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CONGRUENCES AND SUBALGEBRAS

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Varieties whose algebras satisfy several conditions on congruences and/or subalgebras were studied by many authors. E. g., such conditions are: two congruences coincide provided they have a congruence class in common [3]; every two subalgebras have an element in common ([5], Theorem 5.6.); every subalgebra is a class of some congruence [4]. In what follows we shall consider the simplest such conditions and we characterize the varieties whose algebras fulfil them. Among others, we obtain some new Mal'cev-type theorems.

The basic terminology we use is that of [1]. Note, however, that we shall write "term" instead of "polynomial symbol". Furthermore, we say that a term f is essentially n-ary ($n \geq 0$) on the variety $\mathcal{V}$, if the polynomial $\mathfrak{f}$ on the countably generated free $\mathcal{V}$-algebra depends on (exactly) n variables. "Operation" means basic operation, and "trivial operation" (or "trivial term") means projection. "Translation" is the same as "unary algebraic function". The set of all translations of the algebra $\mathfrak{A}$ will be denoted by $T(\mathfrak{A})$, and $P_n(\mathfrak{A})$ designates the set of all (formally) n-ary polynomials of $\mathfrak{A}$.

Consider a non-empty set M. We say that a subset S and an equivalence ϑ of M are *connected* if S is a class of ϑ. Let M be equipped with an algebraic structure; then the subsets and equivalences compatible with this structure (i.e., the subalgebras and congruences) will be referred to shortly as *compatible* subsets and equivalences. Observe that the above conditions are, in fact, requirements on connectedness, compatibility and equality of subsets and equivalences of the considered algebras (e. g., the first of them can be formulated as "if a subset is connected with two equivalences, which are compatible, then they are equal". It is not hard to construct a two-sorted first order logic with equality suitable to express all such requirements.) Now we introduce a series of very simple conditions (including also the above examples) based on the notions just introduced. For this purpose, let us agree in the following notations:

1. α denotes anyone of the words "subset" and "equivalence". If α means "subset" in a given context, then $\bar{\alpha}$ means "equivalence" and vice versa.

2. $\mathbf{Q}$ denotes anyone of the expressions "at most one", "at least one" and "every".

The conditions we shall treat of are of the following form:

($*$) *For any (compatible)* α, $\mathbf{Q}$ $\bar{\alpha}$ *connected with it is compatible.*

Performing all the meaningful substitutions for α and $\mathbf{Q}$, we get six distinct conditions if the word "compatible" in parentheses is neglected, and six further ones if it is considered effective. We shall describe those varieties whose members satisfy one particular (but arbitrary) condition. Denote the conditions of form ($*$) by (1)–(12) according to the following table:

α \ Q	at most one	at least one	every
subset	(1)	(3)	(5)
equivalence	(2)	(4)	(6)

The condition $(i+6)$ for $i = 1, \ldots, 6$ arises from (i) by the requirement that α is compatible.

THEOREM 1. *All algebras of the variety* $\mathcal{V}$ *satisfy* (1) *iff there exists a natural number n such that for some 5-ary, resp. ternary terms* $\mathbf{p}_1, \ldots, \mathbf{p}_n$ and $\mathbf{q}_1, \ldots, \mathbf{q}_n$

$$\mathbf{q}_1(\mathbf{x}, \mathbf{x}, \mathbf{y}) = \ldots = \mathbf{q}_n(\mathbf{x}, \mathbf{x}, \mathbf{y}) = \mathbf{y}$$

$$\mathbf{x} = \mathbf{p}_1(\mathbf{q}_1(\mathbf{x}, \mathbf{y}, \mathbf{z}), \mathbf{z}, \mathbf{x}, \mathbf{y}, \mathbf{z})$$

$$\mathbf{p}_i(\mathbf{z}, \mathbf{q}_i(\mathbf{x}, \mathbf{y}, \mathbf{z}), \mathbf{x}, \mathbf{y}, \mathbf{z}) = \mathbf{p}_{i+1}(\mathbf{q}_{i+1}(\mathbf{x}, \mathbf{y}, \mathbf{z}), \mathbf{z}, \mathbf{x}, \mathbf{y}, \mathbf{z}) \qquad (i = 1, \ldots, n-1)$$

$$\mathbf{p}_n(\mathbf{z}, \mathbf{q}_n(\mathbf{x}, \mathbf{y}, \mathbf{z}), \mathbf{x}, \mathbf{y}, \mathbf{z}) = \mathbf{y}$$

hold identically in $\mathcal{V}$.

