Life is functionally complete

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Life, the popular no-player game invented by J. H. Conway (see [1], Ch. 25), is played on an infinite squared board. At any time \( t \) (a non-negative integer), the state of each cell can be 1 (live) or 0 (dead). Let the states of the cells of a solid \( 3 \times 3 \) square at time \( t \) be

\[
\begin{array}{c}
s_8 \\
s_7 \\
s_6 \\
s_5 \\
s_4 \\
s_3 \\
s_2 \\
s_1 \\
s_0
\end{array}
\]

\((s_i \in \{0, 1\}; \ i = 0, \ldots, 8)\), then the state of the middle cell at time \( t + 1 \) is 1 if

\( s_0 = 1 \) and \( 2 \leq \sum_{i=1}^{8} s_i \leq 3 \),

or

\( s_0 = 0 \) and \( \sum_{i=1}^{8} s_i = 3 \),

and it is 0 otherwise.

Observe that the state of the middle cell at \( t + 1 \) is a Boolean function

\[ f = f(x_0, x_1, \ldots, x_8) \]

with the states of cells of the whole square at \( t \) as variables; hence we have an algebra \( L = (\mathbb{Z}_2, f) \), providing a description of Life. The algebra \( L \) is functionally complete; indeed, by definition,

\[
\begin{aligned}
f(0, 0, 0, x, x, y, y, y) &= x + y \mod 2, \\
f(0, 0, 0, 0, 0, x, x, y) &= xy \mod 2,
\end{aligned}
\]

i.e., the basic operations of \( GF(2) \) — which is functionally complete — are polynomial operations of \( L \).

Following C. Bays, the rules of Life can be generalized by postulating \( a \leq \sum_{i=1}^{8} s_i \leq b \) in (1) and \( c \leq \sum_{i=1}^{8} s_i \leq d \) in (2) with \( 0 < a, b, c, d < 8 \) (see [2]; these constraints mirror the principles of “death by exposure or overcrowding”, emphasized in [1]). The corresponding Boolean functions \( f_{abcd} \) give rise to algebras \( L_{abcd} = (\mathbb{Z}_2, f_{abcd}) \). Our remark on the functional completeness of \( L \) extends to all \( L_{abcd} \). By Post’s classical result, we have to show only that \( f_{abcd} \) is neither monotonic, nor linear. We have \( f_{abcd}(1, \ldots, 1, 0, \ldots, 0) = 1 \geq f_{abcd}(1, \ldots, 1) = 0 \), whenever the number of units on the left side is between \( a + 1 \) and \( b + 1 \); hence \( f_{abcd} \) is not monotonic. It is not linear, either: if

\[
f_{abcd}(x_0, x_1, \ldots, x_8) = t_0 x_0 + t_1 x_1 + \cdots + t_8 x_8 + t \quad (t_t, t \in \mathbb{Z}),
\]

then \( t = f_{abcd}(0, \ldots, 0) = 0, t_0 = f_{abcd}(1, 0, \ldots, 0) = 0 \); further, \( t_t = \cdots = t_8 \) by symmetry, and hence \( f_{abcd}(x_0, x_1, \ldots, x_8) = x_1 + \cdots + x_8 \), as \( f_{abcd} \) does not vanish identically. Now, \( f_{abcd}(1, 1, 0, \ldots, 0) = f_{abcd}(1, 1, 1, 1, 0, \ldots, 0) = 1 \), whence \( a = 1, b \geq 3 \), implying \( f_{abcd}(1, 1, 1, 0, \ldots, 0) = 1 \); however, \( f_{abcd}(1, 1, 1, 0, \ldots, 0) = 0 \) by (3), a contradiction. Notice that a further generalization is possible, namely the use of non-trivial threshold conditions of the form \( u < \sum \gamma_i s_i \leq v \) (with \( u, v, \gamma_i \) positive real and \( 0 < u, v < 1 \)), instead of (1) and (2), yet providing all functionally complete algebras.

It must be said that our result is exactly what could be expected \textit{a priori}. Indeed, for any \( n \)-ary Boolean functions, the proportion of non-monotonic non-linear functions tends to 1 rapidly as \( n \) increases and, on the other hand, it seems unlikely that a cellular automaton with quite simple local behavior would have very complex global behavior.

REFERENCES
