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## Finite homogeneous algebras. I

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**1. Preliminaries.** Following MARCZEWSKI [7], an operation  $f: A^k \rightarrow A$  is called *homogeneous* if  $h(f(x_1, \dots, x_k)) = f(h(x_1), \dots, h(x_k))$  for every permutation  $h$  and any elements  $x_1, \dots, x_k$  of  $A$ . An algebra  $\langle A; F \rangle$  is said to be *homogeneous* if each operation  $f \in F$  is homogeneous.

In this paper, we shall describe all finite homogeneous algebras up to equivalence. This is the same as determining all clones of homogeneous operations on finite sets. In the present Part I we shall

(1) list all minimal clones consisting of homogeneous operations (it turns out that this list contains at most three items on any finite set, and the dual discriminator function  $d$ , introduced by E. Fried and A. F. Pixley, always generates such a minimal clone);

(2) determine all clones of homogeneous operations containing the minimal clone generated by the dual discriminator.

Let us start with notions and notations. The symbol  $\mathbf{n}$  means the set  $\{0, 1, \dots, n-1\}$ . For the sake of simplicity, we shall consider algebras of the form  $\langle \mathbf{n}; F \rangle$  only. The following description of homogeneous operations was given by MARCZEWSKI [7]: for a homogeneous  $k$ -ary operation  $f$  on  $\mathbf{n}$ ,  $f(a_1, \dots, a_k) = a_i$  where  $1 \leq i \leq k$ ; or, possibly,  $f(a_1, \dots, a_k) = a_{k+1}$  if  $a_{k+1}$  is the unique element of  $\mathbf{n}$  distinct from  $a_1, \dots, a_k$ , in such a way that the index of the value of  $f(a_1, \dots, a_k)$  depends upon the pattern of equalities in the sequence  $\langle a_1, \dots, a_k \rangle$  only. A homogeneous operation  $f$  is called a *pattern function* provided  $f(a_1, \dots, a_k)$  always belongs to  $\{a_1, \dots, a_k\}$ .

Several kinds of homogeneous operations will play an important role in the sequel: Pixley's ternary discriminator  $p$ , the dual discriminator  $d$ , the switching function  $s$ , the  $k$ -ary near-projection  $l_k$  where  $k \geq 3$  (they are defined on any set); further, the  $(n-1)$ -ary operation  $r_n$ , defined on  $\mathbf{n}$  for  $n \geq 2$ , and Świerczkowski's ternary function  $f_0$ , defined on 4. Let us recall their definitions:

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$$\begin{aligned}
p(a, b, c) &= c \text{ if } a = b, \text{ and } p(a, b, c) = a \text{ otherwise;} \\
d(a, b, c) &= a \text{ if } a = b, \text{ and } d(a, b, c) = c \text{ otherwise;} \\
s(a, b, c) &= c \text{ if } a = b, \text{ } s(a, b, c) = b \text{ if } a = c \text{ and } s(a, b, c) = a \\
&\text{otherwise;} \\
l_k(a_1, \dots, a_k) &= a_1 \text{ if } a_1, \dots, a_k \text{ are pairwise distinct and } l_k(a_1, \dots, a_k) = a_k \\
&\text{otherwise;} \\
r_n(a_1, \dots, a_{n-1}) &= a_n \text{ if } \{a_1, \dots, a_{n-1}, a_n\} = \mathbf{n} \text{ and } r_n(a_1, \dots, a_{n-1}) = a_1 \\
&\text{otherwise;}
\end{aligned}$$

finally,

$$f_0(1, 2, 3) = f_0(0, 1, 1) = f_0(1, 0, 1) = f_0(1, 1, 0) = f_0(0, 0, 0) = 0$$

(see [8], [7], [9], [3], [2]).