PROOF. As (1) means regularity [3] it is enough to notice that the identities in the theorem are equivalent to the corresponding ones in [3], and can be obtained by using Fichtner's symmetrization technique [2], which will be illustrated in the course of the proof of Theorem 6.

THEOREM 2. *All algebras of the variety* $\mathcal{V}$ *satisfy* (2) *iff there exists an at most unary term* $\mathbf{h}$ *which is essentially nullary in* $\mathcal{V}$.

PROOF. (2) means that no disjoint subsets can be subalgebras simultaneously, i.e., every two subalgebras (of any algebra in $\mathcal{V}$) have an element in common. Hence, as Taylor proved ([5], Therorem 5.6), the existence of the required $\mathbf{h}$ follows. The converse is obvious.

Note that (1) and (2) (in other words, regularity and Taylor's condition on subalgebras) are "dual" in the sense that they can be obtained from each other by replacing "subset" and "equivalence" by each other.

THEOREM 3. *For any variety* $\mathcal{V}$, *the following four conditions are equivalent:*

I. *All algebras of* $\mathcal{V}$ *satisfy* (3).
II. *All algebras of* $\mathcal{V}$ *satisfy* (5).
III. *All algebras of* $\mathcal{V}$ *satisfy* (11).
IV. *Each non-trivial operation in* $\mathcal{V}$ *is nullary.*

PROOF. Observe that IV→II→III and IV→I are trivial. Hence we have to prove I→IV and III→IV only.

I→IV. Let $\mathbf{f}$ be an n-ary ($n \geq 1$) term which depends essentially on its first variable in $\mathcal{V}$. Denote by $\mathfrak{F}_n$ the $\mathcal{V}$-free algebra freely generated by $x_0, x_1, \ldots, x_n$. Then $f(x_0, x_2, \ldots, x_n) \neq f(x_1, x_2, \ldots, x_n)$. Further, (3) implies that any subset of $\mathfrak{F}_n$ is a class of some congruence on $\mathfrak{F}_n$, whence so does also the subset $\{x_0, x_1, f(x_0, x_2, \ldots, x_n)\}$. Now we have $f(x_1, x_2, \ldots, x_n) = x_0$ or x_1. The first case implies that $\mathbf{f}$ depends on none of its variables, a contradiction. Hence $f(x_1, \ldots, x_n) = x_1$, i.e., $\mathbf{f}$ is trivial. Thus if an operation in $\mathcal{V}$ is nontrivial, it must be nullary.

III→IV. (11) means that in any algebra of $\mathcal{V}$ an equivalence having a subalgebra among its classes is a congruence. Let $\mathfrak{F}_2$ be freely generated in $\mathcal{V}$ by $\{x_0, x_1, x_2\}$ and denote the subalgebra $[x_0, x_1]$ by $\mathfrak{E}$. If $\mathcal{V}$ has an essentially at least binary term $\mathbf{f}$, then $f(x_0, x_2) \notin \mathfrak{E}$. Define an equivalence ϑ on $\mathfrak{F}_2$ as follows: for $u, v \in \mathfrak{F}_2$ let $u \equiv v(\vartheta)$ if $u = v$ or $u, v \in \mathfrak{E}$. Then (11) implies that ϑ is a congruence and from $x_0 \equiv x_1(\vartheta)$ we have $f(x_0, x_2) \equiv$ $\equiv f(x_1, x_2)(\vartheta)$. As $f(x_0, x_2) \notin \mathfrak{E}$, even $f(x_0, x_2) = f(x_1, x_2)$ holds, whence $\mathbf{f}$ is essentially at most unary, a contradiction.

Suppose $\mathcal{V}$ has no essentially nullary operations. Then $[x_0]$ and $[x_1, x_2]$ are disjoint. As the equivalences having $[x_0]$ as a class are congruences by (11), any equivalence on $[x_1, x_2]$ is a congruence. For $u, v \in [x_1, x_2]$ let $u \equiv v(\psi)$ if $u = x_1$, $v = x_1$ or $u = v$. Now ψ is a congruence on $[x_1, x_1]$, and if $\mathbf{f}$ is an essentially unary operation there we have $f(x_1) \equiv f(x_2)(\psi)$ and thus $f(x_1) = f(x_2)$ holds. This means that $\mathbf{f}$ is essentially nullary, a contradiction showing that $\mathcal{V}$ has no non-trivial operations in this case.

