2} = \{0, 1\}$; $i = 0, \ldots, 8$), then the state of the middle cell at time $t + 1$ is 1 if

$$s_0 = 1 \quad \text{and} \quad 2 \le \sum_{i=1}^{8} s_i \le 3, \tag{1}$$

or

$$s_0 = 0 \quad \text{and} \quad \sum_{i=1}^{8} s_i = 3, \tag{2}$$

and it is 0 otherwise.

Observe that the state of the middle cell at $t + 1$ is a Boolean function $f = f(x_0, x_1, \ldots, x_8)$ with the states of cells of the whole square at t as variables; hence we have an algebra $\underline{L} = \langle \underline{2}, f \rangle$, providing a description of Life. *The algebra* $\underline{L}$ *is functionally complete*; indeed, by definition,

$$f(0, 0, 0, x, x, x, y, y, y) = x + y \bmod 2,$$

$$f(0, 0, 0, 0, 0, 0, x, x, y) = xy \bmod 2,$$

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i.e., the basic operations of $GF(2)$ – which is functionally complete – are polynomial operations of $\underline{L}$.

Following C. Bays, the rules of Life can be generalized by postulating $a \le \sum_{i=1}^{8} s_i \le b$ in (1) and $c \le \sum_{i=1}^{8} s_i \le d$ in (2) with $0 < a, b, c, d < 8$ (see [2]; these constraints mirror the principles of "death by exposure or overcrowding", emphasized in [1]). The corresponding Boolean functions f_{abcd} give rise to algebras $\underline{L}_{abcd} = \langle \underline{2}, f_{abcd} \rangle$. Our remark on the functional completeness of $\underline{L}$ extends to all $\underline{L}_{abcd}$. By Post's classical result, we have to show only that f_{abcd} is neither monotonic, nor linear. We have $f_{abcd}(1, \ldots, 1, 0, \ldots, 0) = 1 > f_{abcd}(1, \ldots, 1) = 0$, whenever the number of units on the left side is between $a + 1$ and $b + 1$; hence f_{abcd} is not monotonic. It is not linear, either: if

$$f_{abcd}(x_0, x_1, \ldots, x_8) = t_0x_0 + t_1x_1 + \cdots + t_8x_8 + t \qquad (t_i, t \in \underline{2}), \tag{3}$$

then $t = f_{abcd}(0, \ldots, 0) = 0$, $t_0 = f_{abcd}(1, 0, \ldots, 0) = 0$; further, $t_1 = \cdots = t_8$ by symmetry, and hence $f_{abcd}(x_0, x_1, \ldots, x_8) = x_1 + \cdots + x_8$, as f_{abcd} does not vanish identically. Now, $f_{abcd}(1, 1, 0, \ldots, 0) = f_{abcd}(1, 1, 1, 1, 0, \ldots, 0) = 1$, whence $a = 1$, $b \ge 3$, implying $f_{abcd}(1, 1, 1, 0, \ldots, 0) = 1$; however, $f_{abcd}(1, 1, 1, 0, \ldots, 0) = 0$ by (3), a contradiction. Notice that a further generalization is possible, namely the use of non-trivial threshold conditions of the form $u \le \sum \gamma_i s_i \le v$ (with u, v, γ_i positive real and $0 < u, v < \sum \gamma_i$) instead of (1) and (2), yet providing all functionally complete algebras.

It must be said that our result is exactly what could be expected *a priori*. Indeed, for any n-ary Boolean functions, the proportion of non-monotonic non-linear functions tends to 1 rapidly as n increases and, on the other hand, it seems unlikely that a cellular automaton with quite simple local behavior would have very complex global behavior.

REFERENCES

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