A set of operations on a set  $\mathbf{n}$  is called a *clone* if it contains all trivial operations (i.e., all projections) and it is closed under superposition. For any set  $F$  of operations on  $\mathbf{n}$ , we say that  $F$  produces the operation  $g$  and we use the symbol  $F \rightarrow g$  if  $g$  can be obtained from operations in  $F$  and the projections by superposition (in this case, one can also say that  $g$  is a term function of the algebra  $\langle \mathbf{n}; F \rangle$ ). In the case  $F = \{f\}$  we write  $f \rightarrow g$ . Obviously, the relation  $\rightarrow$  is transitive. For the negation of  $F \rightarrow g$  we write  $F \not\rightarrow g$ . An algebra  $\langle \mathbf{n}; F \rangle$  is *functionally complete* if the set  $F \cup \{0, 1, \dots, n-1\}$  (i.e.,  $F$  together with the constant nullary operations) produces each possible operation on  $\mathbf{n}$ . The clone  $[F]$  generated by  $F$  is the set of all operations  $F$  produces. We write  $[f_1, f_2, \dots]$  instead of  $[\{f_1, f_2, \dots\}]$ . The algebras  $\langle \mathbf{n}; F \rangle$  and  $\langle \mathbf{n}; G \rangle$  are said to be *equivalent* if  $[F] = [G]$ . A clone  $T$  is called *minimal* if the clone of all projections is the unique one which is contained in  $T$  properly; this means that  $T$  contains a non-projection, and any non-projection in  $T$  produces every other non-projection.

In the next lemma we collect the basic facts about how the above-mentioned homogeneous operations produce each other:

Lemma 1. On a finite set  $\mathbf{n}$ , the following hold:

- (1)  $p \rightarrow f$  for any pattern function  $f$ .
- (2)  $l_j \rightarrow l_k$  for  $j \leq k$ .
- (3)  $r_n \rightarrow l_{n-1}$  for  $n > 3$ .
- (4)  $l_k \rightarrow d$  for  $n > 1$ .
- (5)  $d \rightarrow l_k$  for  $n > 2$ ,  $n \geq k$ .
- (6)  $l_j \rightarrow l_k$  for  $j > k$ ,  $n \geq k$ .

Proof. (1) is a result in [4].

- (2). It is sufficient to establish  $l_j \rightarrow l_{j+1}$ , and this is given by the identity  $l_{j+1}(x_1, \dots, x_j, x_{j+1}) = l_j(l_j(x_1, x_2, \dots, x_j, x_{j+1}), l_j(x_2, x_3, \dots, x_j, x_{j+1}), x_4, \dots, x_{j+1})$ .
- (3).  $l_{n-1}(x_1, \dots, x_{n-1}) = r_n(x_{n-1}, \dots, x_3, x_2, r_n(x_{n-1}, \dots, x_2, x_1))$ .

To prove (4)–(6), we use the following fact. Let  $f, g$  be operations on  $\mathbf{n}$  and  $f \rightarrow g$ ; then, for any natural number  $t$ , the subalgebras of  $\langle \mathbf{n}; f \rangle^t$  are closed under the (componentwise performed) operation  $g$ .

(4). Observe that  $\sigma = \{\langle 1, 0, 0 \rangle, \langle 0, 1, 0 \rangle, \langle 0, 0, 1 \rangle\}$  is a subalgebra of  $\langle \mathbf{n}; l_k \rangle^3$  but  $d(\langle 1, 0, 0 \rangle, \langle 0, 1, 0 \rangle, \langle 0, 0, 1 \rangle) = \langle 0, 0, 0 \rangle \notin \sigma$ . Hence  $l_k \rightarrow d$  is impossible.