Now assume that $\mathcal{V}$ has essentially nullary operations. Denote by $\mathfrak{D}$ the subalgebra of $\mathfrak{F}_2$ generated by their values. If $\mathcal{V}$ has an essentially unary (non-trivial) operation $\mathbf{g}$, then $x_0 \neq g(x_0)$. If, moreover, $g(g(x_0)) \neq$ $\neq x_0, g(x_0)$, then the equivalence on $\mathfrak{F}_2$ whose classes are D, $\{x_0, g(x_0)\}$ and the rest, is not a congruence there, which contradicts (11). If, finally,

$g(g(x_0)) = x_0$ or $g(g(x_0)) = g(x_0)$, then the equivalence with classes D, $\{x_0, g(x_0), x_1\}$ and the rest furnishes a contradiction. Thus, IV is proved.

THEOREM 4. *All algebras of the variety $\mathcal{V}$ satisfy* (4) *iff $\mathcal{V}$ has at most one at most nullary term (i. e., $\mathcal{V}$ is equivalent to the variety of sets or pointed sets).*

PROOF. Consider the free $\mathcal{V}$-algebra $\mathfrak{F}$ with the free generating set $X = \{x_0, x_1, \dots\}$. Suppose that there exists a term $\mathbf{f}$ which depends essentially on $n(>1)$ variables in $\mathcal{V}$. Then $f(x_n, \dots, x_{2n-1}) \notin X$ and $f(x_n, \dots, x_{2n-1}) \notin [x_0, \dots, x_{n-1}]$. Denote the set $\{x_0, \dots, x_{n-1}, f(x_n, \dots, x_{2n-1})\}$ by D. We have $x_n, \dots, x_{2n-1} \in F \setminus D$ and $f(x_n, \dots, x_{2n-1}) \in F \setminus D$, whence $F \setminus D$ is not a subalgebra in $\mathfrak{F}$. As (4) means that among the classes of any equivalence there is a subalgebra, we may assert that D is a subalgebra in $\mathfrak{F}$. Hence either $f(x_0, \dots, x_{n-1}) = x_i$ $(0 \le i \le n-1)$ or $f(x_0, \dots, x_{n-1}) = f(x_n, \dots, x_{2n-1})$. Both possibilities contradict, however, to the assumption on the essential arity of $\mathbf{f}$.

Let $\mathfrak{F}_1$ be the free $\mathcal{V}$-algebra with free generators x and y. If $\mathfrak{F}_1 = \{x, y\}$, then $\mathcal{V}$ is equivalent to the variety of sets. In the remaining case there exists an at most unary polynomial f of $\mathfrak{F}_1$ with $f(y) \neq y$. The set $\{x, f(y)\}$ is a subalgebra in $\mathfrak{F}_1$; indeed, its complement is not closed under f, and we can apply (4). Hence the unique at most unary term in $\mathcal{V}$ is $\mathbf{f}$, and it identically fulfils $\mathbf{f(x)} = \mathbf{f(y)}$. This means that $\mathcal{V}$ is equivalent to the variety of pointed sets.

On the other hand, sets and pointed sets satisfy (4) obviously.

THEOREM 5. *All algebras of the variety $\mathcal{V}$ satisfy* (6) *iff $\mathcal{V}$ is equivalent to the variety of sets.*

PROOF. (6) means exactly that in any algebra of $\mathcal{V}$ all subsets are subalgebras. Especially, the free generating set of a countably generated free $\mathcal{V}$-algebra is a subalgebra there, whence it follows that $\mathcal{V}$ has no non-trivial terms, i.e., it is equivalent to the variety of sets.

