Algebra univers. **54** (2005) 73–89
0002-5240/05/010073 – 17
DOI 10.1007/s00012-005-1924-2

Algebra Universalis

Minimal clones—a minicourse

BÉLA CSÁKÁNY

ABSTRACT. This paper provides an elementary introduction to minimal clones, as well as a survey of recent trends and results.

1. Clones

Clone is an English word of Greek origin, widely used, mainly due to its importance in biology and, as a consequence, in science fiction. In Greek, it means "slip, twig". In the recent nonmathematical usage, it stands for "population of genetically identical organisms derived originally from a single individual by asexual methods" (Encyclopaedia Britannica), or, in other contexts, "a person or thing that duplicates, imitates, or closely resembles another in appearance, function, etc." (Webster's College Dictionary). In a mathematical monograph, to my knowledge, it appeared for the first time in 1965, namely in the monograph of P. M. Cohn [4], who attributed the notion to Ph. Hall. Among the predecessors, E. Post and K. Menger should be mentioned. Instead of *clone*, we find *function algebra* or *iterative Post algebra* in several essential publications.

The notion of a clone generalizes that of a monoid. By a *monoid* we mean a set of selfmaps of a set S that is closed under composition and contains the identical mapping. Similarly, *a clone is a set of (finitary) operations on a set S that is closed under composition and contains all the projection mappings.*

Composition forms from one k-ary operation f and k n-ary operations $g_1, \dots, g_k$, an n-ary operation $f(g_1, \dots, g_k)$ defined by

$$f(g_1, \dots, g_k)(a_1, \dots, a_n) = f(g_1(a_1, \dots, a_n), \dots, g_k(a_1, \dots, a_n)),$$

for all $a_i \in S$. Here k and n are arbitrary positive integers. For $k = n = 1$, composition is the usual product of selfmaps.

Presented by G. Grätzer.
Received September 27, 2002; accepted in final form October 10, 2002.
2000 *Mathematics Subject Classification*: 08-02, 08A02, 08A05, 08A62.
Key words and phrases: Clones, finite algebras, composition of operations.
Research supported by OTKA of Hungary under grant number T037877.

Projections are operations $(a_1, \ldots, a_m) \mapsto a_i$ with fixed $(1 \leq)$ i $(\leq m)$. For $i = m = 1$, the unique projection is the identical mapping. Projections are often referred to as trivial operations.

Basic examples:

(1) The set O_S of all operations on the set S (the full clone).
(2) The set P_S of all projections on the set S (the trivial clone).
(3) All term operations of an algebra (e.g., all linear combinations over a vector space).
(4) All polynomial operations of an algebra (e.g., all polynomials over a field).
(5) All continuous operations on a topological space (e.g., all continuous real functions).
(6) All operations monotone in each variable on a partially ordered set (called *the clone of* that *partial order*).
(7) All idempotent operations on a set (f is *idempotent* if $f(x, \ldots, x) = x$ holds identically).
(8) All conservative operations on a set S (f is *conservative* if, for $a_i \in S$, $f(a_1, \ldots, a_n) \in \{a_1, \ldots, a_n\}$); such an operation is called also a *quasiprojection*).
(9) All operations commuting with every operation of an algebra (called the *centralizer* of the algebra; it generalizes the endomorphism monoid).
(10) All operations preserving some (finitary) relations on a set (e.g., all operations invariant under some permutations of the set; (10) is a generalization of (8) and (9)).

For finite sets, all clones are of form (10). This basic fact may be formulated as follows: *preserving a relation induces a Galois correspondence between operations and relations, in which the closed classes of operations are exactly the clones*; see [1], [19].

People who prefer to define monoids as associative groupoids with unit element may ask whether clones could also be defined abstractly, e.g., by means of identities. To accomplish this, we may consider a clone C a "many-sorted algebra" with countably many base sets $C_1, C_2, \ldots$, $k+1$-ary basic operations with $s^k_n \colon C_k \times C^k_n \to C_n$ called *superpositions* (for k, $n = 1, 2, \ldots$), n-ary basic operations, $p^i_n \in C_n$ (for all n, i with $1 \leq i \leq n < \infty$) called projections, and identities that reflect the behavior of composition of operations and projection operations. (E.g., $s^n_k(p^i_k, f_1, \ldots, f_k) = f_i$ holds identically, for $f_1, \ldots, f_k \in C_n$.) Such a many-sorted algebra is an *abstract clone* [4], [57]. Clones of operations as introduced above are sometimes called *concrete clones*. They can be considered as abstract clones: for $i = 1, 2, \ldots$, the base set C_i is the set of all i-ary operations.

Isomorphism of clones can be introduced in a natural way. Every abstract clone is isomorphic to a concrete clone of operations [4]. The proof is essentially the same as that of the Cayley representation theorem for monoids; however, it is intrinsically clumsy. We can also define subclones, quotient clones, etc. There is a close connection between varieties and clones: term operations of an algebra generating a variety V form a clone C_V, and the correspondence $V \leftrightarrow C_V$ is one-to-one between varieties and clones (up to term equivalence of varieties and isomorphism of clones; cf. [58]). If $V' \subseteq V$, then $C_{V'}$ is a quotient clone of C_V.

