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B. Csákány, József Attila University, Bolyai Institute, Aradi vértanuk tere 1., Szeged, Hungary 6720

A. A. Iskander, Mathematics Department, University of Southwestern Louisiana, Lafayette, Louisiana 70504, USA

Acta Math. Univ. Comenianae
Vol. LXII, 2(1993), pp. 169–178

SOLVABILITY OF EQUATIONS IN VARIETIES OF UNIVERSAL ALGEBRAS

B. CSÁKÁNY AND A. A. ISKANDER

ABSTRACT. We study classes of universal algebras in a variety that are closed under the formation of Cartesian products, homomorphic images and extensions. We prove a Birkhoff type theorem for such axiomatic classes.

Let $\mathcal{V}$ be a variety of algebras of a fixed type τ and let S be a set of first order sentences in the language of τ of the form $(\exists x_1)\dots(\exists x_n)((u_1 = v_1) \wedge \dots \wedge (u_m = v_m))$, where $u_1, v_1, \dots, u_m, v_m$ are terms in the language of type τ. We are interested in the study of classes of algebras of $\mathcal{V}$ in which every member of S is true. Let $\mathcal{M}$ be the class of all models of S belonging to $\mathcal{V}$. It is clear that $\mathcal{M}$ is closed under the formation of Cartesian products, homomorphic images (cf. [9], [12]) and extensions in $\mathcal{V}$; i.e., if $A \in \mathcal{M}$ and A is isomorphic to a subalgebra of some $B \in \mathcal{V}$, then $B \in \mathcal{M}$. We will study a more general case and give the following

Definition 1. Let $\mathcal{V}$ be a variety of universal algebras of type τ and let $\mathcal{K}$ be a subclass of $\mathcal{V}$. We call $\mathcal{K}$ a manifold of $\mathcal{V}$ if $\mathcal{K}$ is not void, and $\mathcal{K}$ is closed under the formation of Cartesian products, homomorphic images and extensions in $\mathcal{V}$.

General references in universal algebra are Cohn [**3**], Grätzer [**7**], Henkin, Monk and Tarski [**8**], Mal'cev [**14**], and McKenzie, McNulty and Taylor [**15**]; also Bell and Slomson [**1**] for models and ultraproducts. Unless stated otherwise, the notations are those of Grätzer [**7**]. The word "algebra" will mean "universal algebra". We will use the same symbol for a term (polynomial) symbol and its value in a given algebra. We will also use the same symbol for an algebra and its carrier set.

Every manifold of $\mathcal{V}$ contains all trivial algebras of $\mathcal{V}$, and consequently contains all members of $\mathcal{V}$ containing a singleton subalgebra. The intersection of any class of manifolds of $\mathcal{V}$ is a manifold of $\mathcal{V}$. Thus the class of all manifolds of $\mathcal{V}$ is a "complete lattice" under inclusion. The least manifold is the class of all algebras in $\mathcal{V}$ containing a singleton subalgebra and the largest manifold is $\mathcal{V}$. Thus we can talk about the manifold of $\mathcal{V}$ generated by a class $\mathcal{L} \subseteq \mathcal{V}$ as the intersection of

Received April 6, 1993.
1980 *Mathematics Subject Classification* (1991 *Revision*). Primary 08B05; Secondary 08C10, 08C15.

all manifolds of $\mathcal{V}$ containing $\mathcal{L}$. As usual, let P, H, O denote the operations of forming Cartesian products, homomorphic images and extensions in $\mathcal{V}$ respectively. Then we have

Theorem 1. *Let $\mathcal{V}$ be a variety algebras of type τ and let $\mathcal{L}$ be a subclass of $\mathcal{V}$. Then the manifold of $\mathcal{V}$ generated by $\mathcal{L}$ is $OHP\mathcal{L}$.*

