[5] Csákány, B., Homogeneous algebras are functionally complete, *Algebra Universalis*, v. 11, 1980, pp. 149—158.
[6] Csákány, B., Three-element groupoids with minimal clones, *Acta Sci. Math. (Szeged)*, v. 45, 1983, pp. 111—117.
[7] Csákány, B., T. Gavalcová, Finite homogeneous algebras I, *Acta Sci. Math. (Szeged)*, v. 42, 1980, pp. 57—65.
[8] Fried, E., A. F. Pixley, The dual discriminator function in universal algebra, *Acta Sci. Math. (Szeged)*, v. 41, 1979, pp. 83—100.
[9] Grätzer, G., *Universal Algebra*, Van Nostrand, Princeton, 1968.
[10] Jablonskiĭ, S. V., Functional constructions in k-valued logic (Russian), *Trudy Mat. Inst. Steklov*, v. 51, 1958, pp. 5—142.
[11] Kepka, T., Quasi-trivial groupoids and balanced identities, *Acta Univ. Carolin.*, v. 22, 1981. pp. 49—64.
[12] Marčenkov, S. S., J. Demetrovics, L. Hannák, On the closed classes of selfdual functions in P_3 (Russian), *Diskret. Analiz.*, v. 34, 1980, pp. 38—73.
[13] Płonka, J., On algebras with n distinct essentially n-ary operations, *Algebra Universalis*, v. 1, 1971, pp. 73—79.
[14] Płonka, J., R-prime idempotent reducts of groups, *Arch. Math.*, v. 24, 1973, pp. 129—132.
[15] Post, E., *The two-valued iterative systems of mathematical logic*, Ann. of Math. Studies, No. 5, Princeton, 1941.
[16] Pöschel, R., L. A. Kalužnin, *Funktionen- und Relationenalgebren*, DVW, Berlin, 1979.
[17] Rosenberg, I. G., Über die funktionale Vollständigkeit in den mehrwertigen Logiken, *Rozpravy Československé Akad. Věd Ser. Math. Nat. Sci.* v. 80(4), 1970, pp. 3—93.
[18] Świerczkowski, S., Algebras which are independently generated by every n elements, *Fund. Math.*, v. 49, 1960, pp. 93—104.
[19] Takasaki, M., Abstraction of symmetric transformations (Japanese), *Tôhoku Math. J.*, v. 49, 1943, pp. 145—207.
[20] Werner, H., *Discriminator-Algebras*, Akademie-Verlag, Berlin, 1978.

(Received Dec. 12, 1982)

83-2054 — Szegedi Nyomda — F. v.: Dobó József igazgató

Acta Cybernetica, Tom. 6, Fasc. 3, Szeged, 1983

All minimal clones on the three-element set

By B. Csákány

1. Preliminaries

A *clone* on a set M is a set of finitary operations on M which is closed under composition and contains all projections. The clones on M form an algebraic lattice; the atoms and the dual atoms of this lattice are called *minimal clones* and *maximal clones* on M, respectively. A full description of all clones, hence of all minimal and maximal clones for $|M|=2$ was given by Post; a complete list of all maximal clones was found by Jablonskiĭ for $|M|=3$ and by Rosenberg for any finite M (see [15], [10], and [17]). Until now, only special examples of minimal clones were known for the case $|M|>2$. In this paper we determine all minimal clones on a three-element M.

We use the standard universal algebraic terminology [9] except that *function* stands for operation and *term function* for polynomial. All functions (and hence all clones) are defined on the base set $\mathbf{3}=\{0, 1, 2\}$. If f is a function, $[f]$ is the clone generated by f i.e. the clone of all term functions of the algebra $\langle \mathbf{3}; f\rangle$. Projections will also be called *trivial functions*. We use the notation σ for the set of triplets consisting of distinct entries from $\mathbf{3}$ and ι for $\mathbf{3}^3\setminus\sigma$.

In what follows we often make use of functions of the following types 1)—4).

1) *Unary functions.* Such a function f is denoted by u_n, where $n=9.f(0)+3.f(1)+f(2)$.

2) *Binary idempotent functions.* Such a function with the Cayley table

	0	1	2
0	0	n_5	n_4
1	n_3	1	n_2
2	n_1	n_0	2

will be denoted by b_n, where $n=\Sigma_{i=0}^{5} 3^i n_i$.

3) *Majority functions.* A ternary function m satisfying $m(x, x, y)=m(x, y, x)=m(y, x, x)=x$ for any $x, y\in\mathbf{3}$ is called a majority function.

4) *Semiprojections.* A ternary function s is called a semiprojection if there exists a $k\in\mathbf{3}$ such that $s(x_0, x_1, x_2)=x_k$ for arbitrary $\langle x_0, x_1, x_2\rangle\in\iota$.

A function belonging to one of the above four classes will be referred to as a *special function*. For $n>1$ we call an n-ary nontrivial function f *sharp* if $f(x_0, \dots, x_{j-1}, x_i, x_{j+1}, \dots, x_{n-1})=x_{m_{ij}}$ where $0\leqq m_{ij}<n$ for all $i\neq j$, $0\leqq i, j<n$. A binary function is sharp iff it is idempotent. Majority functions and nontrivial semiprojections are sharp ternary. As a trivial consequence of the definition, an n-ary function f is sharp iff all k-ary functions in $[f]$ are trivial provided $k<n$.

