Then the algebraic function φ of $\mathfrak{A}$, defined by $x\varphi = f(0, 1, 2, x)$ is a near-transposition of **4**. By Lemma 2, $\mathfrak{A}$ is functionally complete, a contradiction again, showing that $P_4(\mathfrak{A}) = 0$.

It remains to prove $p_3(\mathfrak{A}) \neq 0$. In the opposite case, the polynomial of $\mathfrak{A}$ with minimal number of essential variables turns out to be a near-projection, making use of Lemma 4, whence the functional completeness of $\mathfrak{A}$ follows. Claim 4 is proved, completing the proof of the theorem.

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Homogeneous algebras are functionally complete

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A finite algebra $\mathfrak{A}$ with base set A is called *functionally complete* if every (finitary) operation on A is an algebraic function of $\mathfrak{A}$. A well-known theorem of H. Werner asserts that any algebra having the ternary discriminator function as an algebraic function is functionally complete [17]. Recently, E. Fried and A. F. Pixley proved [2] that an algebra $\mathfrak{A}$ with $3 \leq |A| < \infty$ is functionally complete provided the dual discriminator function d, defined by

$$d(x, y, z) = \begin{cases} x, & \text{if } x = y, \\ z & \text{otherwise,} \end{cases}$$

is an algebraic function of $\mathfrak{A}$.

The above functions are pattern functions in the sense of R. W. Quackenbush ([11], p. 74). Two n-tuples $\langle a_1, \ldots, a_n \rangle$, $\langle b_1, \ldots, b_n \rangle$ formed from arbitrary elements are said to be of the same pattern if, for $1 \leq i, j \leq n$, $a_i = a_j$ implies $b_i = b_j$ and vice versa. An n-ary function f is called a *pattern function* if, for any possible choice of $a_1, \ldots, a_n$,

$$f(a_1, \ldots, a_n) = a_i (1 \leq i \leq n) \tag{$*$}$$

where i depends upon $\langle a_1, \ldots, a_n \rangle$ in such a manner that $(*)$ implies $f(b_1, \ldots, b_n) = b_i$ (with the same i) whenever $\langle b_1, \ldots, b_n \rangle$ is of the same pattern as $\langle a_1, \ldots, a_n \rangle$. Obviously, projections are pattern functions; we call a function (in particular, a pattern function) *non-trivial* if it is not a projection (and an algebra is non-trivial if it has a non-trivial operation). Another example of non-trivial pattern function is the normal transform, introduced by M. I. Gould and G. Grätzer for studying Boolean extensions [4].

Following E. Marczewski [8], we call a function $f : A^n \to A$ *homogeneous* if every permutation of A is an automorphism of the algebra $\langle A; f \rangle$. Marczewski

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showed that a function $f: A^n \to A$ is homogeneous iff, for every n-tuple $\langle a_1, \dots, a_n\rangle \in A^n$, either $(*)$ or

$$f(a_1, \dots, a_n) = a(\notin \{a_1, \dots, a_n\};\ A = \{a_1, \dots, a_n, a\}) \qquad (**)$$

holds in such a way that $(*)$ implies $f(b_1, \dots, b_n) = b_i$ (with the same i) and $(**)$ implies $f(b_1, \dots, b_n) = b$ $(\notin \{b_1, \dots, b_n\};\ A = \{b_1, \dots, b_n, b\})$ whenever $\langle a_1, \dots, a_n\rangle$ and $\langle b_1, \dots, b_n\rangle$ are of the same pattern. In particular, pattern functions are exactly those homogeneous functions whose value is given by $(*)$ for every n-tuple $\langle a_1, \dots, a_n\rangle$. In what follows, we shall use this description of homogeneous functions without further reference. An algebra $\mathfrak{A}$ is called *homogeneous*, if every permutation of the base set of $\mathfrak{A}$ is an automorphism of $\mathfrak{A}$. Clearly, $\mathfrak{A}$ is homogeneous, iff every operation of $\mathfrak{A}$ is homogeneous, and this means that every polynomial of $\mathfrak{A}$ is homogeneous.