There exists another way to define clones of operations ([22], [39], [45]). Consider the following procedures which yield a new operation from given operation(s):

- *Substitution* forms from any m-ary f and n-ary g on a set S, an $m+n-1$-ary operation $f * g$ defined by
$$(f * g)(a_1, \ldots, a_n, a_{n+1}, \ldots, a_{m+n-1}) = f(g(a_1, \ldots, a_n), a_{n+1}, \ldots, a_{m+n-1})),$$
for $a_1, \ldots, a_{m+n-1} \in S$.
- *Permutations of variables* form from any n-ary f, the n-ary operations f_π defined by
$$f_\pi(a_1, \ldots, a_n) = f(a_{\pi(1)}, \ldots, a_{\pi(n)}),$$
for $a_1, \ldots, a_n \in S$; here π is an arbitrary permutation of the set $\{1, \ldots, n\}$.
- *Identification of variables* forms from any n-ary f an $n-1$-ary operation $\triangle f$ defined by
$$(\triangle f)(a_1, a_2, \ldots, a_{n-1}) = f(a_1, a_1, a_2, \ldots, a_{n-1}),$$
for $a_1, \ldots, a_n \in S$.
- *Introduction of a fictitious variable* forms from any n-ary f, an $n+1$-ary operation $\triangledown f$ defined by
$$(\triangledown f)(a_1, a_2, \ldots, a_{n+1}) = f(a_2, \ldots, a_{n+1}),$$
for $a_1, \ldots, a_n, a_{n+1} \in S$.

Clones are closed under the above four procedures; a fact we shall often use in our considerations. Moreover, *a set of operations is a clone if and only if it is closed under these procedures and contains the identical mapping.* The proof consists of producing the results of the four procedures by the use of compositions and projections, and vice versa. It is a nice introductory exercise. Note that these procedures are binary and unary operations; thus, clones can be considered also as usual (not many-sorted) algebras.

In what follows, if not stated otherwise, clone always means a clone of operations on a finite set. A set consisting of n elements will be identified with $\mathrm{n} = \{0, 1, \ldots, n-1\}$.

2. Lattices of clones

Let f and g be operations on a set S such that g can be obtained from f and projections by (finitely many) compositions. (This means that g is a term operation of the algebra $(S; f)$.) In this case, we say that f *generates* g. All operations f generates form a clone C; we also say that f *generates* C, and we write $[f]$ for C.

This terminology can be extended to sets F of operations, instead of a single operation f. Generating clones is an algebraic closure operator on the set O_S of all operations on S. Thus, *the clones of operations on S form an algebraic lattice.*

There are countably many clones on the two-element set, and their lattice is completely known (Post [44]). There is a continuum of clones on a set containing at least three elements and a full description of this lattice seems to be hopeless. Nevertheless, several intervals—mainly in the top and bottom regions—have been determined. In particular, a complete list of the dual atoms—called *maximal clones*—was given for the two-element set by Post ([44]), for the three-element set by Jablonskiĭ ([23]), and finally, for every finite set, by Rosenberg ([49]). There is a short proof by Kuznecov for the fact that there are only finitely many maximal clones ([27]). Every clone is contained in a maximal one (easy: on $\mathbf{n}$, the Sheffer-Webb operation $\max(x,y)+1 \pmod n$ generates all operations; thus, the assertion follows from Zorn's Lemma).

The atoms of the lattice of all clones on a set are called *minimal clones.* In the following sections we provide an overview of our knowledge of minimal clones.

3. The basics of minimal clones

The definition implies immediately that *a clone C is minimal if and only if each nontrivial $f \in C$ generates any nontrivial $g \in C$.* Equivalently, a clone C is minimal if and only if there exists a nontrivial $f \in C$ such that f generates any nontrivial $g \in C$ and any nontrivial $g \in C$ generates f. It follows that in order to find all distinct minimal clones contained in a clone D, it is enough to find a set of nontrivial operations $F \subseteq D$ with the following properties:

- For every nontrivial $g \in D$, there exists an $f \in F$, such that g generates f.
- If $f_1, f_2 \in F$ are distinct, then f_1 does not generate f_2.

The distinct minimal clones in D are generated by the distinct operations $f \in F$.

A standard technique to prove that f does not generate g is to find a relation ρ such that f preserves ρ and g does not preserve ρ (since the property of preserving a relation is inherited by composition of operations).

Usual tricks to prove that an operation f on a set S does not generate a minimal clone:

- To find a clone C and a nontrivial $g \in C$ such that $f \notin C$ and f generates g.
- To find an operation g such that f generates g, and $[g]$ is not minimal.
- To find subalgebras $(A; f)$, $(B; f)$ of $(S; f)$, and an operation g such that $g \in [f]$, $(A; g) \cong (B; g)$, but $(A; f) \ncong (B; f)$.

Nontrivial operations of minimal arity in a minimal clone are called *minimal operations.* Examples of minimal operations:

- A unary operation which is a retraction or a cyclic permutation of prime order.
- A semilattice operation.
- The *dual discriminator* d of Fried and Pixley [18], defined by $d(x,y,z) = x$, if $x = y$, and $d(x,y,z) = z$, if $x \neq y$ (it is a *majority operation*: $d(x,x,y) = d(x,y,x) = d(y,x,x)$, identically).
- The operation $m(x,y,z) = x+y+z \pmod 2$ on $\mathbf{2}$ (it is a *minority operation*: $m(x,y,y) = m(y,x,y) = m(y,y,x) = y$, identically).
- An n-ary nontrivial *nearprojection* s on an n-element set; nearprojections are defined by $s(x_1, x_2, \dots, x_n) = x_1$, if $x_1, x_2, \dots, x_n$ are *not* pairwise distinct (in general, operations with this property are called *semiprojections*), and $s(x_1, x_2, \dots, x_n) = x_i$, with fixed i, otherwise [5]. Note that semiprojections and nearprojections are meant up to permutations of variables.

Rosenberg's Classification Theorem. *Every minimal operation is of one of the following types:*

(1) *a unary operation,*
(2) *a binary idempotent operation,*
(3) *a ternary majority operation,*
(4) *a ternary minority operation,*
(5) *an n-ary semiprojection $(n > 2)$.*

Remarks. We prove Rosenberg's Classification Theorem (RCT in the sequel) in the above form. The original formulation in Rosenberg [50] also states that in case (1) the operation is a retraction or a cyclic permutation of prime order (easy!), and in case (4) the minority operation is necessarily $x + y + z$ in an elementary 2-group (not easy!).

The preceding examples show that Rosenberg's classes are nonempty.

Usually, we call a minimal clone generated by a k-ary minimal operation a *k-ary minimal clone.* Applying this terminology, RCT can be formulated as a classification of minimal clones.