A sharp ternary function f is uniquely determined by the values $f(0, 0, 1)$, $f(0, 1, 0)$, $f(0, 1, 1)$, and all $f(\varphi)$ with $\varphi\in\sigma$. Call the numbers

$$\chi(f) = 4.f(0, 0, 1)+2.f(0, 1, 0)+f(0, 1, 1)$$

and

$$\mu(f) = 3^5.f(0, 1, 2)+3^4.f(0, 2, 1)+3^3.f(1, 0, 2)+3^2.f(1, 2, 0)+$$
$$+3.f(2, 0, 1)+f(2, 1, 0)$$

the *characteristic* and *mantissa* of f, and let the pair $\chi(f), \mu(f)$ stand for f. For example 4,44 is Pixley's ternary discriminator function ([20], p. 8) and 1,624 is the dual discriminator [8] on **3**. Observe that $1, t$ $(t=0, \dots, 728)$ are the majority functions, and $0, t$ are the semiprojections with $i=0$. As these ternary functions will play an important role, we also use an alternative notation m_t for $1, t$ and s_t for $0, t$; e.g. m_{728} is the majority function on **3** whose value is 2 on each $\varphi\in\sigma$. Clearly, every majority function or semiprojection f is uniquely determined by the sequence of its values on σ, called the *range* of f; further, the number $v(f)$ of distinct entries in the range of f is called the *variance* of f.

Let φ be a permutation of **3**. To each n-ary function f we assign f^φ, called a *conjugate* of f, defined by $f^\varphi(x_0, \dots, x_{n-1})=(f(x_0\varphi^{-1}, \dots, x_{n-1}\varphi^{-1}))\varphi$. The map $f\to f^\varphi$ carries each clone $\mathscr{C}$ onto the clone $\mathscr{C}^\varphi$; in particular $[f]^\varphi=[f^\varphi]$, and

$$(*) \qquad g\in[f] \quad \text{implies} \quad g^\varphi\in[f^\varphi].$$

We can permute the variables of f as well: for a permutation ψ of $\mathbf{n}$ $(=\{0, \dots, n-1\})$ put $f_\psi(x_0, \dots, x_{n-1})=f(x_{0\psi}, \dots, x_{(n-1)\psi})$. Remark that always $(f^\varphi)_\psi=(f_\psi)^\varphi$; hence we can write simply f_ψ^φ. Note also that $[f_\psi]=[f]$ for any ψ.

The conjugations and permutations of variables generate a permutation group T_n of order $3!n!$ on the set of all n-ary functions on **3**. The classes 1)—4) are closed with respect to T_1, T_2, T_3, and T_3 respectively. Two functions are said to be *essentially distinct* if they have different arities, or belong to distinct orbits of T_n.

We conclude the introduction with the following immediate observation: a nontrivial clone $\mathscr{C}$ is minimal iff $\mathscr{C}=[f]$ for each nontrivial $f\in\mathscr{C}$.

2. The list of minimal clones

First we approximately locate the functions generating minimal clones.

Proposition. *Let $\mathscr{C}$ be a minimal clone and m the minimum arity of nontrivial functions from $\mathscr{C}$. Then $1\leqq m\leqq 3$. If $f\in\mathscr{C}$ is m-ary then f is special; moreover if $f, g\in\mathscr{C}$ are m-ary then both f and g are of the same type i $(1\leqq i\leqq 4)$.*

Proof. The statement is clear if $m=1$. If $m=2$, then each nontrivial binary function $g\in\mathscr{C}$ is sharp and hence idempotent.

Let $m\geqq 3$. First we show that $\mathscr{C}$ contains a sharp ternary function. Indeed, to each sharp at least quaternary f on **3** there exists an i such that $f(x_0, \dots, x_{n-1})=x_i$ $(i\in\mathbf{n})$ whenever $x_0, \dots, x_{n-1}$ are not all distinct ([18]; see also the proof of Thm. 1, § 33 in [9]). This proves that on **3** each sharp function is at most ternary.

Let t be a sharp ternary function from $\mathscr{C}$. We show that $[t]$ contains a nontrivial semiprojection or a majority function. Indeed, if $\chi(t)=0$, 3 or 5 then t itself is a semiprojection, and if $\chi(t)=1$ then t is a majority function. Thus let $\chi(t)\notin\{0, 1, 3, 5\}$. We show that $\chi(g)=6$ for some $g\in[t]$. First if $\chi(t)=2$ $(\chi(t)=7)$ then exchanging the last (first) two variables we obtain r with $\chi(r)=4$. Thus let $\chi(t)=4$. Then the characteristic of

$$t(t(x, y, z),\ t(x, y, t(x, z, y)),\ t(x, z, y))$$

equals 6 (as direct verification or Lemma 1.10 in [20] would show). Let $\chi(t)=6$. Write $t'(x, y, z)$ for $t(t(x, y, z), y, z)$. Then $\chi(t')=0$ and if $t'\neq e_1^3$ $(=0,44)$ we are done. Now if $\mu(t)\neq 44$ then $t(a, b, c)=d\neq a$ for some $\langle a, b, c\rangle\in\sigma$ and therefore $t'(a, b, c)=t(d, b, c)=d\neq a$. It remains to consider $\mu(t)=44$. In this case $t(y, t(z, y, x), z)=0,424$ is the required semiprojection.

We thus have that there is a majority function or a nontrivial semiprojection in $\mathscr{C}$. Taking into account that for g a semiprojection each ternary $f\in[g]$ is a semiprojection and that a similar assertion is valid for majority functions, this proves the first part of the proposition. The second part is implied by the following simple observation: if $i<j$ and the functions f and g are of type i) and j) then $f\notin[g]$.