Proof. It is well known that for any class $\mathcal{K} \subseteq \mathcal{V}$, $PH\mathcal{K} \subseteq HP\mathcal{K}$. So, we need to show that $HO\mathcal{K} \subseteq OH\mathcal{K}$ and $PO\mathcal{K} \subseteq OP\mathcal{K}$. Let $A \in \mathcal{K}$, $B \in \mathcal{V}$ and let A be isomorphic to a subalgebra of B. Suppose $h\colon B \to C$ is a homomorphism, where $C \in \mathcal{V}$. If ι is an injective homomorphism of A into B, then C is an extension of $h(\iota(A))$; i.e., $HO\{A\} \subseteq OH\{A\}$. Let $B \in PO\mathcal{K}$. Then there is a set $\{B_i : i \in I\}$ of $\mathcal{V}$-algebras where for every $i \in I$, B_i contains a subalgebra $A_i \in \mathcal{K}$ and B is the Cartesian product of $\{B_i : i \in I\}$. The Cartesian product of $\{A_i, i \in I\}$ is a subalgebra of B. So, $B \in OP\mathcal{K}$. □

Corollary 1. *Let $\mathcal{V}$ be a variety of algebras of type τ and let $A \in \mathcal{V}$. Then the manifold of $\mathcal{V}$ generated by A is the class of all algebras of $\mathcal{V}$ containing a homomorphic image of A.*

Proof. It is sufficient to show that if $B \in OHP\{A\}$, then B contains a homomorphic image of A. This is true since every Cartesian power of A contains the diagonal which is a subalgebra isomorphic to A. □

The following theorems describe the infimums and supremums in the "complete lattice" of manifolds:

Theorem 2. *Let $\mathcal{V}$ be a variety of algebras of type τ and let $\mathcal{M}_i$, $i \in I$ be a set of manifolds of $\mathcal{V}$. Then $\inf\{\mathcal{M}_i : i \in I\}$ is the class of all homomorphic images of coproducts in $\mathcal{V}$ of $\{A_i : i \in I\}$, where for every $i \in I$, $A_i \in \mathcal{M}_i$.*

Proof. For any family $\{A_i : i \in I\}$, there are homomorphisms of A_i into $\coprod\{A_i : i \in I\}$ — the coproduct in $\mathcal{V}$. Thus if for every $i \in I$, $A_i \in \mathcal{M}_i$, $\coprod\{A_i : i \in I\} \in \mathcal{M}_i$ for every $i \in I$. Let $A \in \mathcal{M}_i$ for every $i \in I$. Then A is a homomorphic image of the coproduct in $\mathcal{V}$ of I copies of A. □

Theorem 3. *Let $\mathcal{V}$ be a variety of algebras of type τ and let $\mathcal{M}_i$, $i \in I$ be a set of manifolds of $\mathcal{V}$. Then $\sup\{\mathcal{M}_i : i \in I\}$ is the class of all extensions in $\mathcal{V}$ of homomorphic images of Cartesian products of $\{A_i : i \in I\}$, where, for every $i \in I$, $A_i \in \mathcal{M}_i$.*

Proof. In view of Theorem 1, this follows from the observation that every Cartesian product of a family of algebras from $\cup\{\mathcal{M}_i : i \in I\}$ is isomorphic to a Cartesian product of $\{A_i : i \in I\}$, where for every $i \in I$, $A_i \in \mathcal{M}_i$ by forming Cartesian products in every $\mathcal{M}_i$ and adding trivial algebras if necessary. □

The existence of homomorphisms between algebras gives a preorder in any variety $\mathcal{V}$ of algebras. That is if $A\rho B$ means that there is a homomorphism of A into B, then ρ is reflexive and transitive. Thus $\sigma = \rho \cap \rho^{-1}$ is an equivalence relation on $\mathcal{V}$ and ρ induces a partial order on the quotient class $\mathcal{V}/\sigma$. The equivalence classes of σ are called the **Mal'cev classes** in $\mathcal{V}$. If $A \in \mathcal{V}$, we will denote by $[A]$ the Mal'cev class of A in $\mathcal{V}$. Thus $[A]$ is the class of all $B \in \mathcal{V}$ such that there are a homomorphism of A into B and a homomorphism of B into A. The partial order on $\mathcal{V}/\sigma$, namely: for $[A]$, $[B] \in \mathcal{V}/\sigma$, $[A] \le [B]$ iff there is a homomorphism of A into B, gives a "complete lattice" structure on $\mathcal{V}/\sigma$, in the sense that, for any set of Mal'cev classes $[A_i] \in \mathcal{V}/\sigma$, $i \in I$, there exist an infimum $[\prod\{A_i : i \in I\}]$ and a supremum $[\coprod\{A_i : i \in I\}]$, where $\prod$ and $\coprod$ denote the Cartesian product and the coproduct in $\mathcal{V}$, respectively. These considerations were made first by W. D. Neumann for the variety of clones in order to study Mal'cev conditions (see [**16**], cf. [**6**]).