RCT shows that all nonunary minimal operations are idempotent.

In general, a minimal operation contained in a minimal clone is not unique.

Proof. Suppose that f is an at least binary minimal operation. Then f is idempotent, else it generates (by identification of variables) the nonidentical unary operation $f(x, \ldots, x)$, which cannot generate f.

Now suppose that f is ternary. Then by any identification of two variables, we obtain a projection. There are eight possibilities:

$$\begin{array}{lcl} f(x,x,y) & = & x \ x \ x \ x \ y \ y \ y \ y \\ f(x,y,x) & = & x \ x \ y \ y \ x \ x \ y \ y \\ f(y,x,x) & = & x \ y \ x \ y \ x \ y \ x \ y \end{array}$$

In the first and eighth cases f is a majority, resp., minority operation. In the second, third, and fifth cases, f is a ternary semiprojection. The remaining cases cannot occur. Indeed, then f generates a majority operation; e.g., in the sixth case $m(x, y, z) = f(x, f(x, y, z), z)$ will do. However, a nontrivial ternary t generated by a majority operation m is also a majority operation (use induction on the number of occurrences of m in t).

Finally, let f be at least quaternary. Again, by identifying any two of its variables, f turns into a projection. Then, by the Świerczkowski Lemma, f is a semiprojection. □

The Świerczkowski Lemma [51]. *Given an at least quaternary operation, if every operation arising by identification of its variables is a projection, then these projections coincide.*

For a proof, see the Appendix.

From RCT we can deduce that *on a given finite set there are only finitely many minimal clones.* Indeed, if $m > n$, then an m-ary semiprojection on an n-element set is necessarily a projection. It follows that on a finite set there are only finitely many minimal operations, and this implies the statement. (It is trivial that there are infinitely many minimal clones on any infinite set.)

From the Świerczkowski Lemma it follows that *every clone on a finite set contains at least one minimal clone.* Here is a proof [24]: Call a nontrivial clone C *n-special*, if every nontrivial operation in C is at least n-ary, and there exists an n-ary operation in C that generates C. Any nontrivial clone B on a finite set M contains n-special clones for some n: a nontrivial operation of minimal arity in B generates such a clone. By the Świerczkowski Lemma, here n cannot exceed the number of elements of M. This implies that B contains only finitely many special clones. They are partially ordered by inclusion. A clone which is minimal with respect to this ordering is a minimal clone.

This is a proof without tools. An equally simple proof with some tools: in the Galois connection mentioned in Section 1, clones correspond to closed classes of relations (it is not necessary to know what they are!). Hence it suffices to prove that every closed class of relations is contained in a maximal one. As there exist relations that are contained only in the closed class of all relations, this follows from Zorn's Lemma. (On the set $\mathbf{n}$ $(n > 2)$, $\{(a, b, c) \mid a \leq b \neq c\}$ is the needed relation; for $n = 2$, the proposition is included in Post's description of all clones.)

On an infinite set there exist clones containing no minimal clone. A trivial example is the clone generated by the (unary) successor function on the set of positive integers. Another example is the clone generated by any nontrivial nearprojection.

4. Some history, objective and subjective (1941–1982)

The full list of minimal operations generating distinct minimal clones on $\mathbf{2}$: the unary constants 0 and 1, the transposition $(0, 1)$, the binary maximum and minimum operations, and the ternary majority and minority operations (Post [44]).

The first nontrivial series of minimal clones was described in Płonka [41]. (He called algebras with minimal clone of term operations *R-prime*, that is, reduct-prime.) His example is the clone of the full idempotent reduct of a(ny) vector space (in short: *the clone of an affine space*) over a(ny) finite prime field. Up to isomorphism, this is the clone C_p of all convex linear combinations, that is, operations of form

$$a_1x_1 + \cdots + a_nx_n,$$

with $\sum a_i = 1$, on the set of all residue classes modulo p. For the proof of minimality, note that it is easy for $p = 2$: all nontrivial convex linear combinations are of form $x_1 + \cdots + x_{2n+1}$ $(n \geq 1)$, and they generate each other. For $p > 2$, it suffices to verify the following propositions:

(1) $m = m(x, y, z) = x - y + z$ generates all binary operations in C_p.
(2) For any $0 < k < p$, the set B of all binary operations in C_p generates $x - ky + kz$.
(3) B generates C_p.
(4) Any nontrivial operation in C_p generates a binary nontrivial operation.
(5) If $f \in C_p$ is binary and nontrivial, then f generates m.

Hints: 1. $ax + (1 - a)y = m(x, y, m(x, y, \ldots m(x, y, x) \ldots))$, with $a - 1$ occurrences of m. 2. $x - ky + kz = u(vx + (1 - v)y) + (1 - u)(wy + (1 - w)z)$, for appropriate u, v, w. 3. Induction on arity; apply the preceding observation. 4. Identify variables. 5. Write xy for $f(x, y) = ax + (1 - a)y$, and use the rightwards rule for parentheses: $xyz = (xy)z$. Then $x - y + z = xy^{c-1}(zy^{d-1})$, where $c \equiv o(a) \pmod p$, $d \equiv o(1 - a) \pmod p$.