In virtue of the Proposition our task is to find the different minimal clones generated by functions of the four types above.

1) *Unary functions.* A nontrivial unary function f generates a minimal clone iff either f is a retraction of M (i.e. $f\circ f=f$) or a permutation of prime order ([16], Theorem 4.4.1). The functions u_0 and u_2 are representatives of the two orbits of retractions while u_7 and u_{15} are representatives of the two orbits of prime order permutations. The table below shows the minimal clones generated by unary functions on **3**. The clone standing at the meet of the row starting with $[u]$ and column marked by the permutation φ is $[u]^\varphi$. The place of $[u]^\varphi$ is empty if $[u]^\varphi$ is equal to some $[u]^\psi$ which appeared earlier. One may check directly that the clones in the table are pairwise distinct.

Table 1

	(01)	(02)	(12)	(012)	(021)
$[u_0]$	$[u_{13}]$	$[u_{26}]$			
$[u_2]$	$[u_{14}]$	$[u_8]$	$[u_3]$	$[u_4]$	$[u_{23}]$
$[u_7]$	$[u_{21}]$	$[u_{11}]$			
$[u_{15}]$					

2) *Binary functions.* We proved in [6]: every minimal clone on **3** containing an essentially binary operation is a conjugate of exactly one of the following twelve clones: $[b_i]$ with $i\in\{0, 8, 10, 11, 16, 17, 26, 33, 35, 68, 178, 624\}$.

Table 2 (which is constructed on the same principle as Table 1) displays the minimal clones generated by binary functions on **3**. The clones are pairwise distinct because each $[b]$ in the table contains no nontrivial binary function other than $b(x, y)$ and its dual $b(y, x)$. In other words, the free algebra with two free generators in the variety generated by $\langle \mathbf{3}; b\rangle$ consists of no more than four elements; in this form, our observation may be found in Berman's paper on three element algebras [2]. Thus, it remains to check that Table 2 has no pair of dual functions.

Table 2

	(01)	(02)	(12)	(012)	(021)
$[b_0]$	$[b_{324}]$	$[b_{728}]$			
$[b_8]$	$[b_{368}]$	$[b_{80}]$	$[b_{36}]$	$[b_{40}]$	$[b_{692}]$
$[b_{10}]$	$[b_{280}]$	$[b_{458}]$	$[b_{20}]$	$[b_{448}]$	$[b_{188}]$
$[b_{11}]$	$[b_{286}]$	$[b_{215}]$			
$[b_{16}]$	$[b_{281}]$	$[b_{296}]$	$[b_{47}]$	$[b_{205}]$	$[b_{179}]$
$[b_{17}]$	$[b_{287}]$	$[b_{53}]$	$[b_{38}]$	$[b_{43}]$	$[b_{206}]$
$[b_{26}]$	$[b_{449}]$		$[b_{37}]$		
$[b_{33}]$	$[b_{122}]$	$[b_{557}]$			
$[b_{35}]$	$[b_{125}]$	$[b_{71}]$	$[b_{42}]$	$[b_{41}]$	$[b_{530}]$
$[b_{68}]$	$[b_{528}]$	$[b_{116}]$			
$[b_{178}]$	$[b_{290}]$				
$[b_{624}]$					

The next lemma serves as a tool for handling the remaining two cases. It is a direct consequence of the definition of a minimal clone.

Lemma 1. *Let G, H be subsets of the set F of special functions such that*

I. $H \subseteq G$;

II. $[g] \cap F \subseteq G$ *for every* $g \in G$;

III. $[g] \cap H \neq \emptyset$ *for every* $g \in G$;

IV. *If* $h_1, h_2 \in H$ *and* $h_1 \neq h_2$ *then* $h_1 \notin [h_2]$.

Then $\{[h]: h \in H\}$ *is the set of pairwise distinct minimal clones generated by* $g \in G$.

3. *Majority functions.* For a majority function f generating a minimal clone we have two possibilities:

a) $v(f)=3$. We prove that up to permutation of variables f is the dual discriminator d (i.e. the majority function with $d(a, b, c)=c$ if $\langle a, b, c\rangle \in \sigma$). Following [8], a non-empty binary relation C on $\mathbf{3}$ is called *p-rectangular* if for every pair $\langle i, j\rangle \notin C$ there are no more than two elements in C which have the form $\langle i, x\rangle$ or $\langle y, j\rangle$. First we show that the subalgebras of $\langle \mathbf{3}; f\rangle^2$ are p-rectangular relations on $\mathbf{3}$. In the opposite case, we may suppose without loss of generality (renaming the elements and taking C^{-1} if necessary) that there is a subalgebra C of $\langle \mathbf{3}; f\rangle^2$ such that $\langle 0, 0\rangle \notin C$ but $\langle 0, 1\rangle, \langle 0, 2\rangle, \langle 1, 0\rangle \in C$. By assumption $v(f)=3$, the range of f contains 0, and we may permute the variables of f so that $f(2, 0, 1)=0$. Then $\langle 0, 0\rangle = \langle f(0, 1, 0), f(2, 0, 1)\rangle = f(\langle 0, 2\rangle, \langle 1, 0\rangle, \langle 0, 1\rangle) \in C$, a contradiction. However, d preserves the p-rectangular relations on $\mathbf{3}$ (see [8]), hence it preserves all subalgebras of $\langle \mathbf{3}; f\rangle^2$. Now we can apply the following theorem of Baker and Pixley ([1], see also [20], Theorem 1.2): if a finite algebra $\mathbf{A}$ has a term function which is a majority function, then every function preserving all subalgebras of $\mathbf{A}^2$ is a term function of $\mathbf{A}$. We obtain that d is a term function of $\langle \mathbf{3}; f\rangle$, hence $d \in [f]$. On the other hand, $[d]$ is a minimal clone ([7], Theorem 1), hence $f \in [d]$.