THEOREM 6. *All algebras of the variety $\mathcal{V}$ satisfy* (7) *iff for some natural number n there exist terms $\mathbf{p}_1, \dots, \mathbf{p}_n$; $\mathbf{q}_1, \dots, \mathbf{q}_n$; $\mathbf{f}_1, \dots, \mathbf{f}_n$ (5-ary, ternary and unary, respectively) for which the identities*

$$(\beta) \qquad \mathbf{q}_1(\mathbf{x}, \mathbf{x}, \mathbf{y}) = \dots = \mathbf{q}_n(\mathbf{x}, \mathbf{x}, \mathbf{y}) = \mathbf{y}$$

$$(\beta_0) \qquad \mathbf{x} = \mathbf{p}_1(\mathbf{q}_1(\mathbf{x}, \mathbf{y}, \mathbf{z}), \mathbf{f}_1(\mathbf{z}), \mathbf{x}, \mathbf{y}, \mathbf{z})$$

$$(\beta_i) \qquad \mathbf{p}_i(\mathbf{f}_i(\mathbf{z}), \mathbf{q}_i(\mathbf{x}, \mathbf{y}, \mathbf{z}), \mathbf{x}, \mathbf{y}, \mathbf{z}) = \mathbf{p}_{i+1}(\mathbf{q}_{i+1}(\mathbf{x}, \mathbf{y}, \mathbf{z}), \mathbf{f}_{i+1}(\mathbf{z}), \mathbf{x}, \mathbf{y}, \mathbf{z}) \quad (i = 1, \dots, n-1)$$

$$(\beta_n) \qquad \mathbf{p}_n(\mathbf{f}_n(\mathbf{z}), \mathbf{q}_n(\mathbf{x}, \mathbf{y}, \mathbf{z}), \mathbf{x}, \mathbf{y}, \mathbf{z}) = \mathbf{y}$$

hold in $\mathcal{V}$.

PROOF. (7) means that two congruences coincide provided they have a congruence-class in common, which is a subalgebra. Especially, if an algebra $\mathfrak{A}$ has the property (7) and c is an idempotent element (i.e., an

one-element subalgebra) in $\mathfrak{A}$, then c is a class for the equality relation only. Hence using an argument due to Mal'cev [6], we get that if $a, b \in \mathfrak{A}$ are such that for any translation τ of $\mathfrak{A}$, the equality $\tau(a) = c$ is equivalent to $\tau(b) = c$, then $a = b$.

Now let the algebras of $\mathcal{V}$ fulfil (7) and consider the free $\mathcal{V}$-algebra $\mathfrak{F}_2$ with free generators x, y, z. Call a congruence ϑ on $\mathfrak{F}_2$ good, if all elements of $[z]$ are congruent under ϑ, and whenever anyone of x and y is translated into z by some translation π of $\mathfrak{F}_2$, the other one is translated by π into an element congruent with z under ϑ. The intersection of good congruences is also good; hence the intersection of all good congruences of $\mathfrak{F}_2$ is good, too; denote it by ψ. The class $\bar{z}$ of ψ containing z is a subalgebra in $\mathfrak{F}_2$. Thus $\bar{z}$ is an idempotent element of $\mathfrak{F}_2/\psi$. By the definition of good congruences, the elements $\bar{x}$ and $\bar{y}$ of $\mathfrak{F}_2/\psi$ (containing x and y, respectively) are such that for any translation τ of $\mathfrak{F}_2/\psi$, the equality $\tau(\bar{x}) = \bar{z}$ is equivalent to $\tau(\bar{y}) = \bar{z}$. Hence we get $\bar{x} = \bar{y}$, i.e., $x \equiv y(\psi)$.

Let σ^* be the congruence on $\mathfrak{F}_2$ generated by the relation

$$\sigma = \{\langle z, f(z)\rangle \mid f \in P_1(\mathfrak{F}_2)\} \cup$$
$$\cup \{\langle z, \tau(x)\rangle \mid \tau \in T(\mathfrak{F}_2) \wedge \tau(y) = z\} \cup \{\langle z, \tau(y)\rangle \mid \tau \in T(\mathfrak{F}_2) \wedge \tau(x) = z\}.$$

This definition guarantees that σ^* is good and any congruence of $\mathfrak{F}_2$ properly included in σ^* is not good. Thus, $\sigma^* = \psi$, whence $x \equiv y(\sigma^*)$. Consequently, σ has a finite subset

$$\sigma_1 = \{\langle z, \bar{f}_1(z)\rangle, \dots, \langle z, \bar{f}_k(z)\rangle, \langle z, \tau_1(x)\rangle, \dots, \langle z, \tau_l(x)\rangle,$$
$$\langle z, \varrho_1(y)\rangle, \dots, \langle z, \varrho_m(y)\rangle\} \; (f_i \in P_1(\mathfrak{F}_2); \tau_i, \varrho_i \in T(\mathfrak{F}_2))$$

such that $x \equiv y(\sigma_1^*)$ also holds, where σ_1^* is the congruence generated by σ_1 in $\mathfrak{F}_2$.