Concerning (5) and (6), we present the crucial subalgebras only:

$$(5) \quad \{\langle k-1, 0 \rangle, \dots, \langle 2, 0 \rangle, \langle 1, 0 \rangle, \langle 0, 0 \rangle, \langle 0, 1 \rangle\} \subset \langle \mathbf{n}; d \rangle^2,$$

$$(6) \quad \{\langle j-2, 0 \rangle, \dots, \langle 2, 0 \rangle, \langle 1, 0 \rangle, \langle 0, 0 \rangle, \langle 0, 1 \rangle\} \subset \langle \mathbf{n}; l_j \rangle^2.$$

**2. Minimal clones of homogeneous operations.** In this section, our main tool is the following fact:

Lemma 2. For  $n \geq 3$ , every non-trivial pattern function on  $\mathbf{n}$  produces  $d$  or some  $l_k$  with  $k \leq n$ .

Proof. It was proved in [2] (see the proof of Lemma 5 there) that any non-trivial pattern function on  $\mathbf{n}$  produces  $d$  or an  $l_k$  which is non-trivial; but  $l_k$  is trivial if  $k > n$ .

The clones in the title of this paragraph are given by

Theorem 1. The minimal clones consisting of homogeneous operations on a finite set  $\mathbf{n}$  ( $n > 1$ ) are the following:

- $[l_n]$  and  $[d]$ , if  $n \geq 5$ ;
- $[l_4]$ ,  $[d]$  and  $[f_0]$ , if  $n = 4$ ;
- $[l_3]$ ,  $[d]$  and  $[r_3]$ , if  $n = 3$ ;
- $[s]$ ,  $[d]$  and  $[r_2]$ , if  $n = 2$ .

Proof. First we prove that, for  $n \geq 3$ ,  $[l_n]$  is minimal on  $\mathbf{n}$ . Take a non-trivial  $f$  with  $l_n \rightarrow f$ ; it is sufficient to show  $f \rightarrow l_n$ . As pattern functions produce pattern functions only, by Lemma 2 we have  $f \rightarrow d$  or  $f \rightarrow l_k$  for a suitable  $k \leq n$ . From  $f \rightarrow d$  it follows  $l_n \rightarrow d$ , contradicting Lemma 1(4); therefore  $f \rightarrow l_k$  holds. Now  $k < n$  is impossible by Lemma 1(6), i.e.,  $f \rightarrow l_n$ , which was needed.

For  $n \geq 3$ , the minimality of  $[d]$  can be proved by an analogous argument; here we have to apply Lemma 1(5) instead of (4).

For  $n \geq 5$ , there is no other minimal clone of operations on  $\mathbf{n}$ . In order to show this, we shall verify that each non-trivial homogeneous operation  $g$  on  $\mathbf{n}$  produces  $l_n$  or  $d$ . There are two possibilities:

a)  $g \rightarrow r_n$ . Then, by Lemma 1(3) and (2), we have  $g \rightarrow l_n$ .

b)  $g \rightarrow r_n$ . If, in addition,  $g$  is a pattern function, then Lemma 2 applies in the above manner. If  $g$  is not a pattern function, then we can identify variables of  $g$  (if necessary) so that we obtain an  $(n-1)$ -ary  $g'$  satisfying  $g'(a_1, \dots, a_{n-1}) = a_n$ ,

whenever  $\{a_1, \dots, a_{n-1}, a_n\} = \mathbf{n}$ , i.e.,  $a_n$  is the unique element of  $\mathbf{n}$  distinct from  $a_1, \dots, a_{n-1}$ . Now, if there exist two variables of  $g'$  whose identification furnishes a non-trivial pattern function, then, applying Lemma 2 for  $g'$  again, our claim follows. Suppose that  $g'$  turns into a projection by identifying any two of its variables. By a result of Świerczkowski,  $g'$  always turns into the same projection ([8]; see also [5], pp. 206–207; note that  $g'$  is at least quaternary). Hence  $g'$  equals  $r_n$  up to permutation of variables, implying  $g \rightarrow r_n$ , contrary to the hypothesis.