As d is a homogeneous function (i.e. a function preserving all permutations), f must be a homogeneous majority function. Thus f coincides with d up to ordering of variables.

b) $v(f)<3$. There exists $3\cdot 2^6-3=189$ majority functions with this property, and they belong to 10 distinct orbits of T_3. Here we list the representatives having the least index for these orbits (the number in brackets indicates the number of functions constituting the represented orbit):

Table 3

$m_0(3)$	$m_{13}(18)$	$m_{109}(6)$
$m_1(36)$	$m_{28}(36)$	$m_{120}(18)$
$m_4(18)$	$m_{39}(18)$	
$m_{10}(18)$	$m_{85}(18)$	

We prove that the minimal clones generated by majority functions having variance less than 3 are exactly $[m_0]$, $[m_{324}]$ $(m_{324}=(m_0)^{(01)})$, $[m_{728}]$ $(m_{728}=(m_0)^{(02)})$, $[m_{109}]$, $[m_{473}]$ $(m_{473}=(m_{109})^{(02)})$, and $[m_{510}]$ $(m_{510}=(m_{109})^{(12)})$; note that the remaining functions in the orbit of m_{109} may be obtained from the listed ones by permutations of variables and hence do not generate further clones.

Apply Lemma 1. Let F be the set of special functions, G the set of majority functions with variance <3, and $H=\{m_i: i=0, 364, 728, 109, 473, 510\}$. The requirement I is fulfilled by definition. As for II, it is a consequence of the following lemma which will be used once more later.

Lemma 2. *Let f be a majority function whose range does not contain the element a $(\in \mathbf{3})$. Then the same holds for every non-trivial ternary function in $[f]$.*

Proof. Clearly, a non-trivial ternary function in $[f]$ is a majority function. Assume that $[f]$ contains a non-trivial ternary function whose range includes a, and let g be such a function with a shortest f-term: $g(x, y, z)=f(g_0(x, y, z), g_1(x, y, z), g_2(x, y, z))$. For a suitable $\varphi \in \sigma$ we have $g(\varphi)=f(g_0(\varphi), g_1(\varphi), g_2(\varphi))=a$. Thus $\langle g_0(\varphi), g_1(\varphi), g_2(\varphi)\rangle \in \iota$, and hence $g_i(\varphi)=a$ for at least two distinct $i \in \mathbf{3}$. By the minimality of g, these g_i must be trivial and hence both of them equal the same projection. But then g also equals that projection, i.e. g is trivial, a contradiction.

Next we prove III. In virtue of (*) it is enough to show that for each function m_i in Table 3 there is an $m_j \in H$ such that $m_j \in [m_i]$. Write $\hat{f}=\hat{f}(x, y, z)$ for $f(f, f_{(012)}, f_{(021)}) = f(f(x, y, z), f(y, z, x), f(z, x, y))$. Then

$$m_0 = \hat{m}_1 = \hat{m}_4 = \hat{m}_{10} = \hat{m}_{120}$$

and

$$m_{109} = (\hat{m}_{13})_{(01)} = \hat{m}_{28} = (\hat{m}_{39})_{(01)} = \hat{m}_{85},$$

i.e. $m_0 \in [m_1], [m_4], [m_{10}], [m_{120}]$ and $m_{109} \in [m_{13}], [m_{28}], [m_{39}], [m_{85}]$, as required.

Finally, we have to prove that IV is fulfilled, i.e. that none of the functions in H is contained in the clone generated by another one. The unique non-trivial permutation of $\mathbf{3}$ preserved by m_{105} and m_{728} is (01), that preserved by m_{324} and m_{510} is (02), and that preserved by m_0 and m_{473} is (12). Hence no function in one of these three pairs is included in the clone generated by a function appearing in another pair. Furthermore, the ranges of m_{105} and m_{728} have no common entry; thus, by Lemma 2, $m_{109} \notin [m_{728}]$ and $m_{728} \notin [m_{109}]$. Concerning the remaining pairs we can argue in the same way.

Now we see that the conclusion of Lemma 1 also holds. This combined with the result in the case $v(f)=3$ allows us to summarize the minimal clones generated by majority functions as follows:

Table 4

	(01)	(02)	(12)
$[m_0]$	$[m_{324}]$	$[m_{728}]$	
$[m_{109}]$		$[m_{473}]$	$[m_{510}]$
$[m_{624}]$			

4) *Semi-projections.* There are $3\cdot 3^6=2187$ semiprojections, and they belong to 74 distinct orbits of T_3. Table 5 shows representatives of these orbits (the number of functions in the orbit is added in brackets if it differs from 36). Every orbit is represented by its member of characteristic 0 having the least mantissa. For the sake of brevity we write down the mantissa only.

Table 5

0(9)	21(18)	86	108	150(12)
1	22	87	109(18)	153
2	23	88	110	154
4(18)	25	90	111(18)	156
5	26(18)	91	113	157
8(18)	44(3)	92	126	324(18)
10	49	96	127	325(18)
11(18)	50(18)	99	128	342
12	52(18)	100	135	343
13	76(9)	101	136	345
14	81	102	138	346(18)
15(18)	82	103	139(18)	396(6)
16	83	104(18)	140	424(6)
17	84	105	141	426(18)
19(18)	85	106	144	

We prove that the semi-projections generating minimal clones are exactly $s_0, s_8, s_{26}, s_{76}, s_{424}$, and the functions in their orbits. This means that the minimal clones generated by semi-projections are exactly those in the following table (which is constructed in the usual manner).