Let $\mathbf{t}_i$ $(i = 1, \dots, l)$ and $\mathbf{r}_j$ $(j = 1, \dots, m)$ be such terms that for any $u \in \mathfrak{F}_2$, $\tau_i(u) = t_i(u, x, y, z)$ and $\varrho_j(u) = r_j(u, x, y, z)$. Furthermore, let $\bar{\mathbf{q}}_i(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \mathbf{t}_i(\mathbf{x}, \mathbf{x}, \mathbf{y}, \mathbf{z})$ $(i = 1, \dots, l)$ and $\bar{\mathbf{q}}_{l+j}(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \mathbf{r}_j(\mathbf{y}, \mathbf{x}, \mathbf{y}, \mathbf{z})$ $(j = 1, \dots, m)$. Now for the $\bar{\mathbf{q}}$'s just defined $\bar{\mathbf{q}}(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \mathbf{z}$ holds. Indeed, for $1 \le i \le l$ we have $z = \tau_i(y) = t_i(y, x, y, z)$. Then the identity $\mathbf{z} = \mathbf{t}_i(\mathbf{y}, \mathbf{x}, \mathbf{y}, \mathbf{z})$ is satisfied in $\mathcal{V}$, whence $\bar{\mathbf{q}}_i(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \mathbf{t}_i(\mathbf{x}, \mathbf{x}, \mathbf{x}, \mathbf{z}) = \mathbf{z}$. A similar argument is valid for $\bar{\mathbf{q}}_{l+1}, \dots, \bar{\mathbf{q}}_{l+m}$.

Return to the formula $x \equiv y(\sigma_1^*)$. It implies the existence of elements $x_0 = x, x_1, \dots, x_{n-1}, x_n = y \in \mathfrak{F}_2$ and translations $\gamma_1, \dots, \gamma_n \in T(\mathfrak{F}_2)$ such that for $i = 1, \dots, n$ the pair $\langle x_{i-1}, x_i\rangle$ arises from a pair in σ_1 (where the order of components is now irrelevant) by the translation γ_i. For any $w \in \mathfrak{F}_2$, let $\gamma_i(w) = g_i(w, x, y, z)$, where $\mathbf{g}_i$ is a suitable quaternary term. Now, for $i = 1, \dots, n$, the terms $\mathbf{p}_i, \mathbf{q}_i, \mathbf{f}_i$ will be defined as follows:

If $x_{i-1} = \gamma_i(z)$, $x_i = \gamma_i(\bar{f}_j(z))$, then

$$\mathbf{p}_i(\mathbf{u}, \mathbf{v}, \mathbf{x}, \mathbf{y}, \mathbf{z}) = \mathbf{e}_1^2(\mathbf{g}_i(\mathbf{u}, \mathbf{x}, \mathbf{y}, \mathbf{z}), \mathbf{v}), \quad \mathbf{q}_i(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \mathbf{z}, \quad \mathbf{f}_i(\mathbf{z}) = \bar{\mathbf{f}}_j(\mathbf{z}).$$

If $x_{i-1} = \gamma_i(\tilde{f}_j(z))$, $x_i = \gamma_i(z)$, then

$$\mathfrak{p}_i(\mathfrak{u}, \mathfrak{v}, \mathfrak{x}, \mathfrak{y}, \mathfrak{z}) = \mathfrak{e}_2^2(\mathfrak{u}, \mathfrak{g}_i(\mathfrak{v}, \mathfrak{x}, \mathfrak{y}, \mathfrak{z})), \quad \mathfrak{q}_i(\mathfrak{x}, \mathfrak{y}, \mathfrak{z}) = \mathfrak{z}, \quad \mathfrak{f}_i(\mathfrak{z}) = \tilde{\mathfrak{f}}_j(\mathfrak{z}).$$

If $x_{i-1} = \gamma_i(z)$, $x_i = \gamma_i(\tilde{q}_j(x, y, z))$, then

$$\mathfrak{p}_i(\mathfrak{u}, \mathfrak{v}, \mathfrak{x}, \mathfrak{y}, \mathfrak{z}) = \mathfrak{e}_2^2(\mathfrak{u}, \mathfrak{g}_i(\mathfrak{v}, \mathfrak{x}, \mathfrak{y}, \mathfrak{z})), \quad \mathfrak{q}_i(\mathfrak{x}, \mathfrak{y}, \mathfrak{z}) = \tilde{\mathfrak{q}}_j(\mathfrak{x}, \mathfrak{y}, \mathfrak{z}), \quad \mathfrak{f}_i(\mathfrak{z}) = \mathfrak{z}.$$