The least element of $\mathcal{V}/\sigma$ is the Mal'cev class of any $\mathcal{V}$-free member of positive rank. The largest Mal'cev class in $\mathcal{V}$ is the class of all members of $\mathcal{V}$ containing a singleton subalgebra; i.e., containing a singleton subalgebra. We will need the following definition:

Definition 2. Let $\mathcal{V}$ be a variety of algebras of type τ. A complete filter of $\mathcal{V}/\sigma$ is a non-void subclass $\mathcal{F} \subseteq \mathcal{V}/\sigma$ such that whenever $[A] \in \mathcal{F}$, $[B] \in \mathcal{V}/\sigma$ and $[A] \le [B]$, then $[B] \in \mathcal{F}$ and if $[A_i]$, $i \in I$ is a set of elements of $\mathcal{F}$, then $\inf\{[A_i] : i \in I\} \in \mathcal{F}$.

The following theorem describes the connection between filters of $\mathcal{V}/\sigma$ and manifolds of $\mathcal{V}$:

Theorem 4. *Let $\mathcal{V}$ be a variety of algebras of type τ and let $\mathcal{M}$ be a subclass of $\mathcal{V}$. Then $\mathcal{M}$ is a manifold of $\mathcal{V}$ iff there is a complete filter $\mathcal{F}$ of $\mathcal{V}/\sigma$ such that $\mathcal{M} = \cup\mathcal{F}$.*

Proof. If $\mathcal{M}$ is a manifold of $\mathcal{V}$, then $\mathcal{M}$ is the union of Mal'cev classes as it is closed under the formation of homomorphic images and extensions. Let $\mathcal{F} = \mathcal{M}/\sigma$. Then $\mathcal{M} = \cup\mathcal{F}$. We need to show that $\mathcal{F}$ is a complete filter of $\mathcal{V}/\sigma$. As $\mathcal{M}$ is not void, $\mathcal{F}$ is not empty. If $[A] \in \mathcal{F}$, $[B] \in \mathcal{V}/\sigma$ and $[A] \le [B]$, there is a homomorphism of A into B. As $A \in \mathcal{M}$ and $\mathcal{M}$ is a manifold, $B \in \mathcal{M}$. Thus $[B] \in \mathcal{F}$. If $[A_i] \in \mathcal{F}$, for an index set I, then $\prod\{A_i : i \in I\} \in \mathcal{M}$ as $A_i \in \mathcal{M}$ for every $i \in I$ and so $\inf\{[A_i] : i \in I\} \in \mathcal{F}$. Conversely if $\mathcal{F}$ is a complete filter of $\mathcal{V}/\sigma$ and $\mathcal{M} = \cup\mathcal{F}$, then $\mathcal{M}$ is not void and is closed under the formation of Cartesian products, homomorphic images and extensions in $\mathcal{V}$. □

We will need the following definitions:

Definition 3. Let $\mathcal{V}$ be a variety of algebras of type τ and let $\mathcal{M}$ be a manifold of $\mathcal{V}$. The manifold $\mathcal{M}$ is said to be principal if there is an algebra $A \in \mathcal{M}$ such that $\mathcal{M}$ is generated by $\{A\}$.

Definition 4. Let $\mathcal{V}$ be a variety of algebras of type τ and let $\mathcal{F}$ be a class of Mal'cev classes in $\mathcal{V}$. Then $\mathcal{F}$ is said to be a principal filter if there is $A \in \mathcal{V}$ such that $\mathcal{F}$ is the class of all Mal'cev classes $[B] \in \mathcal{V}/\sigma$ satisfying $[A] \leq [B]$.

It is clear that for every principal filter of $\mathcal{V}/\sigma$ is a complete filter.

The existence of non-principal manifolds is open; nevertheless we formulate some facts concerning principal manifolds.