The aim of this article is to prove that the functional completeness results mentioned before are corollaries of a more general fact; namely, we have the following

THEOREM. *All but six non-trivial homogeneous finite algebras are functionally complete. The exceptional algebras are equivalent to the following ones:*

(1) $\langle \mathbf{2}; n\rangle = \langle\{0, 1\}; n\rangle$ *with* $n(x) \equiv x + 1 \pmod 2$;
(2) $\langle \mathbf{2}; s\rangle$ *with* $s(x, y, z) \equiv x + y + z \pmod 2$;
(3) $\langle \mathbf{2}; \bar{s}\rangle$ *with* $\bar{s}(x, y, z) \equiv x + y + z + 1 \pmod 2$;
(4) $\langle \mathbf{2}; d\rangle$ *with* $d(x, y, z) \equiv xy + xz + yz \pmod 2$;
(5) $\langle \mathbf{3}; \circ\rangle$ *with* $x \circ y \equiv 2x + 2y \pmod 3$;
(6) $\langle \mathbf{4}; f_0\rangle$ *where* f_0 *is the ternary homogeneous function defined by* $f_0(1, 2, 3) = f_0(0, 1, 1) = f_0(1, 0, 1) = f_0(1, 1, 0) = f(0, 0, 0) = 0$.

Here n is the negation, s is the minority function, or switching function [18], d is the majority function, or dual discriminator function [2]. Further, (5) is the three-element idempotent, commutative, non-associative groupoid in [5]; finally, (6) is called the four-element Świerczkowski algebra (see [15], [3]); it is the direct square of (2).

COROLLARY 1. *Let $\mathfrak{A}$ be an algebra with finite base set A such that there exists a non-trivial homogeneous function f on A which is an algebraic function of $\mathfrak{A}$. Then $\mathfrak{A}$ is functionally complete provided $\langle A; f\rangle$ is not equivalent to any of the algebras* (1)–(6).

The following fact was first proved by B. Ganter, J. Płonka, and H. Werner [3]:

COROLLARY 2. *All but one non-trivial homogeneous algebras are simple. The exceptional algebra is the four-element Świerczkowski algebra.*

As functionally complete algebras are simple, for finite algebras this is immediate from our theorem. In order to settle the infinite case, it is enough to remark that finitely generated homogeneous algebras are finite, and locally simple algebras are simple.

Let us recall the basic notions we shall use in the sequel. A set of operations on a set A is a *clone* if it contains all the projections, and it is closed under superposition. The *clone generated by a set S* of operations is the minimal clone containing S. The algebraic functions of an algebra $\mathfrak{A}$ with base set A form the clone generated by the set of operations of $\mathfrak{A}$ and constant functions on A. For operations f, g on A, we shall say that f *produces* g if g is contained in the clone generated by f. A finite algebra $\mathfrak{A}$ is called *primal* if every possible operation on A is a polynomial of $\mathfrak{A}$ or, equivalently, the set of operations of $\mathfrak{A}$ generates the clone of all operations on A. Thus, an algebra $\langle A; F\rangle$ is functionally complete iff, for the set F' of all algebraic functions on A, the algebra $\langle A; F'\rangle$ is primal. A function $f: A^n \to A$ is *essential* if it essentially depends on at least two variables and takes on all values from A. A set S of unary operations on A (i.e. transformations of A) is *basic* if, for any essential function f, the algebra $\langle A; S \cup \{f\}\rangle$ is primal.

The proof of the theorem is based on the following lemmas which can be easily derived from corresponding primality criteria, due to J. Słupecki [14], S. V. Jablonskiĭ [6], and A. Salomaa [13] (see also [12]).