The first monograph to summarize some facts on minimal clones was the Pöschel-Kalužnin book [45]. It contains, e.g., the minimality of the (lower) median term operation in lattices, a version of the Świerczkowski Lemma, finiteness of the set of minimal clones, and the fact that the lattice union of all minimal clones is the clone of all operations. We recall Problem 12 in that book: *Describe all minimal clones and determine how many there are.*

In 1982, I succeeded in determining all minimal clones on the three-element set. (It was the topic of my Czechoslovakian Summer School talk in 1982.) I started with the notion of a pattern operation due to Quackenbush [47], and the functional completeness theorems for algebras with polynomial ternary discriminator or dual discriminator operations (Werner [61] and Fried-Pixley [18]). Recall that

- two n-tuples $(a_1, \dots, a_n)$, $(b_1, \dots, b_n)$ are of the same *pattern* if, for $i, j \in \{1, \dots, n\}$, $a_i = a_j$ if and only if $b_i = b_j$;
- an operation f is a *pattern operation*, if $f(a_1, \dots, a_n) = a_i$, $i \in \{1, \dots, n\}$, (i.e., f is *conservative*), and this i depends only upon the pattern of $a_1, \dots, a_n$ (note that nearprojections are pattern operations, and all pattern operations on a set form a clone);
- the *ternary discriminator* t is defined by the rule: $t(a, b, c) = c$, if $a = b$; $t(a, b, c) = a$, otherwise.

The theorems of Werner and Fried-Pixley state that the ternary discriminator together with all constant operations generate all operations, and so does the dual discriminator. The two discriminators are pattern operations. I extended these theorems to all pattern operations by proving the following [5]:

(1) Any pattern operation generates either the dual discriminator or some nontrivial nearprojection.
(2) Any nontrivial nearprojection together with all constant operations generate all operations.

Later, I noticed that a nontrivial nearprojection generates any other nearprojection of greater or equal arity [9], and then realized that property (1) of pattern operations provides all minimal clones consisting of pattern operations. Encouraged by this observation, I then took up the study of minimal clones on the three-element set. Knowing RCT, this may seem not too hard; however, there was no RCT at that time. Thus, I had to discover its special case for a three-element set, and examine each operation in order to find the minimal ones. As there are 729 binary idempotent operations, the same number of majority operations, and 2,187 semiprojections on **3**, the task looked overwhelming. So I was compelled to learn some Pascal programming (first version, not Turbo!). As a result, I have found all the 152 minimal operations, the 84 minimal clones, and the 24 essentially distinct minimal clones: 4 unary, 12 binary, and 8 ternary clones (3 generated by majority operations and 5 by semiprojections). (Here we call two clones on a set S *essentially distinct*, if no permutation of S induces isomorphism between them. In this sense, e.g., there are five essentially distinct minimal clones on the two-element set.)

In the last quarter of 1982, we had close co-operation with Rosenberg. In this period he proved RCT; I determined all minimal conservative binary operations and majority operations [8].

A minimal conservative operation f on S restricted to any subset A of S provides a minimal conservative operation f_A on A. If f is n-ary, it is completely determined by its restrictions to all n-element subsets of S. In the binary case, each restriction to a two-element subset is either a semilattice operation (say, $\vee$) or one of the binary projections (up to isomorphism). Call the system of these restrictions the *spectrum* of f. *A conservative binary operation is minimal if and only if its spectrum contains* $\vee$ *but not both projections.*

The number of possible restrictions of a minimal conservative majority operation to a three-element subset is 12 (up to isomorphism). They can be subdivided into three classes, C_0, C_1, C_2, of one, three, and eight elements, respectively. A majority operation f on **3** is determined by the sequence

$$f(0,1,2), f(0,2,1), f(1,0,2), f(1,2,0), f(2,0,1), f(2,1,0).$$

The single operation in the class C_0 is one with constant determining sequence; C_1 consists of the majority nearprojections; C_2 is the set of all majority operations f such that

- the determining sequence of f does not contain 2;
- the transposition (01) is an automorphism of the algebra $(\mathbf{3}; f)$.

A conservative majority operation is minimal if and only if its spectrum contains at most one operation from each C_i $(i = 0, 1, 2)$. As Quackenbush noted [46], this description provides us with a polynomial time algorithm which decides whether a conservative majority operation generates a minimal clone.

5. Some history (continued)

Unary or minority minimal operations cannot be conservative. Minimal conservative semiprojections have not yet been completely determined, except for n-ary ones on n-element sets (Ježek-Quackenbush [24]). Call a binary relation ρ *bitransitive*, if either ρ is an equivalence relation with all nonsingleton classes of the same size, or ρ is a directed bipartite graph whose automorphism group is transitive on the set of arrows. *For any bitransitive relation* ρ *on* $\mathbf{n}$*, the semiprojection* s *defined by*

$$s(a_1, \dots, a_n) = \begin{cases} a_n, & \text{if } a_1, \dots, a_n \text{ are distinct and } (a_1, a_n) \in \rho; \\ a_1, & \text{otherwise,} \end{cases}$$

generates a minimal clone, and every minimal clone consisting of conservative operations on $\mathbf{n}$ *is generated by such an operation. Distinct bitransitive relations determine distinct minimal clones.*

For $\mathbf{4}$, this result provides all quaternary minimal semiprojections. We have no practical description of the ternary minimal clones on the four-element set, not even in the conservative case. This is the last (and very big) gap in determining all minimal clones on $\mathbf{4}$. Indeed, the unary and the minority case are trivial, while the remaining parts of the problem were solved not long ago.

The binary case was settled by Szczepara in his 210 page long Ph.D. thesis [53]. There are 2,182 binary minimal clones on $\mathbf{4}$, and 120 of them are essentially distinct. Szczepara used Pálfy's method ([40], see later) by exhibiting six systems of identities such that a binary operation on $\mathbf{4}$ is minimal if it satisfies one of these systems, and every minimal clone on $\mathbf{4}$ is generated by an operation satisfying one of these systems. The simplest such system of identities is that of the rectangular bands; another simple example is $x^2 = x,\ x(yz) = y(xz),\ (xy)z = yz,\ x(xy) = y$.

All minimal clones of majority operations on $\mathbf{4}$ were determined by Waldhauser in [59]. It is a bit surprising that, up to isomorphism, there are only 12 nonconservative operations among their generators. Even these operations preserve three out of the four three-element subsets of $\mathbf{4}$. They are constants on pairwise distinct triplets on two three-element subsets while on the remaining one subset they are exactly the 12 distinct majority operations on $\mathbf{3}$.