If $x_{i-1} = \gamma_i(\tilde{q}_j(x, y, z))$, $x_i = \gamma_i(z)$, then

$$\mathfrak{p}_i(\mathfrak{u}, \mathfrak{v}, \mathfrak{x}, \mathfrak{y}, \mathfrak{z}) = \mathfrak{e}_1^2(\mathfrak{g}_i(\mathfrak{u}, \mathfrak{x}, \mathfrak{y}, \mathfrak{z}), \mathfrak{v}), \quad \mathfrak{q}_i(\mathfrak{x}, \mathfrak{y}, \mathfrak{z}) = \tilde{\mathfrak{q}}_j(\mathfrak{x}, \mathfrak{y}, \mathfrak{z}), \quad \mathfrak{f}_i(\mathfrak{z}) = \mathfrak{z}.$$

The validity of the identities (β), $(\beta_0)-(\beta_n)$ may be checked immediately. Thus the necessity of the above (Mal'cev-type) condition is proved.

To prove the sufficiency, first notice that $(\beta_0)-(\beta_n)$ imply the validity of the identical implication

$$(\beta') \quad \mathfrak{f}_1(\mathfrak{z}) = \ldots = \mathfrak{f}_n(\mathfrak{z}) = \mathfrak{q}_1(\mathfrak{x}, \mathfrak{y}, \mathfrak{z}) = \ldots = \mathfrak{q}_n(\mathfrak{x}, \mathfrak{y}, \mathfrak{z}) = \mathfrak{z} \Rightarrow \mathfrak{x} = \mathfrak{y}$$

in $\mathcal{V}$. Now suppose that for suitable terms $\mathfrak{f}_i$ and $\mathfrak{q}_i$ (β) and (β') hold identically in $\mathcal{V}$. Let $\mathfrak{A}\in\mathcal{V}$ and assume that $\mathfrak{A}$ has a subalgebra $\mathfrak{C}$ which is a class of two different congruences φ_1, φ_2 on $\mathfrak{A}$. We can also suppose $\varphi_1<\varphi_2$. The class C is an idempotent element in $\mathfrak{A}/\varphi_1$. Furthermore, there exist elements $a, b\in\mathfrak{A}$, such that $a\equiv b(\varphi_1)$ but $a\equiv b(\varphi_2)$. Thus the classes $\bar{a}, \bar{b}$ (containing a, resp. b) of the congruence φ_1 are distinct; and the congruence generated by the pair $\langle\bar{a}, \bar{b}\rangle$ on $\mathfrak{A}/\varphi_1$ is less than the congruence induced by φ_2 there. Hence the singleton $\{C\}$ is a class of φ_2. Therefore, any translation of $\mathfrak{A}/\varphi_1$, translating one of the elements $\bar{a}, \bar{b}$ into C translates the other one into C too.

Let us consider the translations $\pi_i(i = 1, \ldots, n)$ of $\mathfrak{A}/\varphi_1$, defined by $\pi_i(w) = q_i(\bar{a}, w, C)$ $(w\in\mathfrak{A}/\varphi_1)$. In virtue of (β), we have $\pi_i(\bar{a}) = q_i(\bar{a}, \bar{a}, C) = C$, whence also $q_i(\bar{a}, \bar{b}, C) = \pi_i(\bar{b}) = C$. By the idempotency of C we infer $f_1(C) = \ldots = f_n(C) = C$. Applying (β'), we get $\bar{a} = \bar{b}$, which contradicts the definition of a and b. Hence all algebras of $\mathcal{V}$ satisfy (7).