Theorem 5. *Let $\mathcal{V}$ be a variety of algebras of type τ and let $\mathcal{M}$ be a manifold of $\mathcal{V}$. The following conditions are equivalent:*

1) *The manifold $\mathcal{M}$ is principal.*
2) *There is $A \in \mathcal{M}$ such that $\{[B] : [B] \leq [A], B \in \mathcal{M}\}$ is a set.*
3) *The filter $\mathcal{M}/\sigma$ is principal.*
4) *There is a variety $\mathcal{W}$ of algebras of type $\tau \cup \tau'$ where τ' is a sequence of zeros and $\mathcal{W}$ is defined by a set of identities $W_1 \cup W_2$ where W_1 is a set of identities defining $\mathcal{V}$ and W_2 is a set of identities that do not contain any free variables, such that $\mathcal{M}$ is the class of all τ-reducts of members of $\mathcal{W}$.*

Proof. Let $\mathcal{M}$ be a principal manifold of $\mathcal{V}$. If $\mathcal{M}$ is generated by $\{A\}$, then $B \in \mathcal{M}$, $[B] \leq [A]$ implies $[B] = [A]$. Also, the filter $\mathcal{M}/\sigma$ is principal. Thus 1) implies 2) and 3). Let $\mathcal{M}$ be a manifold of $\mathcal{V}$ and let $A \in \mathcal{M}$ be such that $\{[B] : B \in \mathcal{M}, [B] \leq [A]\}$ is a set, and let $[C] = \inf\{[B] : B \in \mathcal{M}, [B] \leq [A]\}$. Then $[C] \in \mathcal{M}/\sigma$ is the least elements of $\mathcal{M}/\sigma$. Indeed, if $D \in \mathcal{M}$, then $[D] \wedge [A] \in \mathcal{M}/\sigma$. Thus $[D] \wedge [A] \leq [A]$ and so, $[C] \leq [D] \wedge [A] \leq [D]$. Thus $\mathcal{M}/\sigma$ is principal; i.e., 2) implies 3). If $\mathcal{M}/\sigma$ is a principal filter, then there is $A \in \mathcal{M}$ such that $[A] = \inf \mathcal{M}/\sigma$. Hence for every $B \in \mathcal{M}$, $[A] \leq [B]$; i.e., there is a homomorphism of A into B. Thus $\mathcal{M}$ is principal. Hence 3) implies 1). And so, 1), 2) and 3) are equivalent.

Let $A \in \mathcal{V}$ and let $\langle X; R\rangle$ be a presentation of A in $\mathcal{V}$. For every $x \in X$, let a_x be a symbol for a nullary operation. For every term w in the first order language of type τ, let w' be the expression obtained from w by replacing every $x \in X$ by the corresponding a_x. Let W_1 be a defining set if identities for the variety $\mathcal{V}$, and let W_2 be the set of all $u' = v'$ where $u = v \in R$. Let $\mathcal{W}$ be the variety of algebras of type $\tau \cup \tau'$ defined by the set of identities $W_1 \cup W_2$, where τ' is a type for $\{a_x : x \in X\}$. Let $B \in \mathcal{W}$. Then the τ-reduct of B contains a homomorphic image of A and belongs to $\mathcal{V}$. Conversely, let $C \in \mathcal{V}$ and let h be a homomorphism of A into C. Then the algebra C can be expanded to an algebra of $\mathcal{W}$: for every a_x assign the value $h(x) \in C$. Thus, the manifold of $\mathcal{V}$ generated by $\{A\}$ is the class of all algebras that are τ-reducts of algebras from $\mathcal{W}$. Let $\mathcal{W}$ be a variety as in 4) and let D be a free algebra of rank 0 in $\mathcal{W}$ and let A be the τ-reduct of D. Then $A \in \mathcal{V}$ and if $B \in \mathcal{W}$, there is a homomorphism of D into B. Thus, the τ-reduct of B contains a homomorphic image of A. If $C \in \mathcal{V}$ and h is a homomorphism of A into C, then C can be expanded to an algebra of $\mathcal{W}$ as above. Thus the class of all τ-reducts of algebras of $\mathcal{W}$ is a principal manifold of $\mathcal{V}$ generated by $\{A\}$.□

It is interesting to ask how many manifolds are there in a given variety? For instance, the variety of Boolean algebras contains precisely two manifolds: the manifold of all Boolean algebras and the manifold of all trivial Boolean algebras. Any variety in which every algebra has a singleton subalgebra, such as varieties of groups, lattices, monoids, bands, semilattices and rings not necessarily with identity, contains precisely one manifold: the whole variety. Thus the study of manifolds may be of interest in the case of varieties in which there are algebras without singleton subalgebras. These are the varieties in which the free algebra of rank 1 has no singleton subalgebras; e.g., varieties of semigroups in which every free member is infinite, and the semi-degenerate varieties, i.e. those having no singleton subalgebras in non-singleton algebras (see [**11**]; also cf. [**17**], [**4**]), including varieties of rings with identity (as a nullary operation), and varieties of cylindric algebras.