We have observed that if a k-ary operation is minimal on $\mathbf{n}$ ($n \geq 3$), then $k \leq n$. The converse is also true: *if* $k \leq n$ *and* $n \geq 3$, *then there exists a* k*-ary minimal operation on* $\mathbf{n}$ (Pálfy [40]). For $k = n$, consider a clone generated by an n-ary semiprojection. It contains a minimal clone which is generated by an n-ary minimal operation. If $k < n$, let $b_1, \dots, b_k, b_{k+1}$ be distinct elements of $\mathbf{n}$, then the k-ary semiprojection f defined by

$$f(a_1, \dots, a_k) = \begin{cases} b_{k+1}, & \text{if } a_1 = b_1 \text{ and } \{a_2, \dots, a_k\} = \{b_2, \dots, b_k\}; \\ a_1, & \text{otherwise,} \end{cases}$$

is a minimal operation.

Pálfy exhibited some identities valid in the algebra $(\mathbf{n}; f)$ which guarantee that

(∗) for every nontrivial term operation g on $(\mathbf{n}; f)$, the operation f is a term operation on the algebra $(\mathbf{n}; g)$

(i.e., every g generated by f generates f). He also observed that the fulfilment of (∗), that is, the minimality of $[f]$, means that an appropriate system of identities (one identity for every nontrivial term operation) is valid on $(\mathbf{n}; f)$. This implies:

- *Minimality is invariant under isomorphism, hence minimal clones—in contrast with the maximal ones—can be investigated as abstract clones.*

- *If* $[f]$ *is a minimal clone, then each nontrivial algebra in the variety generated by* $(\mathbf{n}; f)$ *has a minimal clone of term operations.*

Pálfy's observation on identities has impacted the investigation of minimal clones very favorably, because it has connected the preceding research (of somewhat combinatorial flavor) with the main trends of universal algebra.

Another example of k-ary clones on n-element sets is a subclone of the clone of a suitable partial order. Let $\mathbf{n}$ be the union of an $n - k + 2$-element chain L and a $k - 2$-element antichain $\{b_3, \dots, b_k\}$. Then the monotone semiprojection f defined by

$$f(a_1, \dots, a_k) = \begin{cases} a_2, & \text{if } a_1 \neq a_2,\ a_1, a_2 \in L \text{ and } \{a_3, \dots, a_k\} = \{b_3, \dots, b_k\}; \\ a_1, & \text{otherwise,} \end{cases}$$

is a minimal operation (Lengvárszky [29]).

In [30], Lévai and Pálfy applied the method of identities to the problem of determining all minimal binary clones with exactly n binary operations. Here $n \geq 3$ because the two projections are included. For $n = 3$, there are two such clones: that of Płonka's affine space over the three-element field, and the clone generated by a commutative binary operation satisfying the two-variable identities of semilattices (in short: the clone of 2-semilattices). The cases $n = 4$, 6 are also settled (for $n = 4$, the simplest example is the clone of rectangular bands). Dudek proved that, for $n = 5$, 7, there exists a unique minimal clone with n binary operations (that of affine spaces over the n-element field), and conjectured that this is the case for all prime numbers ([15], [16]). However, Lévai and Pálfy exhibited very tricky minimal clones with $3k+2$ ($k \geq 2$) binary operations, distinct from the clone of affine spaces in case $3k + 2$ is a prime. It is not known whether there exists a minimal abstract clone with infinitely many binary operations.

The fact that the lattice union of all minimal clones on $\mathbf{n}$ is the full clone ([45]) was improved first by Szabó [52]: for prime n, those generated by any full cycle and the median operation (with respect to the lattice operations max and min) are enough, while for composite n we can choose three minimal clones generated by the dual discriminator and the cycles $(0\ 1\ \cdots\ p-1)$ and $(n-p\ \ n-p+1\ \cdots\ n-1)$, with $n/2 < p < n$. However, this unexpected appearance of number theory can be avoided: two minimal operations are enough for any n (Czédli–Halaš–Kearnes–Pálfy–Szendrei [12]). We always need a fixpoint-free permutation π of prime order, and, for a composite $n \neq 4$, 8, an appropriate rectangular band operation, for $n = 8$, some 2-semilattice operation; and Szczepara's list helps in case $n = 4$: up to permutations of $\mathbf{4}$, there exists a unique minimal binary operation f such that π and f together generate the full clone on $\mathbf{4}$. In [38], Machida and Rosenberg gave a necessary and sufficient condition for a pair of minimal operations to generate the full clone.

The minimal operation $m(x,y,z) = x - y + z$ of an affine space is a *Mal'cev operation*, that is, it fulfils the identities $m(x,x,y) = m(y,x,x) = y$ (which, by a fundamental observation of Mal'cev, guarantee permutable congruences in varieties). The converse is also true: if a Mal'cev operation m generates a minimal clone on a set S, then this clone is that of an affine space on S, and $m(x,y,z) = x-y+z$. (Szendrei [55], for finite sets, but it is true also for infinite S, Kearnes [25]). Of course, $|S|$ is a prime power in the finite case. The result of Szendrei implies also the hardest part of RCT: a minimal minority operation is $x+y+z$ on an affine space over the two-element field. By RCT, a group operation is not minimal. A quasigroup operation can be minimal, e.g., binary operations of affine spaces are minimal quasigroup operations. The Szendrei-Kearnes result shows that this is the unique example because quasigroups have Mal'cev operations (namely, $m(x,y,z) = (x/(y\backslash y))(y\backslash z)$).

Kearnes' result on Mal'cev operations is a byproduct of his characterization theorem for Abelian minimal clones [25]. A clone A is *Abelian* if for every n-ary operation $f \in A$ and elements a, b, $c_2, \ldots, c_n$, $d_2, \ldots, d_n$

$$f(a, c_2, \ldots, c_n) = f(a, d_2, \ldots, d_n)$$

implies that

$$f(b, c_2, \ldots, c_n) = f(b, d_2, \ldots, d_n).$$

Up to isomorphism, Abelian minimal clones are the same as minimal clones of term operations of finite cyclic groups. In [60], Waldhauser showed that in this theorem the Abelian property can be replaced by a weaker-but-similar-in-spirit requirement.