THEOREM 7. *All algebras of the variety $\mathcal{V}$ satisfy* (8) *iff for some natural number n there exist terms $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$; $\mathfrak{f}_1, \ldots, \mathfrak{f}_n$ (6-ary, and unary, respectively), for which the identities*

$$\mathfrak{x} = \mathfrak{p}_1(\mathfrak{f}_1(\mathfrak{x}), \mathfrak{f}_1(\mathfrak{y}), \mathfrak{x}, \mathfrak{y}, \mathfrak{x}, \mathfrak{y})$$

$$\mathfrak{p}_i(\mathfrak{x}, \mathfrak{y}, \mathfrak{f}_i(\mathfrak{x}), \mathfrak{f}_i(\mathfrak{y}), \mathfrak{x}, \mathfrak{y}) = \mathfrak{p}_{i+1}(\mathfrak{f}_{i+1}(\mathfrak{x}), \mathfrak{f}_{i+1}(\mathfrak{y}), \mathfrak{x}, \mathfrak{y}, \mathfrak{x}, \mathfrak{y})$$

$$(i = 1, \ldots, n-1)$$

$$\mathfrak{p}_n(\mathfrak{x}, \mathfrak{y}, \mathfrak{f}_n(\mathfrak{x}), \mathfrak{f}_n(\mathfrak{y}), \mathfrak{x}, \mathfrak{y}) = \mathfrak{y}$$

hold in $\mathcal{V}$.

PROOF. It is analogous to that of the preceding theorem, and therefore will be sketched only. (8) means that no distinct subalgebras may be classes of the same congruence. If this condition is fulfilled by the algebra $\mathfrak{F}_1\in\mathcal{V}$, freely generated by $\{x, y\}$, then the minimal congruence ϑ on $\mathfrak{F}_1$, under which any two elements of $[x]$ as well as any two elements of $[y]$ are congruent, fulfils $x\equiv y(\vartheta)$, too. Hence there exists a finite set of unary terms $\tilde{f}$ such that the set ϑ_1 of all pairs of form $\langle x, \tilde{f}(x)\rangle$, $\langle f_i(x), x\rangle$, $\langle y, \tilde{f}(y)\rangle$ and $\langle\tilde{f}(y), y\rangle$ generate a congruence ϑ_1^* for which $x\equiv y(\vartheta_1^*)$. The further steps are left to the reader.

THEOREM 8. *All algebras of the variety $\mathcal{V}$ satisfy* (9) *iff for any n-ary $(n\geq 1)$ term $\mathfrak{f}$ there exists a ternary term $\mathfrak{h}_f$ such that the identity*

$$\mathfrak{f}(\mathfrak{x}_1, \ldots, \mathfrak{x}_n) = \mathfrak{h}_f(\mathfrak{x}_0, \mathfrak{x}_1, \mathfrak{f}(\mathfrak{x}_0, \mathfrak{x}_2, \ldots, \mathfrak{x}_n))$$

holds in $\mathcal{V}$.

As (9) is just the Hamiltonian property, Theorem 8 coincides with Theorem 1 in KLUKOVITS' article [4].

THEOREM 9. *All algebras of the variety $\mathcal{V}$ satisfy* (10) *iff there exists an at most unary term $\mathfrak{f}$ such that for any n-ary $(n\geq 0)$ term $\mathfrak{g}$ the identity*

$$\mathfrak{g}(\mathfrak{f}(\mathfrak{x}), \ldots, \mathfrak{f}(\mathfrak{x})) = \mathfrak{f}(\mathfrak{x})$$

holds in $\mathcal{V}$.

PROOF. (10) means that among the classes of any congruence there is at least one subalgebra. This is the case also for the free $\mathcal{V}$-algebra $\mathfrak{F}_0$, freely generated by x, and for the equality relation. Thus there exists a one-element subalgebra in $\mathfrak{F}_0$ the unique element of which can be written in the form $f(x)$. Obviously, we have $g(f(x), \ldots, f(x)) = f(x)$ for any n-ary g in $\mathfrak{F}_0$, whence the identity in Theorem 9 holds in $\mathcal{V}$.

On the other hand, if the term $\mathfrak{f}$ with the above property exists in $\mathcal{V}$, then in an arbitrary algebra $\mathfrak{A}$ of $\mathcal{V}$ any congruence admits classes which are subalgebras; any class containing an element of form $f(a)$ $(a\in\mathfrak{A})$ will be such a subalgebra.

THEOREM 10. *All algebras of the variety $\mathcal{V}$ satisfy* (12) *iff for any n-ary $(n\geq 0)$ term $\mathfrak{g}$ the identity*

$$\mathfrak{g}(\mathfrak{x}, \ldots, \mathfrak{x}) = \mathfrak{x}$$

holds in $\mathcal{V}$.

PROOF. It is enough to remark that both (12) and the considered identities characterize just the idempotent algebras.