The following theorems give connections between the size of Mal'cev classes and properties of manifolds.

Theorem 6. *Let $\mathcal{V}$ be a variety of algebras of type τ whose Mal'cev classes form a set. Then every manifold of $\mathcal{V}$ is principal.*

Proof. Let $\mathcal{V}/\sigma$ be a set and let $\mathcal{M}$ be a manifold of $\mathcal{V}$. Let $A \in \mathcal{M}$. Then the class $\{[B] : B \in \mathcal{M}, [B] \leq [A]\} \subseteq \mathcal{V}/\sigma$ and hence is a set. By Theorem 5, the manifold $\mathcal{M}$ is principal. □

Theorem 7. *Let $\mathcal{V}$ be a variety of algebras of type τ. Then the following conditions are equivalent:*

1) *The variety $\mathcal{V}$ has only a set of Mal'cev classes.*
2) *The variety $\mathcal{V}$ has only a set of manifolds.*

Proof. In any variety $\mathcal{V}$, $[A] = [B]$ iff the manifold of $\mathcal{V}$ generated by $\{A\}$ coincides with that generated by $\{B\}$. Let $\mathcal{V}$ contain only a set of manifolds. Then $\mathcal{V}/\sigma$ is a set as it is in one-to-one correspondence with the principal manifolds. Conversely, let $\mathcal{V}/\sigma$ be a set. Then, by Theorem 6, every manifold of $\mathcal{V}$ is principal. Thus, the class of manifolds of $\mathcal{V}$ is in one-to-one correspondence with the Mal'cev classes of $\mathcal{V}$. □

There are varieties of algebras whose Mal'cev classes form a proper class. For instance, in the variety of all rings with identity (as a nullary operation), the class of all fields, while in the variety of monadic algebras the special algebras give a proper class of Mal'cev classes. More generally, the Mal'cev classes of a variety $\mathcal{V}$ form a proper class whenever $\mathcal{V}$ contains arbitrarily large simple algebras without singleton subalgebras.

Now we study classes of algebras that are models of sets of sentences of the form $(\exists x_1)\ldots(\exists x_n)((u_1 = v_1) \wedge \cdots \wedge (u_m = v_m))$, where $u_1, v_1, \ldots, u_m, v_m$ are terms in the first order language of type τ.

Theorem 8. *Let $\mathcal{V}$ be a variety of algebras of type τ and let $\mathcal{K}$ be a subclass of $\mathcal{V}$. Then the following conditions are equivalent:*

1) *The class $\mathcal{K}$ is an axiomatic manifold.*

2) *There is a set S of sentences of the form $(\exists x_1)\ldots(\exists x_n)((u_1 = v_1) \wedge \cdots \wedge (u_m = v_m))$ where $u_1, v_1, \ldots, u_m, v_m$ are terms in the first order language of algebras of type τ such that $\mathcal{K}$ is the class of all models of S belonging to $\mathcal{V}$.*

3) *The class $\mathcal{K}$ is a manifold of $\mathcal{V}$ generated by an algebra that is a coproduct of finitely presented algebras in $\mathcal{V}$.*

4) *The class $\mathcal{K}$ is the intersection of a set of manifolds of $\mathcal{V}$ each of which is generated by a finitely presented algebra in $\mathcal{V}$.*

Proof. The proof that 1) implies 2) bears some resemblance to C. C. Chang's proof of Birkhoff's Theorem for varieties [2]. If $\mathcal{K}$ is an axiomatic manifold of $\mathcal{V}$, then $\mathcal{K}$ is defined relative to $\mathcal{V}$ by a set S of first order sentences in the language of τ. As $\mathcal{K}$ is closed under the formation of extensions, we can assume that every sentence in S is existential. As $\mathcal{K}$ is closed under the formation of Cartesian products, we can assume that every sentence in S is existential and Horn type (cf. [**10**], [**13**]). As $\mathcal{K}$ is closed under the formation of homomorphic images, we can assume that every sentence in S is a positive Horn existential sentence [**12**]. Thus 1) implies 2).