The operations f and g (m-ary and n-ary, resp.) *commute*, if, for any matrix

$$\begin{matrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{matrix}$$

the equation

$$g(f(a_{11}, \ldots, a_{m1}), \ldots, f(a_{n1}, \ldots, a_{mn})) = f(g(a_{11}, \ldots, a_{1n}), \ldots, g(a_{m1}, \ldots, a_{mn}))$$

holds (cf. Example (9) in Section 1). An operation is commutative, if it commutes with itself. A commutative operation generates a clone C in which any two operations commute; in this case, C is called a *commutative clone*. Unary minimal clones are commutative; a majority operation cannot be commutative; a minimal minority operation is always commutative. The remaining hard cases of commutative minimal clones are completely determined in Kearnes-Szendrei [26]. In the binary case,

they belong to six types (affine spaces, rectangular bands, and semilattices are the simple types). The semiprojection case is the most difficult; it can be dealt with by introducing a natural semimodule structure on the set of minimal operations of the given clone.

6. Variations and generalizations

A clone is *essentially minimal*, if its nontrivial subclones are generated by unary operations (Machida, [32], [33]). An operation of minimal arity generating a given essentially minimal clone is also called essentially minimal. An at least binary essentially minimal operation f on S is either idempotent—and then it generates a minimal clone—or $a \mapsto f(a, \ldots, a)$ is a nonsurjective selfmap of S. (Otherwise, iterating f, we obtain a nontrivial idempotent operation g which cannot generate f.) A thorough investigation of essentially minimal binary operations was carried out by Machida and Rosenberg in [37].

A set of partial operations on a set is a *partial clone*, if it is closed under composition and it contains all projections. Minimal partial clones on finite sets S are completely determined (Börner–Haddad–Pöschel [2]). A minimal partial clone is either a minimal clone or a partial clone generated by a partial n-ary projection whose domain is a totally symmetric and totally reflexive proper subset R of S^n. (R is *totally symmetric*, if it is invariant under permutations of components of the vectors $(a_1, \ldots, a_n) \in R$; R is *totally reflexive*, if it contains all vectors whose components are not pairwise distinct.) On any finite set S, there are three minimal partial clones whose union generates the partial clone of all partial operations on S (Haddad–Machida–Rosenberg [21]).

Composition is not the unique natural way of forming new operations from given ones. We feel that the unary inverse operation in groups is somehow "generated" by the multiplication, however, not by composition. For group elements a, b, $b = a^{-1}$ means that the formula $\exists x(xab = x)$ is true. Formulae of such type are called *primitive positive*. Thus, over group theory, the inverse operation is expressible from the multiplication by a primitive positive formula. Expressibility by a primitive positive formula (in short: *parametrical expressibility*) is more effective than composition: $f(g_1, \ldots, g_k)(a_1, \ldots, a_n) = b$ is the same as

$$\exists x_1 \ldots \exists x_k (g_1(a_1, \ldots, a_n) = x_1 \wedge \cdots \wedge g_k(a_1, \ldots, a_n) = x_k \wedge f(x_1, \ldots, x_k) = b).$$

It follows that sets of operations containing all projections and closed under parametrical expressibility are clones; we call them *primitive positive clones*. The lattice of primitive positive clones on 2 and 3 element sets—including minimal primitive positive clones—are known (Kuznecov [28]; Danilčenko [13]). This lattice is finite for any finite set $\mathbf{n}$ (Burris-Willard [3]); nevertheless, we do not know all minimal positive primitive clones for $n > 3$.

A minimal operation $[f]$ on S is *doubly minimal*, if also the algebra $(S; f)$ is minimal, that is, it has no proper subalgebras except, possibly, one-element subalgebras. Without requiring finiteness, all such algebras are on the following list: two-element zero-semigroup and semilattice, **2** with majority operation, **p** with a full cycle, and the p-element affine space [10].

For a binary operation $\circ$, the sequences $x_1 \circ x_2 \circ x_3$, $x_1 \circ x_2 \circ x_3 \circ x_4, \dots, x_1 \circ x_2 \circ \dots \circ x_n, \dots$ can be bracketed in various ways, counted by the Catalan numbers 2,5,14,.... The bracketings give rise to various term operations which, however, can coincide for distinct bracketings. The sequence of numbers of distinct term operations arising in this way is the *associative spectrum* of $\circ$; it "measures" how far $\circ$ is from being associative. Binary minimal operations on **3** were investigated from the point of view of associativity: five out of the 12 are associative, four of them are Catalan (i.e., distinct bracketings induce distinct term operations), (2,4,8,...) is the associative spectrum of two, and finally, the associative spectrum of the unique binary operation of the 3-element affine space is $([2^n/3])$ $(n \geq 3)$ (see [11]).

Appendix. The Świerczkowski Lemma

Świerczkowski proved his lemma in the following form: *If $n > 3$, and any n elements of an at least n-element algebra form a basis (i.e., a free generating system of that algebra), then the algebra is trivial.* Since then several people have rediscovered it in the equivalent form, given at the proof of RCF (Harnau, Danilčenko, etc.).

In order to prove the lemma, it is enough to show that, for any at least quaternary operation f that always turns into some projection, if we identify at least two of its variables, the projections $f(x_1, x_1, x_3, x_4, \dots)$ and $f(x_1, x_2, x_1, x_4, \dots)$ coincide. For other identifications of nondistinct pairs of variables, apply permutations of variables; for identification of distinct pairs of variables, apply the above twice.