Table 6

	(01)	(02)	(12)	(012)	(021)
$[s_0]$	$[s_{364}]$	$[s_{728}]$			
$[s_8]$	$[s_{368}]$	$[s_{80}]$	$[s_{36}]$	$[s_{40}]$	$[s_{692}]$
$[s_{26}]$	$[s_{449}]$		$[s_{37}]$		
$[s_{76}]$	$[s_{684}]$	$[s_{332}]$			
$[s_{424}]$					

Denote by S the set of functions appearing in Table 6, and let S_i stand for the set of conjugates of s_i.

For the proof, we again apply Lemma 1. Let F be the set of special functions, G the set of semi-projections, and $H=S$. Clearly, they fulfil I and II. However, the verification of III and IV demands tiresome computations.

Consider two semi-projections, s_i, s_j, which are representatives of orbits of T_3 (i.e. whose indices appear in Table 5). Draw an arrow from s_i to s_j if there exists an s_i-term function which is conjugate to s_j (i.e. $(s_j)^\varphi \in [s_i]$ for some permutation of **3**). To prove III, in view of (*) it is sufficient to produce a set of arrows such that in the resulting oriented graph, for each non-trivial representative s there exists a path which starts from s and ends in one of s_0, s_8, s_{26}, s_{76}, and s_{424}. Such a set of 68 arrows is in the appendix at the end of the paper.

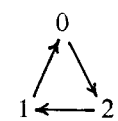
Our concluding task is to prove that for any different functions $s_i, s_j \in S$ $s_j \notin [s_i]$ is valid. We start with a trivial lemma.

Lemma 3. *Let f, g be functions. If there exist a subset K and an element k of some direct power $\mathbf{3}^n$ such that k belongs to the subalgebra of $\langle \mathbf{3}; g\rangle^n$ generated by K but does not belong to the subalgebra of $\langle \mathbf{3}; f\rangle^n$ generated by K, then $g \notin [f]$.*

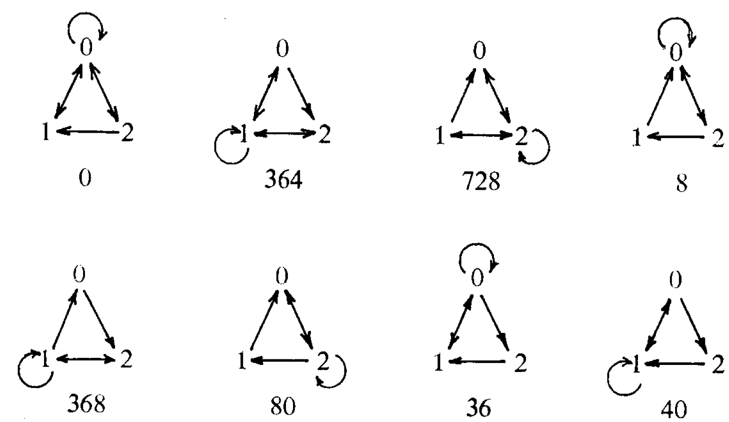
We also need a special case of this lemma.

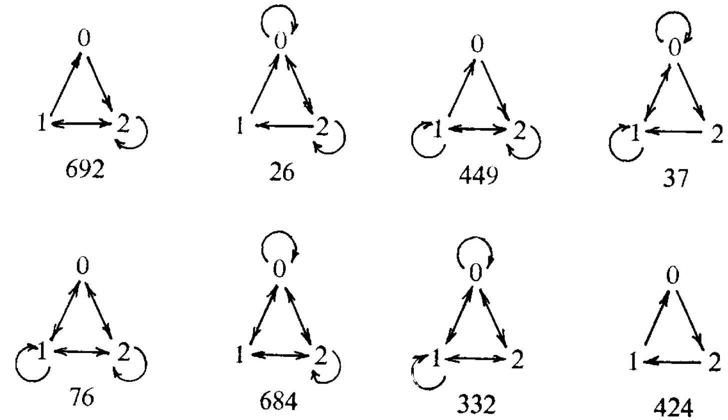
Lemma 3*. *Let f, g be as above. If there exists a permutation of M which is an automorphism of the algebra $\langle M; f\rangle$ but not of the algebra $\langle M; g\rangle$ then $g \notin [f]$* (cf. the proof of IV for majority functions with variance <3).

Apply Lemma 3 for the case $M=\mathbf{3}$, $n=2$, $K=\{\langle 0,2\rangle, \langle 1,0\rangle, \langle 2,1\rangle\}$. The set K may be visualized by means of the figure



Then the subalgebras of $\langle \mathbf{3}; s_i\rangle^2$ generated by K appear in the following figures (i is indicated below the given figure):





It can be read:

α) $[s_{424}] \cap S = \{s_{424}\}$,

β) if $s \in S_8$, $[s] \cap S = \{s\}$,

γ) if $s \in S_0 \cup S_{26}$, then $[s] \cap (S_0 \cup S_{26} \cup S_{76}) = \{s\}$,

δ) if $s \in S_{76}$, then $[s] \cap S_{76} = \{s\}$,

ε) $s_0 \notin [s_{76}]$, $s_{364} \notin [s_{684}]$, $s_{728} \notin [s_{332}]$.