LEMMA 1. *A non-trivial homogeneous finite algebra $\mathfrak{A}$ whose base set A has at least three elements is functionally complete if there exist a transposition as well as a transformation of defect* 1 *of A* (i.e., a transformation whose range consists of $|A| - 1$ elements) *that are algebraic functions of $\mathfrak{A}$.*

Proof. Observe that a non-trivial homogeneous operation on A is essential provided $|A| \geq 3$. Further the assumptions of the lemma imply that all the transformations of A are algebraic functions of $\mathfrak{A}$. Indeed, let (ab) be that transposition of A which is assumed to be algebraic, and (cd) another transposition of A. Take a permutation φ of A satisfying $a\varphi = c$, $b\varphi = d$, and let f be an n-ary polynomial of $\mathfrak{A}$ such that for suitable $a_2, \dots, a_n \in A$, $f(x, a_2, \dots, a_n) = (ab)$. Then φ is an automorphism of A, whence $(cd) = f(x, a_2\varphi, \dots, a_n\varphi)$ is also an algebraic function of $\mathfrak{A}$. However, the set of all transpositions with a transformation of defect 1 added generates the full transformation semigroup of A (see [1], p. 7).

Slupecki's theorem asserts that, for $|A|\geq 3$, the full transformation semigroup of A is basic. Hence A provided with all algebraic operations of $\mathfrak{A}$ is primal, i.e. $\mathfrak{A}$ is functionally complete, as needed.

Call a transformation ψ of A a *near-transposition* if there exist different $a, b, c, d \in A$ such that $a\psi = b$, $b\psi = a$, $c\psi = d$, and $x\psi = x$ if $x \in A$ and $x \neq a, b, c$.

LEMMA 2. *A non-trivial homogeneous algebra $\mathfrak{A}$ with four-element base set A is functionally complete if there exists a near-transposition of A which is an algebraic function of $\mathfrak{A}$.*

Proof. We can proceed as in the proof of the preceding lemma observing that any near-transposition is an algebraic function of $\mathfrak{A}$. It is easy to verify that the set of all near-transpositions generates the semigroup of all non-onto transformations of A. Jablonskiĭ's theorem says that, for $|A|\geq 3$, the semigroup of all non-onto transformations of A is basic. This proves the lemma.

LEMMA 3. *A non-trivial homogeneous finite algebra $\mathfrak{A}$ with at least five-element base set A is functionally complete if there exists a transposition of A which is an algebraic function of $\mathfrak{A}$.*

Proof. We can copy the proof of Lemma 1, applying Salomaa's theorem here: for $|A|\geq 5$, the alternating group of A is basic.

Let $f(x_1, \ldots, x_n)$ be an arbitrary function on a finite set A. Denote by f_{ij} the function arising from f by identification of the variables x_i and x_j. We write $f_{ij} = g$ to indicate that f coincides with g on the set of all n-tuples from A whose i-th and j-th entries coincide. The following technical lemma is essentially due to S. Świerczkowski [15]:

LEMMA 4. *Let f be an at least quaternary operation on A such that, for every distinct i, $j(\leq n)$, there exists a $k = k(i, j)$ with $f_{ij} = e_k^n$. Then k is always the same number*, i.e. *it does not depend upon i and j.*

The main preparatory step to prove the theorem is the following lemma:

LEMMA 5. *If $\mathfrak{A} = \langle A; f\rangle$ is a finite algebra with $3 \leq |A|$ where f is a non-trivial pattern function then $\mathfrak{A}$ is functionally complete.*

Proof. First we remark that if $\langle A; g\rangle$ is functionally complete and f produces g then $\langle A; f\rangle$ is functionally complete as well. Thus, in order to prove the theorem, we have to find a set S of pattern functions on A with the following properties:

1. For each pattern function f on A there exists a $g \in S$ such that f produces g.
2. For every $g \in S$, the algebra $\langle A; g\rangle$ is functionally complete.

Let T be the set of all essentially ternary pattern functions on A. Further, call the n-ary pattern function l_n on A *near-trivial* if $l_n(x_1, \ldots, x_n) = x_1$ whenever $x_1, \ldots, x_n$ are all different and $l_n(x_1, \ldots, x_n) = x_n$ otherwise ([8], p. 83). Under a *near-projection* we shall mean any function which can be obtained from a near-trivial function by a permutation of variables. Let N be the set of all near-trivial functions on A. We shall prove that $S = T \cup N$ has the properties 1 and 2.