Now, if $f(x_1, x_1, x_3, x_4, \dots)$ is the i-th and $f(x_1, x_2, x_1, x_4, \dots)$ is the j-th n-ary projection, then—up to a permutation of variables—we have the following four cases to discuss (in fact, to refute):

$$\begin{array}{rcl} i & = & 1\ 3\ 3\ 4 \\ j & = & 4\ 2\ 4\ 5 \end{array}$$

(of course, the last case only for an at least 5-ary f). Indeed, in each operation under consideration the set V of variables is the union of the following three sets:

- I, the set of variables identified (with some other variable);
- O, the set of variables identified in the other operation;
- R, the set of the remaining variables.

There are six possibilities:

$$\begin{array}{ccccccccc} & & (1) & (2) & (3) & (4) & (5) & (6) \\ i & \in & I & I & I & O & O & R \\ j & \in & I & O & R & O & R & R \end{array}$$

In the cases (1) and (2), the projections obtained by identification coincide. Cases (3), ..., (6) are represented—in this order—by the choice of variables i and j in the above four cases. We show that each of these four cases leads to a contradiction. For the sake of brevity, we write $x_1x_2\dots$ instead of $f(x_1, x_2, \dots)$. Case (3):

$$\begin{array}{lcl} x_1x_1x_3x_4\dots = x_1 & \Rightarrow & x_1x_1x_1x_4\dots = x_1, \\ x_1x_2x_1x_4\dots = x_4 & \Rightarrow & x_1x_1x_1x_4\dots = x_4, \end{array}$$

and the same simple trick works for the cases (4) and (6). Case (5):

$$\begin{array}{lclcl} x_1x_1x_3x_4\dots = x_3 & \Rightarrow & x_1x_1x_3x_1\dots = x_3 & \Rightarrow & x_1x_2x_3x_1\dots = x_3, \\ x_1x_2x_1x_4\dots = x_2 & \Rightarrow & x_1x_2x_1x_1\dots = x_2 & \Rightarrow & x_1x_2x_3x_1\dots = x_2. \end{array}$$

References

[1] V. G. Bodnarčuk, L. A. Kalužnin, V. N. Kotov, and B. A. Romov, *Galois theory for Post algebras I–II.*, Kibernetika (Kiev) (1969), no. 3, 1–10; (1969), no. 5, 1–9. (Russian.)

[2] F. Börner, L. Haddad, and R. Pöschel, *Minimal partial clones*, Bull. Austral. Math. Soc. **44** (1991), 405–415.

[3] S. Burris and R. Willard, *Finitely many primitive positive clones*, Proc. Amer. Math. Soc. **101** (1987), 427–430.

[4] P. M. Cohn, *Universal algebra*, Harper & Row, Publishers, New York-Evanston-London, 1965. (Second edition: Mathematics and Its Applications, 6, D. Reidel, Publishing Co., Dordrecht-Boston, 1981.)

[5] B. Csákány, *Homogeneous algebras are functionally complete*, Algebra Universalis **11** (1980), 149–158.

[6] B. Csákány, *Three-element groupoids with minimal clones*, Acta Sci. Math. (Szeged) **45** (1983), 111–117.

[7] B. Csákány, *All minimal clones on the three-element set*, Acta Cybernet. **6** (1983), 227–238.

[8] B. Csákány, *On conservative minimal operations*, Lectures in universal algebra (Szeged, 1983), Colloq. Math. Soc. János Bolyai, 43, North-Holland, Amsterdam, 1986, pp. 49–60.

[9] B. Csákány and T. Gavalcová, *Finite homogeneous algebras I*, Acta Sci. Math. (Szeged), **42** (1980), 57–65.

[10] B. Csákány and K. A. Kearnes, *Algebras whose subalgebras and reducts are trivial*, Acta Sci. Math. (Szeged), **63** (1997), 379–381.

[11] B. Csákány and T. Waldhauser, *Associative spectra of binary operations*, Mult.-Valued Log. **5** (2000), 175–200.

[12] G. Czédli, R. Halaš, K. A. Kearnes, P. P. Pálfy, and Á. Szendrei, *The join of two minimal clones and the meet of two maximal clones*, Algebra Universalis **45** (2001), 161–178.

[13] A. F. Danilčenko, *On parametrical expressibility of the functions of k-valued logic*, Finite algebra and multiple-valued logic (Szeged, 1979), Colloq. Math. Soc. János Bolyai, 28, North-Holland, Amsterdam, 1981, pp. 147–159.