Next we prove that  $[f_0]$  is minimal on 4. Let  $f_0 \rightarrow f$  and suppose  $f \neq f_0$ . Then  $\langle 4; f_0 \rangle$  and  $\langle 4; f \rangle$  are not equivalent. A homogeneous non-trivial algebra  $\langle 4; F \rangle$  is not functionally complete iff it is equivalent to  $\langle 4; f_0 \rangle$  (see [2]); therefore,  $\langle 4; f \rangle$  is functionally complete. Now,  $\langle 4; f_0 \rangle$  is functionally complete a fortiori, a contradiction.

Similarly, a non-trivial homogeneous functionally incomplete algebra  $\langle 3; F \rangle$  is equivalent to  $\langle 3; r_3 \rangle$  (see [2]), hence the minimality of  $[r_3]$  on 3 follows.

Furthermore, every non-trivial homogeneous operation  $g$  on 4 produces one of  $l_4$ ,  $d$  and  $f_0$ , showing that there are no other minimal clones of homogeneous operations on 4. Indeed, if  $g$  is a pattern function, Lemma 2 applies. If  $g$  fails to be a pattern function, then an appropriate identification of variables of  $g$  leads to a ternary  $g'$  satisfying  $g'(a_1, a_2, a_3) = a_4$ , whenever  $\{a_1, \dots, a_4\} = 4$ . As we have  $g'(a_1, a_2, a_3) = a_i$  ( $1 \leq i \leq 3$ ) if  $\text{card}\{a_1, a_2, a_3\} < 3$ , and the pattern of equalities in  $\langle a_1, a_2, a_3 \rangle$  determines the value of  $i$ , the operation  $g'$  is defined uniquely by the sequence  $\langle g'(0, 1, 1), g'(1, 0, 1), g'(1, 1, 0) \rangle$  (of course,  $g'(0, 0, 0) = 0$  always). Let us denote  $g'$  by  $f_k$  ( $k = 0, 1, \dots, 7$ ) if this sequence is the dyadic form of  $k$  (i.e.,  $4g'(0, 1, 1) + 2g'(1, 0, 1) + g'(1, 1, 0) = k$ ). This notation is consistent with the original definition of  $f_0$ . We have to verify that every  $f_k$  produces one of  $l_4$ ,  $d$  and  $f_0$ .

One can check the following identities:

- (a)  $f_3(x, y, z) = r_4(x, y, z)$ ;
- (b)  $f_5(y, x, z) = f_6(z, y, x) = f_3(x, y, z)$ ;
- (c)  $f_1(y, z, f_1(z, y, x)) = f_4(y, f_4(z, x, y), z) = p(x, y, z)$ ;
- (d)  $f_2(y, f_2(y, z, x), x) = f_7(y, f_7(y, z, x), x) = d(x, y, z)$ .

From (a) and Lemma 1(3) and (2), it follows  $f_3 \rightarrow l_4$ . From (b),  $f_5 \rightarrow l_4$  and  $f_6 \rightarrow l_4$ . Further, (c) together with Lemma 1(1) implies  $f_1 \rightarrow d$  and  $f_4 \rightarrow d$ ; finally, (d) shows  $f_2 \rightarrow d$  and  $f_7 \rightarrow d$ . The case  $n=4$  is settled.

In the case  $n=3$  we can proceed similarly. Any non-trivial homogeneous function  $g$  on 3 is either a pattern function — then we use Lemma 2 — or not. In the latter case  $g$  produces a binary  $g'$  in the usual way such that  $g'(a_1, a_2) = a_3$  whenever  $\{a_1, a_2, a_3\} = 3$ , and  $g'(a, a) = a$ . Clearly,  $g' = r_3$ , hence  $g \rightarrow r_3$ , as required.

All minimal clones we have found are distinct. This is implied by Lemma 1(4) and the fact that pattern functions produce merely pattern functions.