Observe further that for $s_i \in S_0 \cup S_{26} \cup S_{76}$ the algebra $\langle \mathbf{3}; s_i \rangle$ has a non-trivial automorphism, while for $s_j \in S_8$ the algebra $\langle \mathbf{3}; s_j \rangle$ does not. Hence, by Lemma 3*, we have

ζ) if $s_i \in S_0 \cup S_{26} \cup S_{76}$ and $s_j \in S_8$, then $s_j \notin [s_i]$.

The transposition (01) is an automorphism of $\langle \mathbf{3}; s_{332} \rangle$ and not an automorphism of $\langle \mathbf{3}; s_i \rangle$ for $i = 0, 364, 26, 449$. Hence

η) $s_i \notin [s_{332}]$ for $i = 0, 364, 26, 49$.

Similarly, with the aid of (02) we obtain

θ) $s_i \notin [s_{684}]$ for $i = 0, 728, 449, 37$,

and using (12) there follows

ι) $s_i \notin [s_{76}]$ for $i = 364, 728, 26, 37$.

Lemma 3 can be used also to prove

κ) for $s_i \in S$, $i \neq 424$, always $s_{424} \notin [s_i]$.

Indeed, subalgebras of $\langle \mathbf{3}; s_{424} \rangle^2$ must be p-rectangular, since for a subalgebra C and distinct elements $x_1, x_2 \in \mathbf{3}$ it is impossible to have $\langle i, j \rangle \notin C$ and $\langle i, x_1 \rangle$, $\langle i, x_2 \rangle$, $\langle y, j \rangle \in C$ because of $s_{424}(\langle i, x_1 \rangle, \langle y, j \rangle, \langle i, x_2 \rangle) = \langle i, j \rangle$. Now, for $s_i \in S$, $i \neq 424$, consider the subset of $\mathbf{3}^2$ displayed in the above figure with subscript i, and call it K_i. Take this subset for K, s_i for f, and s_{424} for g. Then, by its definition, K_i is a subalgebra of $\langle \mathbf{3}; s_i \rangle^2$ but it is not p-rectangular (see the figure) and thus the subalgebra of $\langle \mathbf{3}; s_{424} \rangle$ generated by K_i contains at least one element $k \notin K_i$. Lemma 3 then applies and gives κ).

The last step is:

λ) $s_{449} \notin [s_{76}]$, $s_{26} \notin [s_{684}]$, $s_{37} \notin [s_{332}]$.

We prove the first assertion, the proof of the others being similar. As the range of s_{449} does not contain 0, we are done if we show that the range of any function in $[s_{76}]$ contains 0. In the contrary case, take the shortest s_{76}-term function t whose range does not contain 0. Let $t(x, y, z) = s_{76}(t_0(x, y, z), t_1(x, y, z), t_2(x, y, z))$. Then $t_0(\varphi) = 0$ for a suitable $\varphi \in \sigma$, and hence $t(\varphi) = 0$, a contradiction.

α)—λ) together mean that for any $s_i, s_j \in S$, $s_j \notin [s_i]$ whenever $i \neq j$. We have verified IV of Lemma 1, hence Table 6 really is the list of all minimal clones on $\mathbf{3}$ generated by semi-projections.

3. Summary

In the preceding part we have proved the following result:

Theorem. *There exist* 84 *minimal clones on* $\mathbf{3}$ *and each of them is a conjugate of exactly one of the following* 24 *clones:*
$[u_0]$, $[u_2]$, $[u_7]$, $[u_{15}]$;
$[b_0]$, $[b_8]$, $[b_{10}]$, $[b_{11}]$, $[b_{16}]$, $[b_{17}]$, $[b_{26}]$, $[b_{33}]$, $[b_{35}]$, $[b_{68}]$, $[b_{178}]$, $[b_{624}]$;
$[m_0]$, $[m_{109}]$, $[m_{624}]$;
$[s_0]$, $[s_8]$, $[s_{26}]$, $[s_{76}]$, $[s_{424}]$.

The other minimal clones may be read from Tables 1, 2, 4, and 6. The Cayley tables of the binary functions in the Theorem may be found in [6]. The ranges of the ternary functions appearing in the Theorem may be seen in the following:

Table 7

f	$f(0,1,2)$	$f(0,2,1)$	$f(1,0,2)$	$f(1,2,0)$	$f(2,0,1)$	$f(2,1,0)$
m_0	0	0	0	0	0	0
m_{109}	0	1	1	0	0	1
m_{624}	2	1	2	0	1	0
s_0	0	0	0	0	0	0
s_8	0	0	0	0	2	2
s_{26}	0	0	0	2	2	2
s_{76}	0	0	2	2	1	1
s_{424}	1	2	0	2	0	1

Several functions occurring here are familiar from earlier research. $\langle \mathbf{3}; b_0 \rangle$ and $\langle \mathbf{3}; b_{10} \rangle$ are the two three-element semilattices. They and $\langle \mathbf{3}; b_{11} \rangle$, $\langle \mathbf{3}; b_{26} \rangle$ (left zero semigroups with an outer zero, resp. unit element) are bands satisfying $xyx = xy$ identically. Hence the minimality of clones they generate is involved by Theorem 4.4.4 in the book of Pöschel and Kalužnin [16]. The minimality of $[b_{178}]$ was proved in [12] by Marčenkov, Demetrovics, and Hannák ($\langle \mathbf{3}; b_{178} \rangle$ is a tournament; it is known as "the paper-stone-scissors algebra"). The algebra $\langle \mathbf{3}; b_{624} \rangle$ is essentially the affine space [4] over GF(3).