Let 2) be true. For every $s \in S$, let X_s be the set of free variables occurring in the kernel of s. We can assume that the family of sets X_s, $s \in S$ are mutually disjoint and let X be their disjoint union. Let R_s be the set of relations $\{u_1 = v_1, \ldots, u_m = v_m\}$ where $((u_1 = v_1) \wedge \cdots \wedge (u_m = v_m))$ is the kernel of s. Let A_s be an algebra of $\mathcal{V}$ whose presentation in $\mathcal{V}$ is $\langle X_s; R_s\rangle$ and let A be an algebra of $\mathcal{V}$ whose presentation in $\mathcal{V}$ is $\langle X; \cup\{R_s : s \in S\}\rangle$. Then $A \in \mathcal{K}$ and A is a coproduct in $\mathcal{V}$ of the algebras A_s, $s \in S$. Thus A is a coproduct of a set of finitely presented algebras in $\mathcal{V}$. Furthermore, let $B \in \mathcal{K}$. As B is a model of S, for every $s \in S$, s is true in B. Thus, there are b_i, $1 \le i \le n$ in B such that $u_j = v_j$, $1 \le j \le m$ for the assignment $x_i = b_i$, $1 \le i \le n$. The mapping $h_s\colon X_s \to B$ defined by $h_s(x_i) = b_i$ can be extended to a homomorphism α_s of A_s into B. The family of mappings α_s, $s \in S$ leads to a homomorphism of A (as a coproduct of A_s, $s \in S$) into B. Thus every $B \in \mathcal{K}$ contains a homomorphic image of A. Conversely, if $B \in \mathcal{K}$ and there is a homomorphism $f\colon A \to B$, then B satisfies S. This shows that $\mathcal{K}$ is the manifold of $\mathcal{V}$ generated by $\{A\}$. Thus 2) implies 3).

Let 3) be true and let $\mathcal{K}$ be the manifold of $\mathcal{V}$ generated by $\{A\}$, where A is a coproduct in $\mathcal{V}$ of a set of algebras B_i, $i \in I$, where for every $i \in I$, the algebra B_i is finitely presented in $\mathcal{V}$. For every $i \in I$, let $\mathcal{M}_i$ be the manifold of $\mathcal{V}$ generated by $\{B_i\}$ and let $\mathcal{M}$ be the intersection of the set of manifolds $\mathcal{M}_i$, $i \in I$. By Theorem 2, $B \in \mathcal{M}$ iff B is a homomorphic image of $\coprod\{C_i : i \in I\}$ for some family of algebras $C_i \in \mathcal{M}_i$, $i \in I$. As the manifold $\mathcal{M}_i$ is generated by $\{B_i\}$, there is a homomorphism of B_i into C_i. As A is a coproduct in $\mathcal{V}$ of B_i, $i \in I$, there is a homomorphism of A into $\coprod\{C_i : i \in I\}$ and consequently, there is a homomorphism of A into B. Thus $\mathcal{M} \subseteq \mathcal{K}$. If $B \in \mathcal{K}$, then there is a homomorphism $h\colon A \to B$. But for every $i \in I$, there is a homomorphism α_i of B_i into A and so there is a homomorphism $h \circ \alpha_i\colon B_i \to B$; i.e., $B \in \mathcal{M}_i$ for every $i \in I$. Thus 3) implies 4).

Let $D \in \mathcal{V}$ be finitely presented in $\mathcal{V}$ and let $\langle x_1, \ldots, x_n; u_1 = v_1, \ldots, u_m = v_m\rangle$ be a finite presentation of D in $\mathcal{V}$. If $C \in \mathcal{V}$, then C satisfies the sentence $(\exists x_1)\ldots(\exists x_n)((u_1 = v_1) \wedge \cdots \wedge (u_m = v_m))$ iff there is a homomorphism of D into C. Thus the manifold of $\mathcal{V}$ generated by $\{D\}$ is axiomatic. Hence 4) implies 1) as the intersection of any family of axiomatic manifolds is an axiomatic manifold. □

In any variety of algebras, every axiomatic manifold is principal and the axiomatic manifolds form a set. Thus, if the variety has a proper class of Mal'cev classes, there are principal manifolds that are not axiomatic. This is certainly the case in the variety of rings with identity and the variety of monadic algebras. Since manifolds are closed under the formation of Cartesian products and homomorphic images, manifolds are closed under the formation of ultraproducts. We will give here an explicit example of a principal manifold that is not axiomatic.