The case  $n=2$  of Theorem 1 can be realized by casting a glance at the diagram of the lattice of all clones on 2, due to POST (see, e.g., [6]; note that  $r_2(x) = x+1 \pmod 2$  and  $d(x, y, z) = xy+xz+yz \pmod 2$  on 2).

**3. Homogeneous dual discriminator algebras.** After WERNER [9], an algebra  $\langle \mathbf{n}; F \rangle$  is said to be a discriminator algebra (or quasi-primal algebra) if  $p \in [F]$ . Analogously, an algebra  $\langle \mathbf{n}; F \rangle$  will be called a *dual discriminator algebra* if  $d \in [F]$ . In this paragraph we determine all homogeneous dual discriminator algebras up to equivalence, i.e., for any  $\mathbf{n}$ , we determine all clones of homogeneous operations on  $\mathbf{n}$  containing  $d$ . From now on,  $n$  is fixed and  $n \geq 3$ .

Call a ternary operation  $m$  on  $\mathbf{n}$  a *majority operation* if, for any  $x, y \in \mathbf{n}$ ,  $m(x, x, y) = m(x, y, x) = m(y, x, x) = x$  holds. The dual discriminator is a majority operation. The following theorem of BAKER and PIXLEY [1; Corollary 5.1] is basic for our considerations (see also [9]):

Let  $\langle \mathbf{n}; F \rangle$  be a finite algebra such that  $F$  produces a majority operation and let  $g$  be an arbitrary operation on  $\mathbf{n}$ . If every subalgebra of  $\langle \mathbf{n}; F \rangle^2$  is closed under the (componentwise performed) operation  $g$ , then  $F$  produces  $g$ .

For a clone  $T$  on  $\mathbf{n}$ , let  $ST$  stand for the set consisting of base sets of all subalgebras of  $\langle \mathbf{n}; T \rangle^2$ . Let  $\mathcal{F}$  be the set of all clones on the set  $\mathbf{n}$  containing  $d$ . We call a set  $P$  of subsets of  $\mathbf{n}^2$  *complete* if there exists a clone  $T \in \mathcal{F}$  such that  $P = ST$  (i.e., if there exists a dual discriminator algebra on  $\mathbf{n}$  such that  $P$  is the set of all subalgebras of the direct square of this algebra). Denote by  $\mathcal{S}$  the set of all complete sets.

**Lemma 3.**  $S$  is an inclusion-reversing one-to-one mapping of  $\mathcal{F}$  onto  $\mathcal{S}$ .

*Proof.* The unique non-trivial part of this assertion is that  $S$  is one-to-one. Suppose  $T_1, T_2 \in \mathcal{F}$  and  $ST_1 = ST_2$ . If  $f \in T_2$  then every set in  $ST_1 (= ST_2)$  is closed under  $f$ , hence, by the Baker—Pixley theorem,  $T_1 \rightarrow f$  follows. This means  $f \in T_1$  as  $T_1$  is a clone. Therefore,  $T_2 \subseteq T_1$  (and by symmetry,  $T_1 \subseteq T_2$ ). We get  $T_1 = T_2$ , which was needed.

By virtue of Lemma 3, we can investigate complete sets instead of clones. First we establish some properties of complete sets. Subsets of  $\mathbf{n}^2$  may be considered as binary relations on  $\mathbf{n}$ . The following lemma is familiar:

**Lemma 4.** Any complete set contains the complete relation; furthermore, it is closed under relation product, intersection and forming the inverse relation.