[14] J. Demetrovics and I. A. Mal'cev, *Essentially minimal TC-clones on three-element base set*, C. R. Math. Rep. Acad. Sci. Canada **8** (1986), no. 3, 191–196.
[15] J. Dudek, *The unique minimal clone with three essentially binary operations*, Algebra Universalis **27** (1990), 261–269.
[16] J. Dudek, *Another unique minimal clone*, preprint, 1993.
[17] J. Dudek, *Minimal clones with commutative operations*, manuscript.
[18] E. Fried and A. F. Pixley, *The dual discriminator function in universal algebra*, Acta Sci. Math. (Szeged) **41** (1979), 83–100.
[19] D. Geiger, *Closed systems of functions and predicates*, Pacific J. Math. **27** (1968), 95–100.
[20] G. Grätzer, *Universal algebra*, D. Van Nostrand, Co., Inc., Princeton-Toronto-London, 1968. (Second edition: Springer-Verlag, New York-Heidelberg, 1979.)
[21] L. Haddad, H. Machida, and I. G. Rosenberg, *Maximal and minimal partial clones*, J. Autom. Lang. Comb. **7** (2002), 83–93.
[22] S. W. Jablonski, G. P. Gawrilow, and W. B. Kudrjawzew, *Boolesche Funktionen und Postsche Klassen*, Akademie-Verlag, Berlin, 1970.
[23] S. V. Jablonskiĭ, *Functional constructions in the k-valued logic*, Tr. Mat. Inst. Steklova **51** (1958), 5–142. (Russian.)
[24] J. Ježek and R. W. Quackenbush, *Minimal clones of conservative functions*, Internat. J. Algebra Comput. **5** (1995), 615–630.
[25] K. A. Kearnes, *Minimal clones with Abelian representations*, Acta Sci. Math. (Szeged) **61** (1995), 59–76.
[26] K. A. Kearnes and Á. Szendrei, *The classification of commutative minimal clones*, Discuss. Math. Algebra Stochastic Methods **19** (1999), 147–178.
[27] A. V. Kuznecov, *Structures with closure and criteria for functional completeness*, Uspekhi Mat. Nauk **16** (1961), 201–202.
[28] A. V. Kuznecov, *On detecting non-deducibility and non-expressibility*, Logical deduction, Nauka, Moscow, 1979, pp. 5–33. (In Russian.)
[29] Zs. Lengvárszky, *A note on minimal clones*, Acta Sci. Math. (Szeged) **50** (1986), 335–336.
[30] L. Lévai and P. P. Pálfy, *On binary minimal clones*, Acta Cybernet. **12** (1996), 279–294.
[31] H. Machida, *Essentially minimal closed sets in multiple-valued logic*, Trans. IECE Japan **E64** (1981), 243–245.
[32] H. Machida, *A theorem on essential minimality in k-valued logic*, Trans. IECE Japan **E65** (1982), 123–124.
[33] H. Machida, *Toward a classification of minimal closed sets in 3-valued logic*, Proc. 12th Int. Symp. Multiple-Valued Logic IEEE (1982), pp. 4–7.
[34] H. Machida and I. G. Rosenberg, *Classifying essentially minimal clones*, Proc. 14th Int. Symp. Multiple-Valued Logic IEEE (1984), pp. 4–7.
[35] H. Machida and I. G. Rosenberg, *Essentially minimal groupoids*, Proc. 15th Int. Symp. Multiple-Valued Logic IEEE (1985), pp. 338–344.
[36] H. Machida and I. G. Rosenberg, *A "large" essentially minimal clone over an infinite set*, Proceedings Int. Conf. Algebra, Part 3 (Novosibirsk, 1989), pp. 159–167. (Contemp. Math., 131, Part 3, Amer. Math. Soc., Providence, 1992).
[37] H. Machida and I. G. Rosenberg, *Essentially minimal groupoids*, Algebras and orders (Montreal, 1991), Kluwer Acad. Publ., Dordrecht, 1993, pp. 287–316.
[38] H. Machida and I. G. Rosenberg, *Gigantic pairs of minimal clones—characterization and existence*, Mult.-Valued Log. **7** (2001), 129–148.
[39] A. I. Mal'cev, *Iterative Post algebras*, Novosibirsk. Gos. Univ., Novosibirsk, 1976. (Russian.)
[40] P. P. Pálfy, *The arity of minimal clones*, Acta Sci. Math. (Szeged) **50** (1986), 331–333.
[41] J. Płonka, *R-prime idempotent reducts of groups*, Arch. Math. (Basel) **24** (1973), 129–132.
[42] J. Płonka, *On groups in which idempotent reducts form a chain*, Colloq. Math. **29** (1974), 87–91.

[43] J. Płonka, *On k-cyclic groupoids*, Math. Japon. **30** (1985), 371–382.
[44] E. Post, *The two-valued iterative systems of mathematical logic*, Annals of Mathematics Studies, no. 5, Princeton University Press, Princeton, 1941.
[45] R. Pöschel and L. A. Kalužnin, *Funktionen- und Relationenalgebren*, Mathematische Monographien, VEB Deutsche Verlag der Wissenschaften, Berlin, 1979.
[46] R. W. Quackenbush, *A survey of minimal clones*, Aequationes Math. **50** (1995), 3–16.
[47] R. W. Quackenbush, *Some classes of idempotent functions and their compositions*, Colloq. Math. **29**(1974), 71–81.
[48] A. Romanowska and B. Roszkowska, *Representations of n-cyclic groupoids*, Algebra Universalis **26** (1989), 7–15.
[49] I. G. Rosenberg, *La structure des fonctions de plusieurs variables sur un ensemble fini*, C. R. Acad. Sci. Paris **260** (1965), 3817–3819.
[50] I. G. Rosenberg, *Minimal clones I: the five types*, Lectures in universal algebra (Szeged, 1983), Colloq. Math. Soc. János Bolyai, 43, North-Holland, Amsterdam, 1986, pp. 405–427.
[51] S. Świerczkowski, *Algebras which are independently generated by every n elements*, Fund. Math. **49** (1960/61), 93–104.
[52] L. Szabó, *On minimal and maximal clones I–II.*, Acta Cybernet. **10** (1992), 323–327; **13** (1998), 405–411.
[53] B. Szczepara, *Minimal clones generated by groupoids*, Ph.D. Thesis, Université de Montréal, 1995.
[54] Á. Szendrei, *Clones in universal algebra*, Presses de l'Université de Montréal, Montreal, 1986.
[55] Á. Szendrei, *Idempotent algebras with restrictions on subalgebras*, Acta Sci. Math. (Szeged) **51** (1987), 251–268.
[56] M. B. Szendrei, G. Czédli, and Á. Szendrei, *Exercises in abstract algebra*, Tankönyvkiadó, Budapest, 1985. (Hungarian.)
[57] W. Taylor, *Abstract clone theory*, Algebras and orders (Montreal, 1991), Kluwer Acad. Publ., Dordrecht, 1993, pp. 507-530.
[58] W. Taylor, *Characterizing Mal'cev conditions*, Algebra Universalis **3** (1973), 351–397.
[59] T. Waldhauser, *Minimal clones generated by majority operations*, Algebra Universalis **44** (2000), 15–26.
[60] T. Waldhauser, *Minimal clones with weakly abelian representations*, Acta Sci. Math. (Szeged) **69** (2003), 505–521.
[61] H. Werner, *Eine Characterisierung funktional vollständiger Algebren*, Arch. Math. (Basel) **21** (1970), 381–385.

Béla Csákány

Béla Csákány
University of Szeged, Bolyai Institute, Szeged H-6720, Hungary