Let $\mathcal{R}$ be the variety of all communicative and associative rings with identity (as a basic nullary operation). Let α be an infinite cardinal and let $\mathcal{L}$ be the class of all rings in $\mathcal{R}$ containing a subring with identity which is a field of cardinality α. Then the manifold $\mathcal{M}$ of $\mathcal{R}$ generated by $\mathcal{L}$ contains no fields of cardinality less than α.

We shall show that all non-trivial members of $\mathcal{M}$ are of cardinality at least α. It will be sufficient to show this for every non-trivial $A \in HP\mathcal{L}$. The ring A is isomorphic to a quotient $\prod\{A_i : i \in I\}/J$, where $A_i \in \mathcal{L}$, for every $i \in I$ and J is a proper ideal of $\prod\{A_i : i \in I\}$. Thus, for every $i \in I$, A_i contains a subfield F_i of cardinality α. Identifying every A_i with the appropriate ideal of $\prod\{A_i : i \in I\}$, we get $J_i = J \cap A_i$ is an ideal of A_i and J contains the direct sum of $\{J_i : i \in I\}$. Either $F_i \cap J_i = \{0\}$ or $F_i \subseteq J_i$, in which case $J_i = A_i$, as F_i contains the identity of A_i. If for some $i \in I$, $J_i \neq A_i$, the quotient A_i/J_i contains an isomorphic copy of F_i and consequently has cardinality at least α. But then $B = \prod\{A_i : i \in I\}/\prod\{J_i : i \in I\} \cong \prod\{A_i/J_i : i \in I\}$ has cardinality at least α. As $J \subseteq \prod\{J_i : i \in I\}$, B is a homomorphic image of A, and A is of cardinality at least α.

Thus, we need to consider the case $A_i = J_i$ for every $i \in I$. In this case, the ring $A \cong \prod\{A_i : i \in I\}/J$ contains the subring $J + \prod\{F_i : i \in I\}/J \cong \prod\{F_i : i \in I\}/(J \cap \prod\{F_i : i \in I\})$. This subring is non-trivial since the identity of $\prod\{F_i : i \in I\}$ does not belong to J. Hence, we need to show that for any family of fields F_i, $i \in I$, each of which is of cardinality at least α, and for any proper ideal J of $\prod\{F_i : i \in I\}$ containing the direct sum of $\{F_i : i \in I\}$, the cardinality of the quotient $\prod\{F_i : i \in I\}/J$ is at least α. It is clear that the set I is infinite. Let $\mathcal{F}$ be the family of all subsets of I defined by: $\lambda \in \mathcal{F}$ iff there is $a \in J$ such that $\lambda = \{i : i \in I, a(i) = 0\}$; i.e., $\lambda \in \mathcal{F}$ iff λ is the complement of the support of some $a \in J$. The set $\mathcal{F}$ is a filter on I. Indeed, if $\lambda \in \mathcal{F}$ and $\lambda \subseteq \mu \subseteq I$, and $a \in J$ is such that $\lambda = \{i : i \in I, a(i) = 0\}$ and t is the element of $\prod\{F_i : i \in I\}$ defined by $t(i) = 0$ if $i \in \mu$ and $t(i) = 1$ if $i \neq \mu$, then $ta \in J$ and μ is the complement of the support of ta and so $\mu \in \mathcal{F}$. For any $a \in J$, let a' be defined by $a'(i) = 0$ if $a(i) = 0$ and $a'(i) = 1$ if $a(i) \neq 0$.. Then $a' \in J$ since $a' = da$ for any $d \in \prod\{F_i : i \in I\}$ such that $d(i) = a(i)^{-1}$ whenever $a(i) \neq 0$. It is clear that a and a' have the same support. Let $a, b \in J$. Then $c = a' + b' + va'b' \in J$ for all $v \in \prod\{F_i : i \in I\}$. Choosing $v(i) = 0$ unless $a'(i) = b'(i) = 1$ and F_i is of characteristic 2, in which case $v(i) = 1$, the complement of the support of c is the intersection of the complements of the supports of a and b.