The functions b_8, b_{11}, b_{35}, and b_{68} appeared in Płonka's paper [13]; the last one goes back to Takasaki [19]. Kepka deals with b_{16} and b_{17} in [11].

As for the ternary functions, m_0 is the median function on the chain $1 < 0 < 2$, and the minimality of $[m_0]$ is a special case of 4.4.5 in [16]. Finally, the minimality of $[m_{624}]$ and $[s_{424}]$ was established in [7].

Acknowledgements. Thanks are due to Joel Berman, Károly Dévényi, Isidore Fleischer, Péter P. Pálfy, László Szabó and Ágnes Szendrei for their interest and comments. I am especially indebted to Ivo G. Rosenberg for his helpful criticism. This research was partly supported by NSERC Canada grant A-5407. The Centre de recherche de mathématiques appliquées, Université de Montréal, provided excellent conditions for the completion of this work. The computing was done at the Kalmár Laboratory of Cybernetics of the József Attila University, Szeged.

Apology. I am aware that this research may be regarded as something in computer-assisted quasi-mathematics (cf. [3], p. 14). However, I am convinced that the determination of minimal clones is important; hence I feel that the result should be published in spite of the fact that my way is tedious and several steps were obtained through a computer search. Let us consider this paper as a modest step towards the description of minimal clones.

Appendix

$$s_1 \to s_0 = s_1\big(s_1, (s_1)_{(12)}, x\big) = s_1\big(s_1(x, y, z), s_1(x, z, y), x\big)$$
$$s_2 \to s_0 = s_2\big(s_2, (s_2)_{(12)}, y\big)$$
$$s_4 \to s_0 = s_4\big(s_4, (s_4)_{(02)}, x\big)$$
$$s_5 \to s_1 = s_5\big(s_5, (s_5)_{(02)}, (s_5)_{(12)}\big)$$
$$s_{10} \to s_1 = s_{10}\big(s_{10}, (s_{10})_{(12)}, y\big)$$
$$s_{11} \to s_0 = s_{11}\big(s_{11}, (s_{11})_{(12)}, (s_{11})_{(01)}\big)$$
$$s_{12} \to s_1 = s_{12}\big((s_{12})_{(12)}, s_{12}, z\big)$$
$$s_{13} \to s_4 = s_{13}\big(s_{13}, (s_{13})_{(12)}, y\big)$$
$$s_{14} \to s_4 = s_{14}\big(s_{14}, (s_{14})_{(02)}, (s_{14})_{(01)}\big)$$
$$s_{15} \to s_0 = s_{15}\big(s_{15}, (s_{15})_{(021)}, (s_{15})_{(012)}\big)$$
$$s_{16} \to s_4 = s_{16}\big(s_{16}, (s_{16})_{(12)}, (s_{16})_{(012)}\big)$$
$$s_{17} \to s_8 = s_{17}\big(s_{17}, (s_{17})_{(12)}, (s_{17})_{(01)}\big)$$
$$s_{19} \to s_0 = s_{19}\big(s_{19}, (s_{19})_{(12)}, (s_{19})_{(01)}\big)$$
$$s_{21} \to s_1 = s_{21}\big((s_{21})_{(12)}, (s_{21})_{(02)}, s_{21}\big)$$
$$s_{22} \to s_{13} = s_{22}\big(s_{22}, (s_{22})_{(12)}, (s_{22})_{(01)}\big)$$
$$s_{23} \to s_{11} = s_{23}\big(s_{23}, (s_{23})_{(02)}, x\big)$$
$$s_{25} \to s_{23} = s_{25}\big(s_{25}, (s_{25})_{(12)}, (s_{25})_{(02)}\big)$$
$$s_{49} \to s_8 = \big(s_{49}(s_{49}, (s_{49})_{(12)}, (s_{49})_{(02)})\big)^{(021)}$$
$$s_{50} \to s_{49} = s_{50}\big(s_{50}, (s_{50})_{(02)}, y\big)$$
$$s_{52} \to s_{76} = s_{52}\big(s_{52}, (s_{52})_{(12)}, (s_{52})_{(01)}\big)$$
$$s_{81} \to s_0 = s_{81}\big(s_{81}, (s_{81})_{(12)}, y\big)$$
$$s_{82} \to s_1 = s_{82}\big(s_{82}, (s_{82})_{(12)}, y\big)$$
$$s_{83} \to s_0 = s_{83}\big(s_{83}, (s_{83})_{(12)}, y\big)$$
$$s_{84} \to s_1 = s_{84}(s_{84}, x, y)$$
$$s_{85} \to s_1 = s_{85}(s_{85}, x, y)$$
$$s_{86} \to s_2 = s_{86}\big(s_{86}, (s_{86})_{(12)}, (s_{86})_{(012)}\big)$$
$$s_{87} \to s_2 = s_{87}\big((s_{87})_{(12)}, s_{87}, (s_{87})_{(021)}\big)$$
$$s_{88} \to s_4 = s_{88}\big(s_{88}, (s_{88})_{(12)}, y\big)$$
$$s_{90} \to s_0 = s_{90}\big(s_{90}, (s_{90})_{(12)}, y\big)$$
$$s_{91} \to s_1 = s_{91}\big(s_{91}, (s_{91})_{(12)}, y\big)$$
$$s_{92} \to s_0 = s_{92}\big(s_{92}, (s_{92})_{(12)}, y\big)$$
$$s_{96} \to s_5 = s_{96}\big((s_{96})_{(12)}, s_{96}, z\big)$$
$$s_{99} \to s_0 = s_{99}\big(s_{99}, (s_{99})_{(12)}, (s_{99})_{(02)}\big)$$