As for Property 1, it suffices to prove that, if g is a non-trivial pattern function on A having the minimal number of essential variables among all non-trivial pattern functions it produces, then g is essentially ternary or it is near-trivial. Notice that g is essentially at least ternary. Hence it is enough to show that if g is essentially at least quaternary and it turns into a projection by identifying any two of its variables, then g is a near-projection. This follows readily from Lemma 4.

In order to prove that S enjoys Property 2, first we show that for any non-trivial ternary pattern function g the algebra $\langle A; g\rangle$ is functionally complete. We can establish without difficulty that there exist 24 distinct ternary pattern functions on A; it is straightforward to show that the following six functions form a maximal subset of them relative to the property that they cannot be obtained from each other by permutation of variables: Pixley's ternary discriminator function p; the dual discriminator function d; the switching function s (see [18], p. 9), defined by

$$s(x, y, z) = \begin{cases} y, & \text{if } x = z, \\ z, & \text{if } x = y, \\ x & \text{otherwise;} \end{cases}$$

the function e, defined by the requirement that $e(x, y, z) = x$ if and only if x, y, z are all distinct or no two of them are distinct; the ternary near-trivial function l_3; and the ternary first projection e_1^3.

All of these functions are produced by p ([3], second theorem). Further, $e(y, e(x, z, y), x) = p(x, y, z)$ and $s(x, s(y, x, z), y) = l_3(x, y, z)$. Hence our claim will be proven if we show that $\langle A; d\rangle$ and $\langle A; l_3\rangle$ are functionally complete. For $\langle A; d\rangle$, this is the result of Fried and Pixley, quoted in the introduction and for $\langle A; l_3\rangle$, this will be done right now.

Secondly, we have to prove that for any non-trivial near-trivial function l_n the algebra $\langle A; l_n\rangle$ is functionally complete. By Lemma 1, the proof will be complete if we show that there exist a transposition as well as a transformation of defect 1 of A which are algebraic functions of $\langle A; l_n\rangle$.

Let $A=\{1,\dots,m\}(m\geq 3)$; then $n\leq m$, because l_n is non-trivial. We use the following notation: for $M\subset A$ and $i, k\in A$, $f^k_{M\to i}(x)$ means the transformation of A defined by

$$f^k_{M\to i}(x)=\begin{cases} i, & \text{if } x\in M,\\ k & \text{otherwise.}\end{cases}$$

If $M=\{i_1,\dots,i_t\}$, we also write $f^k_{i_1,\dots,i_t\to i}$ instead of $f^k_{M\to i}$. We shall construct the transposition (12) from functions of form $f^k_{M\to i}(x)$, and then show that the latter ones are algebraic functions of $(A; l_n)$.

For $n=3$, we have

$$(12)=l_n(x, f^3_{1,2,3\to 1}(x), f^1_{1,3\to 2}(x)).$$

Now, let $n>3$; then

$$(12)=l_n(x, f^3_{3\to 4}(x), f^4_{3,4\to 5}(x),\dots, f^{n-1}_{3,4,\dots,n-1\to n}(x),$$
$$f^n_{1,2,3,\dots,n\to 1}(x), f^1_{1,3,4,\dots,n\to 2}(x)).$$

Thus, it is sufficient to show that, for any meaningful value of k, $F_k(x)=f^2_{2,3,\dots,k\to 1}(x)$ and $G_k(x)=f^n_{1,2,\dots,k\to 1}(x)$ are algebraic functions of $\langle A; l_n\rangle$. This can be done by induction on k. As a preparation, we show that $f^3_{1\to 2}$ is algebraic. This requires a separate induction on m. If $m=n$, we have $f^3_{1\to 2}=l_n(2, x, 4, 5,\dots, n, 3)$. Let $m>n$ and let $h(x)$ be an algebraic function which equals to $f^3_{1\to 2}(x)$ on $\{1,\dots,m-1\}$. Then $l_n(h(x), x, m-n+4, m-n+5,\dots, m, 3)=f^3_{1\to 2}$ on $\{1,\dots,m\}$. Analogous induction starting from the case $m=n$ shows that $f^2_{2\to 1}(x)$ $(=l_n(l_n(1, x, 4, 5,\dots, n, 3), 3, 4,\dots, n, 2)$ if $m=n)$ and $f^n_{1\to 1}(x)$ $(=l_n(x, 2, 3,\dots, n)$ if $m=n)$ are algebraic functions of $\langle A; l_n\rangle$.