For convenience, several kinds of subsets of  $\mathbf{n}^2$  will bear special names. A set of form  $K \times L$  with  $K, L \subseteq \mathbf{n}$ ,  $\text{card } K = k$ ,  $\text{card } L = l$  is a *block of size*  $(k, l)$ . A set of form  $\{\langle i_1, j_1 \rangle, \dots, \langle i_k, j_k \rangle\}$ , where  $i_1, \dots, i_k$  are pairwise distinct as well as  $j_1, \dots, j_k$ , is a *string of size*  $k$ . A set of form  $\{\langle i_k, j_1 \rangle, \dots, \langle i_2, j_1 \rangle, \langle i_1, j_1 \rangle, \langle i_1, j_2 \rangle, \dots, \langle i_1, j_l \rangle\}$

$(k, l \geq 2)$  is called a *cross of size*  $(k, l)$ . Essentially, a string of size  $k$  is a partial permutation with a  $k$ -element domain and a cross of size  $(k, l)$  is the union of two blocks of size  $(k, 1)$  and  $(1, l)$  with a non-empty intersection. *Block of size*  $m$  means a block of size  $(m, l)$  or  $(k, m)$ ; similarly for crosses.

Lemma 5. *Any complete set consists of blocks, strings and crosses; in particular,  $S[d]$  consists of all blocks, strings and crosses.*

Proof. A complete set consists of subsets of  $\mathbf{n}^2$  preserved by  $d$ , and, by result of FRIED and PIXLEY [3; Theorem 2.4],  $d$  preserves a subset  $\sigma$  of  $\mathbf{n}^2$  iff  $\sigma$  is *p-rectangular*, i.e.,

$$\langle i, j_1 \rangle, \langle i, j_2 \rangle, \langle k, l \rangle \in \sigma \text{ implies } \langle i, l \rangle \in \sigma \text{ for } j_1 \neq j_2$$

and

$$\langle i_1, j \rangle, \langle i_2, j \rangle, \langle k, l \rangle \in \sigma \text{ implies } \langle k, j \rangle \in \sigma \text{ for } i_1 \neq i_2.$$

Clearly, blocks, strings and crosses are *p-rectangular* and the converse can also be checked without trouble.

From now on, we shall use the following notations:  $B$  is the set of all blocks and  $B'$  is the set of all blocks of size  $(k, l)$  with  $k, l \neq n-1$ . The set of strings and crosses  $S, S'$  and  $C, C'$ , resp., are defined analogously. Finally, let  $C_m$  be the set of all crosses of size  $(k, l)$  with  $k, l \leq m$ . Now Lemma 5 can be reformulated as follows:

For any complete set  $P$ , the inclusion  $P \subseteq BUSUC$  holds; in particular,  $S[d] = BUSUC$ .

Next we clear up the structure of several further complete sets:

Lemma 6. (1)  $S[d, l_{m+1}] = BUSUC_m$  for  $m=2, \dots, n-1$ .

(2)  $S[p] = BUS$ .

(3)  $S[d, l_{m+1}, r_n] = B'US'UC_m$  for  $m=2, \dots, n-2$ .

(4)  $S[p, r_n] = B'US'$ .

Proof. (1) The following inclusions are obvious:  $BUSUC_m \subseteq S[d, l_{m+1}] \subseteq S[d] = BUSUC$ . Take a set from  $C \setminus C_m$ , i.e., a cross of form  $\{\langle i_k, j_1 \rangle, \dots, \langle i_1, j_1 \rangle, \dots, \langle i_1, j_l \rangle\}$  with  $k > m$  (the case  $l > m$  can be settled similarly). Then  $l_{m+1}(\langle i_{m+1}, j_1 \rangle, \dots, \langle i_2, j_1 \rangle, \langle i_1, j_2 \rangle) = \langle i_{m+1}, j_2 \rangle$  showing that our cross is not closed under  $l_{m+1}$ . Thus, the set of all subalgebras of  $\langle \mathbf{n}; d, l_{m+1} \rangle^2$  is  $BUSUC_m$ , as asserted.