$$s_{100} \to s_1 = s_{100}\big(s_{100}, (s_{100})_{(12)}, (s_{100})_{(02)}\big)$$
$$s_{101} \to s_2 = s_{101}\big(s_{101}, (s_{101})_{(12)}, (s_{101})_{(02)}\big)$$
$$s_{102} \to s_{15} = s_{102}\big(s_{102}, (s_{102})_{(021)}, (s_{102})_{(02)}\big)$$
$$s_{103} \to s_{16} = s_{103}\big(s_{103}, (s_{103})_{(021)}, (s_{103})_{(02)}\big)$$
$$s_{104} \to s_{26} = s_{104}\big(s_{104}, (s_{104})_{(12)}, y\big)$$
$$s_{105} \to s_2 = s_{105}\big((s_{105})_{(12)}, x, (s_{105})_{(021)}\big)$$
$$s_{106} \to s_{25} = s_{106}\big(s_{106}, (s_{106})_{(12)}, (s_{106})_{(01)}\big)$$
$$s_{108} \to s_{17} = s_{108}\big(x, z, (s_{108})_{(02)}\big)$$
$$s_{109} \to s_{26} = \big(s_{109}(s_{109}, y, (s_{109})_{(12)})\big)^{(12)}$$
$$s_{110} \to s_{15} = s_{110}\big((s_{110})_{(12)}, (s_{110})_{(02)}, s_{110}\big)$$
$$s_{111} \to s_{109} = s_{111}\big(s_{111}, x, (s_{111})_{(12)}\big)$$
$$s_{113} \to s_{16} = s_{113}\big((s_{113})_{(12)}, (s_{113})_{(02)}, s_{113}\big)$$
$$s_{126} \to s_{111} = s_{126}\big(s_{126}, (s_{126})_{(021)}, (s_{126})_{(02)}\big)$$
$$s_{127} \to s_{109} = s_{127}\big(s_{127}, (s_{127})_{(012)}, x\big)$$
$$s_{128} \to s_{126} = s_{128}\big(s_{128}, (s_{128})_{(12)}, (s_{128})_{(012)}\big)$$
$$s_{135} \to s_0 = s_{135}\big(s_{135}, (s_{135})_{(021)}, (s_{135})_{(012)}\big)$$
$$s_{136} \to s_{135} = s_{136}\big(s_{136}, (s_{136})_{(012)}, z\big)$$
$$s_{138} \to s_{135} = s_{138}\big(s_{138}, (s_{138})_{(02)}, (s_{138})_{(012)}\big)$$
$$s_{139} \to s_{26} = s_{139}\big(x, s_{139}, (s_{139})_{(12)}\big)$$
$$s_{140} \to s_{26} = s_{140}\big((s_{140})_{(12)}, (s_{140})_{(02)}, s_{140}\big)$$
$$s_{141} \to s_{26} = s_{141}\big(x, s_{141}, (s_{141})_{(12)}\big)$$
$$s_{144} \to s_{90} = s_{144}\big(s_{144}, (s_{144})_{(12)}, (s_{144})_{(02)}\big)$$
$$s_{150} \to s_{396} = s_{150}\big(s_{150}, (s_{150})_{(01)}, (s_{150})_{(12)}\big)$$
$$s_{153} \to s_{135} = s_{153}\big(s_{153}, (s_{153})_{(02)}, (s_{153})_{(021)}\big)$$
$$s_{154} \to s_{150} = s_{154}\big(x, (s_{154})_{(02)}, s_{154}\big)$$
$$s_{156} \to s_{105} = s_{156}\big(s_{156}, (s_{156})_{(02)}, y\big)$$
$$s_{157} \to s_{17} = \big(s_{157}(x, s_{157}, (s_{157})_{(02)})\big)^{(021)}$$
$$s_{324} \to s_{111} = s_{324}(s_{324}, x, z)$$
$$s_{325} \to s_{111} = s_{325}(s_{325}, x, z)$$
$$s_{342} \to s_{111} = s_{342}(s_{342}, x, z)$$
$$s_{343} \to s_{19} = s_{343}\big(x, s_{343}, (s_{343})_{(12)}\big)$$
$$s_{345} \to s_{19} = s_{345}\big(x, s_{345}, (s_{345})_{(12)}\big)$$
$$s_{346} \to s_{26} = s_{346}\big(x, s_{346}, (s_{346})_{(12)}\big)$$
$$s_{396} \to s_{424} = s_{396}\big(s_{396}, (s_{396})_{(02)}, x\big)$$
$$s_{426} \to s_{111} = s_{426}(s_{426}, x, z)$$

BOLYAI INSTITUTE
A. JÓZSEF UNIVERSITY
ARADI VÉRTANÚK TERE 1
SZEGED, HUNGARY
H-6720

References

[1] BAKER, K. A, A. F. PIXLEY, Polynomial interpolation and the Chinese Remainder Theorem for algebraic systems, *Math. Zeitschrift,* v. 143, 1975, pp. 165—174.
[2] BERMAN, J., Free spectra of 3-element algebras, Preprint.
[3] BONSALL, F. F., A down-to-earth view of mathematics, *Amer. Math. Monthly,* v. 89, No. 1, 1982. pp. 8—15.
[4] CSÁKÁNY, B., Affine spaces over prime fields, *Acta Sci. Math. (Szeged)*, v. 37, 1975, pp. 33—36.