Hence, in particular, we have that $F_2(x)$ and $G_1(x)$ are algebraic. Finally, the induction steps we need are the following:

$$F_k(x)=l_n(F_{k-1}(x), 3, 4,\dots, n-1, f^n_{k\to 2}(x), f^2_{k\to 1}(x)),$$
$$G_k(x)=l_n(G_{k-1}(x), 3, 4,\dots, n-1, f^2_{k\to n}(x), f^n_{k\to 1}(x)).$$

A slight modification of the function representing (12), namely, the replacement of $f^n_{1,2,\dots,n\to 1}(x)$ by $f^n_{1,3,4,n\to 1}(x)$ in the next to last argument, gives a

transformation of defect 1, which moves 1 into 2, and leaves the further elements invariant. Thus S has Property 2, completing the proof of Lemma 5.

Now we are ready to prove our theorem which can be decomposed into the following four claims.

CLAIM 1. A non-trivial homogeneous finite algebra is functionally complete provided its base set consists of at least five elements.

CLAIM 2. A two-element non-trivial homogeneous algebra is not functionally complete iff it is equivalent to one of algebras (1)–(4).

CLAIM 3. A three-element non-trivial homogeneous algebra is not functionally complete iff it is equivalent to the algebra (5).

CLAIM 4. A four-element non-trivial homogeneous algebra is not functionally complete iff it is equivalent to the algebra (6).

Proof of Claim 1. It suffices to prove that if $|A|\geq 5$ and f is a non-trivial homogeneous operation on A, then $\langle A; f\rangle$ is functionally complete. If f is a pattern function, we can apply Lemma 5. If not, by a suitable identification of variables of f, if necessary, we get a homogeneous function f' with $n(=|A|-1)$ distinct variables $x_1,\dots,x_n$, whose value is given by $(**)$ on any n-tuple $\langle a_1,\dots,a_n\rangle$ consisting of pairwise different entries. By the assumption, f' is at least quaternary. If it has two variables whose identification turns f' into a non-trivial f'', then f'' is a pattern function and Lemma 5 applies again. Otherwise f' fulfils the conditions of Lemma 4, whence it turns into the same $(n-1)$-ary projection – say, into the first one – by identifying any two of its variables. Put, as usual, $A=\{0, 1,\dots,n\}$. Now, the transposition (01) is an algebraic function of $\langle A; f\rangle$, namely, $(01)=f'(x, 2,\dots,n)$. Hence, by Lemma 3, $\langle A; f\rangle$ is functionally complete.

Proof of Claim 2. An algebra on the set $\mathbf{2}=\{0, 1\}$ is determined up to equivalence by the clone of its polynomials that are Boolean functions; the algebra is homogeneous iff its clone consists of self-dual functions only. Considering the diagram of the lattice of all two-element algebras, due to E. Post (see, e.g. [7]), we check that $\langle\mathbf{2}; R_4\rangle$, $\langle\mathbf{2}; L_4\rangle$, $\langle\mathbf{2}; L_5\rangle$, and $\langle\mathbf{2}; D_2\rangle$ (in notations of Post) are exactly those two-element algebras which are not functionally complete, and their clones can be generated by the functions $x+1$, $x+y+z$, $x+y+z+1$, $xy+xz+yz$ (mod 2), respectively.