The filter $\mathcal{F}$ is proper since it does not contain any of the finite subsets of I. Indeed, if $\gamma \in \mathcal{F}$ is finite, then γ is the complement of the support of some $a \in J$ and so J contains $\prod\{F_i : i \in I, i \notin \gamma\} \times \{0\} \times \cdots \times \{0\}$ and so, $\prod\{F_i : i \in I\}/J \cong \prod\{F_i : i \in \gamma\}/K$ where $K = J \cap \prod\{F_i : i \in \gamma\}$ which is in contradiction with the assumption that $\prod\{F_i : i \in I\}/J$ is non-trivial since J contains the direct sum of the F_i, $i \in I$. The filter $\mathcal{F}$ contains the cofinite filter on I. Hence $\mathcal{F}$ is contained in a non-principal ultrafilter $\mathcal{U}$. Hence, the ultraproduct $\prod\{F_i : i \in I\}/\mathcal{U}$ is a homomorphic image of the reduced product $\prod\{F_i : i \in I\}/\mathcal{F} \cong \prod\{F_i : i \in I\}/J$. The cardinality of this ultraproduct is at least α [5]. The manifold just described is principal. The isomorphism classes of fields of cardinality α form a set. Thus the given manifold is generated by the Cartesian product of a set of representatives of the isomorphism classes of fields of cardinality α.

If α is the first non-denumerable cardinal, then an appropriate ultrapower of the field of rational numbers is of cardinality at least α [5]. Hence the given manifold which does not contain the field of rational numbers is not axiomatic. This also provides yet another example of a pseudo-axiomatic class (cf. Theorem 5) that is not axiomatic.

We can also describe elementary manifolds; i.e., manifolds defined by a single first order sentence.

Theorem 9. *Let $\mathcal{V}$ be a variety of algebras of type τ and let $\mathcal{K}$ be a manifold of $\mathcal{V}$. Then the following conditions are equivalent:*

1) *The manifold $\mathcal{K}$ is elementary relative to $\mathcal{V}$.*

2) *There is an existential sentence s of the form $(\exists x_1)\ldots(\exists x_n)((u_1 = v_1) \wedge \cdots \wedge (u_m = v_m))$, where $u_1, v_1, \ldots, u_m, v_m$ are terms in the first order language of type τ such that $\mathcal{K}$ is the class of all algebras in $\mathcal{V}$ satisfying s.*

3) *The manifold $\mathcal{K}$ is generated by a finitely presented algebra relative to $\mathcal{V}$.*

4) *The class of all algebras in $\mathcal{V}$ not belonging to $\mathcal{K}$ together with all trivial algebras is a subquasivariety of $\mathcal{V}$ defined relative to $\mathcal{V}$ by a finite set of first order sentences in the language of τ.*

Proof. The equivalence of 1), 2) and 3) is essentially given in the proof of Theorem 8. If $\mathcal{K}$ is an elementary manifold relative to $\mathcal{V}$, then its complement in $\mathcal{V}$ is also elementary relative to $\mathcal{V}$. If s is the sentence in 2) defining $\mathcal{K}$ relative to $\mathcal{V}$, then the class described in 4) is the class of all algebras in $\mathcal{V}$ satisfying the sentence $(\forall x_1)\ldots(\forall x_n)(\forall x_{n+1})(\forall x_{n+2})(((u_1 = v_1) \wedge \cdots \wedge (u_m = v_m)) \to (x_{n+1} = x_{n+2}))$. This shows that 1) implies 4). If 4) is true and $\mathcal{L}$ is the complement of $\mathcal{K}$ in $\mathcal{V}$, and if $\mathcal{M}$ is the union of $\mathcal{L}$ and the trivial algebras, then $\mathcal{M}$ is defined relative to $\mathcal{V}$ by a finite set of first order sentences. Hence $\mathcal{L}$ is defined relative to $\mathcal{V}$ by a finite set of first order sentences, one of which is $(\exists x_1)(\exists x_2)(\neg(x_1 = x_2))$. Thus $\mathcal{L}$ is elementary relative to $\mathcal{V}$ and so $\mathcal{K}$ is elementary relative to $\mathcal{V}$. That is 4) implies 1). □

Acknowledgment. This research was supported by Hungarian National Foundation for Scientific Research grant No. 1903, and University of Southwestern Louisiana, Lafayette, LA.

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