Inner automorphisms of universal algebras

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In this short note we introduce a notion of inner automorphism for universal algebras, which is a generalization of the corresponding notion referring to groups. This notion preserves some well-known properties of the usual one. Our terminology is essentially that of [1].

The following definition seems to be natural: a mapping is an inner automorphism if and only if it is an automorphism and a translation \(^*\) in the same time. Restricting, however, ourselves to groups, this notion is more general than the usual one; this is shown by the mapping \(x \rightarrow 2x\) of the additive group of rationals. Further, we can have an automorphism, which is a translation, but its inverse is not a translation. Take, e.g., the set of natural numbers and a single unary operation on it, defined by the permutation \(\varphi = (234)\ldots(n^2 + 1, n^2 + 2, \ldots, (n+1)^2)\ldots\). The mapping \(\varphi\) is an automorphism and a translation, its inverse, however, is no translation. These deficiencies can be eliminated by defining inner automorphisms not for a single algebra, but for classes of algebras as follows:

Let \(\mathcal{A}\) be a primitive class ([1], p.114.) of universal algebras, let \(A \in \mathcal{A}\) and let \(\alpha\) be an automorphism of \(A\). Suppose we have in the class \(\mathcal{A}\) a principal derived operation (shortly operation, [1], p.115.) \(\mu\) depending on \(m\) variables with the properties:

I. There exist elements \(a_1, \ldots, a_m \in A\) such that for every \(a \in A\) the equality \(ax = a_{a_1} \ldots a_{a_m}\) holds.

II. In each algebra \(B\) of \(\mathcal{A}\) the mapping \(b \rightarrow bb_1 \ldots b_{a_m} (b \in B)\) is for every choice of \(b_1, \ldots, b_m \in B\) an automorphism of \(B\).

Such an automorphism \(\alpha\) of \(A\) will be called an inner automorphism.

Examples. 1. For groups the introduced notion coincides with the usual notion of inner automorphism. Indeed, let \(G\) be a group, and suppose \(ga = h^{-1}gh\) for every \(g \in G\), \(h\) being a fixed element of \(G\). Then the operation \(xy = y^{-1}xy\) satisfies I, II. On the other hand, let \(\beta\) be an inner automorphism of \(G\) in the new sense.

We must prove the existence of an element \(h \in G\) such that for every \(g \in G\) the equality \(g\beta = h^{-1}gh\) holds. Set \(g\beta = g \mu_1 \ldots \mu_m (a_1, \ldots, a_m \in G)\) with \(\mu\) satisfying II too. We can suppose, without violating generality, that \(x\mu = \mu X_1 X_2 \ldots X_m\),

\[
Y_i = \prod_{j=1}^{m} x_j^{e_{ij}}, \quad (X_1, \ldots, n), \quad Y_j = \prod_{j=1}^{m} y_{ij}^{e_{ij}}, \quad (y_{ij} \in \{1, \ldots, n\}, i = 0, \ldots, n; j = 1, \ldots, t); \quad \text{here} \ e_{ij} \neq 0, \text{ if } 1 \neq i \neq n - 1).
\]

\(^*\) By translation we mean a derived operation with a single variable [1].
Now let \( x, y_1, \ldots, y_m \) denote the distinct free generators of a free group \( F \). Then according to II \( \xi = x_1, y_2, \ldots, y_m \equiv Y_0, Y_1, \ldots, Y_{m-1} (x = e) \) is an automorphism of \( F \). Especially, we have \( x^{-1} = x^{-1}_2, \ldots, y_m = Y_m \), \( Y_1^{-1} Y_0 Y_1 Y_0^{-1} = x^{-1} = Y_2^{-1} Y_1^{-1} Y_2^{-1} \). Hence \( Y_0^{-1} Y_1 Y_0 Y_1 Y_0 = Y_0^{-1} Y_1 Y_0 Y_1 Y_0 = (x^{-1})^{-1} = (x_2^{-1})^{-1} = Y_2^{-1} Y_1^{-1} Y_2^{-1} \). Therefore \( Y_0 Y_1 Y_0 Y_1 Y_0 = (x^{-1})^{-1} = (x_2^{-1})^{-1} = Y_2^{-1} Y_1^{-1} Y_2^{-1} \).

To prove the third assertion of the Theorem it is sufficient to show, that any class of an arbitrary congruence \( \theta \) of \( A \) maps onto an other class of \( \theta \) under \( \alpha \). If \( a = a(\theta) \), then \( a_2, \ldots, a_m = a_2, \ldots, a_m(\theta) \), that is, \( a = a(\theta) \). If, however, \( a = a(\theta) \), then by a similar argumentation it follows, that \( a = a(\theta) \). This completes the proof of the Theorem.

It would be of interest to investigate the question: for which primitive classes is the converse of the third assertion of the Theorem true? To put it otherwise, supposing that a subalgebra \( N \) is invariant under inner automorphisms, under which conditions does it follow that \( N \) is a class of a congruence \( \theta \) and \( \theta \) has no other class which would be a subalgebra? We have this latter case in the three examples above, but this does not hold, e.g., for the primitive class of semigroups, because they have only trivial inner automorphisms.

References


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