Proof of Claim 3. Let $\mathfrak{A}=\langle \mathbf{3}; F\rangle$ be non-trivial, homogeneous, and functionally incomplete. Let f be a polynomial of $\mathfrak{A}$ having the minimal number of essential variables among all polynomials of $\mathfrak{A}$. If f is essentially at least quaternary, then Lemma 4 shows that f is a projection, in contrary to the hypothesis. Thus, f is essentially at most ternary.

As f cannot be a pattern function, we can suppose that the value of, e.g., $f_{12}(0,1)$ is given by $(**)$, i.e., $f(0,0,1)=2$. Then, the transposition (01) is an algebraic function of $\mathfrak{A}$, namely, $(01)=f(x,x,2)$. Now, no unary algebraic function of $\mathfrak{A}$ can have a two-element image set (i.e., no transformation of defect 1 of $\mathbf{3}$ can be an algebraic function of $\mathfrak{A}$), or else $\mathfrak{A}$ is functionally complete by Lemma 1.

Suppose $f(0,1,1)=2$; then $f(0,2,1)=2$, as the image set of $f(0,x,1)$ cannot have exactly two elements. Now examining $f(x,0,1)$ we obtain $f(1,0,1)=1$. It is an easy computation that $f(x,y,z)\equiv 2x+2z \pmod 3$ for any possible $x, y, x\in\mathbf{3}$. Suppose $f(0,1,1)=1$; we get $f(x,y,z)\equiv 2y+2z \pmod 3$ analogously. Finally, $f(0,1,1)=0$ implies in the same way that $f(x,y,z)\equiv x+y+2z\equiv 2(2x+2y)+2z$ (mod 3).

Anyway, we have a polynomial f of $\mathfrak{A}$, satisfying identically $f(x,y,x)=f(y,x,x)=y$. By the classical theorem of A. I. Malcev, $\langle \mathbf{3}; F\rangle$ generates a congruence permutable variety. Furthermore, $\langle \mathbf{3}; F\rangle$ is simple, and it has trivial subalgebras only. By a result of R. McKenzie ([9], Theorem 4), these three conditions jointly imply that either $\langle \mathbf{3}; F\rangle$ is quasi-primal, or the operations in F are linear functions in a vector space over a prime field. As quasi-primal algebras are functionally complete, we have the second possibility. Thus, for any $g=g(x_1,\dots,x_n)\in F$, $g(x_1,\dots,x_n)\equiv a_1x_1+\cdots+a_nx_n+b \pmod 3$. Here $b=0$, since g is idempotent, i.e., the operations in F are (idempotent) vector space polynomials over GF(3). However, by a result of J. Płonka [10], any two non-trivial idempotent polynomials of a vector space over a prime field generate each other, whence $\langle \mathbf{3}; F\rangle$ is equivalent to $\langle \mathbf{3}; \circ\rangle$, as asserted.

Proof of Claim 4. Let $p_k(\mathfrak{A})$ denote the number of the essentially k-ary polynomials of the algebra $\mathfrak{A}$. In [16], K. Urbanik proved that a finite idempotent algebra $\mathfrak{A}$ with $p_2(\mathfrak{A})=p_4(\mathfrak{A})=0$ and $p_3(\mathfrak{A})\neq 0$ is equivalent to a direct power $\mathfrak{P}$ of the algebra (2), or it can be obtained from $\mathfrak{P}$ by introducing new, essentially at least m-ary ($m\geq 5$) operations f, satisfying the equation $f(x_1,\dots,x_m,\dots,x_k)=x_1$ whenever the elements $x_1,\dots,x_k$ belong to a subalgebra of $\mathfrak{P}$ generated by less than m elements. Hence it follows that a four-element idempotent algebra $\mathfrak{A}$ with $p_2(\mathfrak{A})=p_4(\mathfrak{A})=0\neq p_3(\mathfrak{A})$ is equivalent to the four-element Świerczkowski algebra.

Let $\mathfrak{A}$ be a four-element, non-trivial, homogeneous, functionally incomplete algebra. Obviously, $\mathfrak{A}$ is idempotent, and $p_2(\mathfrak{A})=0$ (as a non-trivial homogeneous essentially binary function can be defined on a three-element set only). Thus, in order to prove Claim 4, it suffices to verify that $p_3(\mathfrak{A})\neq 0$ and $p_4(\mathfrak{A})=0$.