(2)–(4) can be verified in an analogous manner observing that no cross is closed under  $p$ , because we have  $p(\langle i_2, j_1 \rangle, \langle i_1, j_1 \rangle, \langle i_1, j_2 \rangle) = \langle i_2, j_2 \rangle$ ; furthermore, no block, string and cross, each of size  $n-1$ , is closed under  $r_n$ . Indeed, take, e.g., a block  $\{i_1, \dots, i_{n-1}\} \times L$  of size  $n-1$  and a  $j \in L$ ; then  $\langle i_1, j \rangle, \dots, \langle i_{n-1}, j \rangle$  belong to this block but  $r_n(\langle i_1, j \rangle, \dots, \langle i_{n-1}, j \rangle)$  does not.

Lemma 7. *For the clone  $H$  of all homogeneous operations on  $\mathbf{n}$ ,  $SH = B'US'$ .*

Proof. By (4) of the previous lemma,  $SH \subseteq B'US'$ . On the other hand,  $SH$  contains all permutations of  $\mathbf{n}$ , i.e. all strings of size  $n$ , since for any operation  $f$  homogeneity means that each permutation is a subalgebra of  $\langle \mathbf{n}; f \rangle^2$ . Now we can apply Lemma 4 in order to obtain all sets in  $B'US'$ . Namely, every string of size less than  $n-1$  is the intersection of two permutations, every block of size  $(k, n)$  is the (relation) product of a string of size  $k$  and the complete relation, every block of size  $(n, l)$  is the inverse of a block of size  $(l, n)$ , and every block of size  $(k, l)$  is the intersection of blocks of size  $(k, n)$  and  $(n, l)$ .

In view of Lemmas 5 and 7, our task is reduced to determining all complete sets between  $B'US'$  and  $BUSUC$ .

Lemma 8. *All complete sets containing  $B'US'$  and contained in  $BUSUC$  are those listed in Lemma 6.*

Proof. It is sufficient to prove the following two propositions:

(a) If a complete set contains  $B'US'$  and a block, or a string, or a cross, any of them of size  $n-1$ , then it contains  $BUS$ .

(b) If a complete set contains  $B'US'$  and a cross of size  $m$ , then it contains  $C_m$ ; moreover, if  $m \geq n-1$ , it contains even  $BUS$ .

Indeed, suppose (a) and (b) are fulfilled, and let  $P$  be a complete set with  $B'US' \subseteq P \subseteq BUSUC$ . If  $P$  contains no crosses, then (a) implies  $P = B'US'$  or  $P = BUS$ . Otherwise, let  $m$  be the maximum of the sizes of crosses in  $P$ . If there is a block or a string of size  $n-1$  in  $P$ , then in virtue of (a), (b) and the maximality of  $m$  we have  $P = BUSUC_m$ . In the opposite case,  $P = B'US'UC_m$  by the same reason.

It remains to prove (a) and (b). As for (a), one can check easily that all blocks and strings of size  $n-1$  can be obtained from sets in  $B'US'$  and an arbitrary *fixed* block or string or cross, any of them of size  $n-1$ , by product, intersection and formation of inverse relation. Applying Lemma 4, the assertion (a) follows.

(b) First let  $R$  be a complete set containing  $B'US'$  and an arbitrary cross  $\zeta$  of size  $(m, l)$  where  $2 \leq l < m \leq n-1$ . Then any cross of the same size  $(m, l)$  can be obtained in the form  $\pi_1 \zeta \pi_2$  with appropriate strings  $\pi_1, \pi_2$  of size  $n$ ; crosses of size  $(l, m)$  arise as inverses of the previous ones; crosses of size  $(m, m)$  can be represented as  $\zeta_1 \pi \zeta_2$  where  $\zeta_1$  and  $\zeta_2$  are crosses of size  $(m, l)$  and  $(l, m)$ , respectively, and  $\pi$  is a string of size  $n$ ; finally, an arbitrary cross of size  $(k_1, k_2)$  with  $k_1, k_2 \leq m$  is the intersection of a cross of size  $(m, m)$  and an appropriate block of size  $(k_1, k_2)$ . Thus,  $C_m \subseteq R$ , as required. In the case  $m = n-1$ , the second part of (b) is a consequence of (a).