Let $A=\mathbf{4}$, and let f be a non-trivial, essentially ternary, homogeneous function such that $\langle A; f\rangle$ is not functionally complete. We start with establishing that $f=f_0$ (the function that appears in the definition of the four-element Świerczkowski algebra). We can suppose $f(0,1,2)=3$, otherwise f is a pattern function and Lemma 5 leads to a contradiction. Furthermore, suppose $f(0,0,1)=f(1,0,0)=0$. We see that $(01)=f(2,x,3)$ is an algebraic function of $\langle A; f\rangle$. We have $f(2,0,2)=2$, for if not, the algebraic function $f(x,0,2)$ is a transformation of defect 1 of A, and then $\langle A; f\rangle$ is functionally complete by Lemma 1. Using, however, $f(2,0,2)=2$ we get

$$f^2_{0,1\to 0}(x)=f(f(x,0,3),\ f(x,0,2),\ f(x,1,2))$$

whence the function $f(f^2_{1,3\to 1}(x), f^0_{1,2\to 2}(x), f^3_{0,1\to 0}(x))$ is algebraic and at the same time it is a transformation of defect 1 of A. By Lemma 1, $f(0,0,1)=f(1,0,0)=0$ impossible.

Now suppose $f(0,0,1)=f(1,0,0)=1$, the unique essentially distinct possibility. The desired identity $f=f_0$ will be proved if we shall have verified $f(0,1,0)=1$. Assume $f(0,1,0)=0$; then $(01)=f(f(x,2,3), f(0,x,1), f(x,3,2))$ and $f(2,f(x,2,3),3)$ is a transformation of defect 1 of A, in contrary to Lemma 1.

Next we prove $p_4(\mathfrak{A})=0$. Let f be an essentially quaternary polynomial of $\mathfrak{A}$. By identifying two variables of f we obtain a ternary polynomial f_{ij} which is either essentially ternary, or – as it is homogeneous – trivial. If f_{ij} is essentially ternary, it coincides with f_0, as we have shown just before. Denote by P the set of all two-element subsets $\{i,j\}$ of $\{1,2,3,4\}$ such that the identification of the i-th and j-th variables of f furnishes a projection. If $\{1,2\}\notin P$ then $f(x,x,y,y)=x$ for $x\neq y$, whence $\{3,4\}\in P$. Thus, $|P|\geq 3$. An equation $f(x,x,x,y)=x$ implies $\{1,2\}$, $\{1,3\}$, $\{2,3\}\in P$. Assume $\{1,2\}$, $\{3,4\}\in P$. As $f(x,x,y,y)$ equals x or y the projections f_{12} and f_{34} are the same, say e^4_1. Now $f(x,x,x,y)=f(x,x,y,x)=f(x,y,x,x)=x$ follows. Thus, it is shown that f turns into a projection by identifying each two of its variables, whence, by Lemma 4, f is either a near-projection or a projection. In the first case, $\mathfrak{A}$ is functionally complete, and in the second one, f is not essentially quaternary. The contradiction we obtained shows that $|P|=3$ holds. We shall distinguish two essentially different possibilities:

1. $P=\{\{1,4\},\ \{2,4\},\ \{3,4\}\}$. Now $f(x,x,y,x)=y$, implying $f_{24}=e^4_3$, and $f(x,y,y,y)=x$, implying $f_{24}=e^4_1$, a contradiction.

2. $P=\{\{2,3\},\ \{2,4\},\ \{3,4\}\}$. This gives $f_{23}=e^4_4$, $f_{24}=e^4_3$, $f_{34}=e^4_2$. If $f(0,1,2,3)=0$, then $f(w,x,y,z)=f_0(x,y,z)$ can be verified by a straightforward computation, i.e., f is not essentially quaternary. Thus, we can assume $f(0,1,2,3)=3$, as the second, third and fourth variables behave analogously now.