Secondly, let  $R$  be complete with  $R \supseteq B'US'$  and let  $R$  contain a cross of

size  $n$ . The preceding considerations show that we have two possibilities only, namely,  $R=B'US'UC'$  or  $R=BUSUC$ . The proof will be complete if we deduce that  $B'US'UC'$  is not a complete set. Assume  $SF=B'US'UC'$  for some homogeneous dual discriminator algebra  $\langle n; F \rangle$ . As  $SF$  is closed under  $r_n$ , we have  $F \rightarrow r_n$  by the Baker—Pixley theorem, hence, according to Lemma 1(3) and (2),  $F \rightarrow l_n$  follows. However, as we have seen in the proof of Lemma 6(1), our cross of size  $n$  is not closed under  $l_n$ , a contradiction.

Now we are ready to formulate the main result of this paragraph.

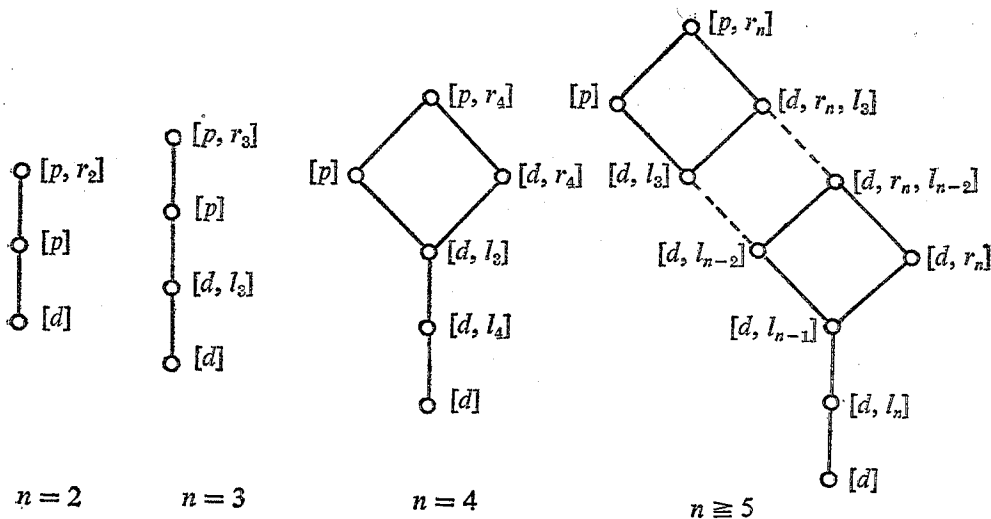
**Theorem 2.** *The finite homogeneous dual discriminator algebras with more than one element are the following (up to equivalence):*

- $\langle 2; d \rangle, \langle 2; p \rangle, \langle 2; p, r_2 \rangle;$
- $\langle 3; d \rangle, \langle 3; p \rangle, \langle 3; p, r_3 \rangle, \langle 3; d, l_3 \rangle;$
- $\langle 4; d \rangle, \langle 4; p \rangle, \langle 4; p, r_4 \rangle, \langle 4; d, l_3 \rangle, \langle 4; d, l_4 \rangle, \langle 4; d, r_4 \rangle$

and for  $n \geq 5$

- $\langle n; d \rangle, \langle n; p \rangle, \langle n; p, r_n \rangle, \langle n; d, l_k \rangle (k = 3, \dots, n),$
- $\langle n; d, r_n \rangle, \langle n; d, r_n, l_k \rangle (k = 3, \dots, n-2).$

The interval of clones between  $[d]$  and  $H=[p, r_n]$  on  $n$  is the lattice with the diagram presented below:



**Proof.** For  $n > 2$ , this follows immediately from Lemmas 6, 7 and 8. The case  $n=2$  can be found in Post's work ([6], pp. 72—76).

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